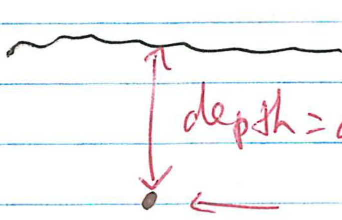
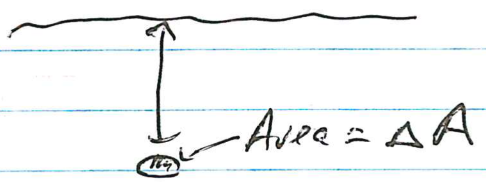
Nov. 5, 2019

Sect. 13.4. Fluid Pressure.

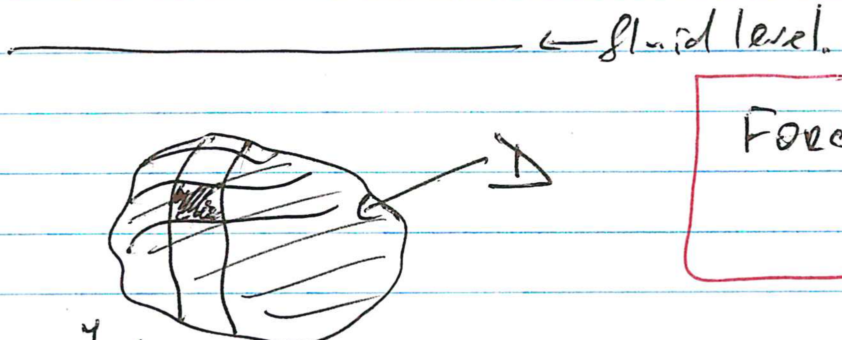


Pressure $p = g \rho d$

density (ρ), depth (d), $g = 9.81\ m/s^2$

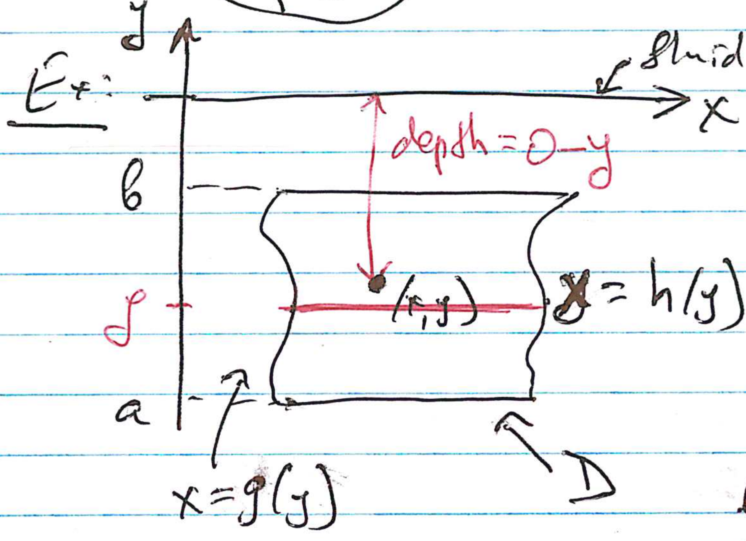


$$\text{Force} \approx p \Delta A = g \rho d \Delta A$$



$$\boxed{\text{Force} = \iint_D p \, dA}$$

Ex:



(ρ - uniform density).

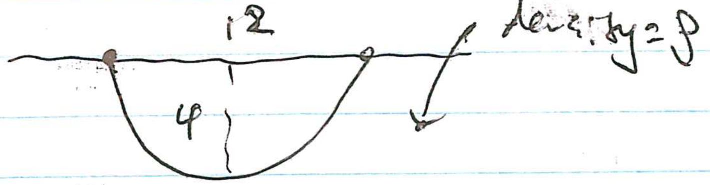
$$F = \iint_D p(x,y)\, dA$$

$$p(x,y) = g \rho (-y)$$

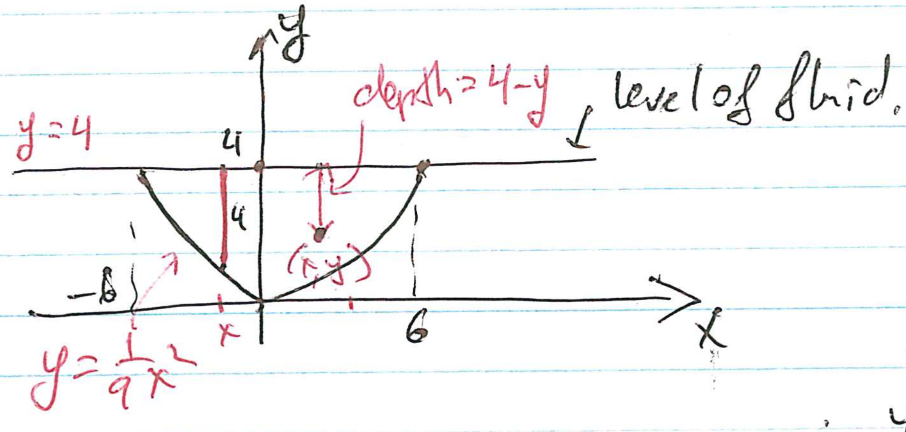
$$F = \iint_D g \rho (-y)\, dA = \int_a^b \int_{g(y)}^{h(y)} g \rho (-y) \, dx \, dy =$$

$$= \underbrace{g \rho \int_a^b (-y)[h(y) - g(y)]\, dy}$$

#2, p. 915 A parabolic segment of base 12 and height 4 with the base in the surface.



Choose the system of coordinates (that you are comfortable working with).



$y = ax^2$ s.t. it passes through $(6,4)$

$\therefore\ 4 = a\cdot 6^2 \iff a = \frac{4}{36} = \frac{1}{9}$

$\therefore\ y = \frac{1}{9}x^2$

$$F = \iint_D p(x,y)\,dA = \left\{ \begin{aligned} p(x,y) &= g\rho\,\text{depth} = \\ &= g\rho(4-y) \end{aligned} \right\} =$$

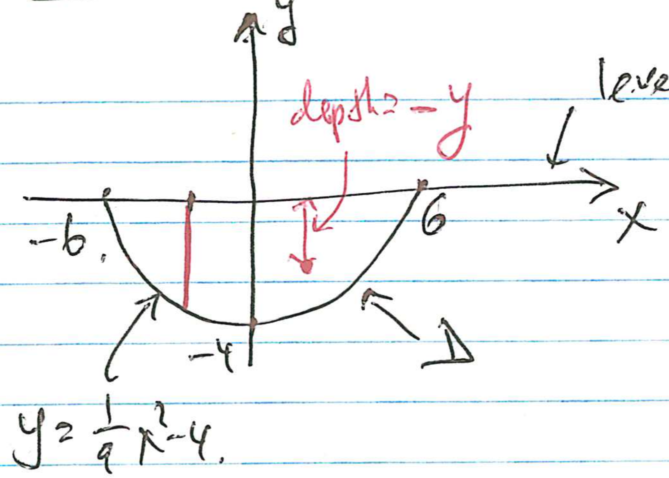
$$= \iint_D g\rho(4-y)\,dA =$$

$$= \int_{-6}^{6} \int_{\frac{1}{9}x^2}^{4} g\rho(4-y)\,dy\;dx = \left\{ \begin{matrix} \text{By} \\ \text{symmetry} \end{matrix} \right\}$$

$$= 2\int_0^6 g\rho\left[-\frac{(4-y)^2}{2}\Bigg|_{\frac{1}{9}x^2}^{4}\right] \cancel{dy}\,dx =$$

$$= 2g\rho \int_0^6 \frac{1}{2}\left(4 - \frac{1}{9}x^2\right)^2 dx = \ldots$$

Method II (the x-axis is at the surface of fluid)



$y = ax^2 - 4$, $a \in \mathbb{R}$, passes through $(6,0)$.

$\therefore\ 0 = a \cdot 6^2 - 4 \iff a = \frac{4}{36} = \frac{1}{9}$

$\therefore\ \boxed{y = \frac{1}{9}x^2 - 4}$

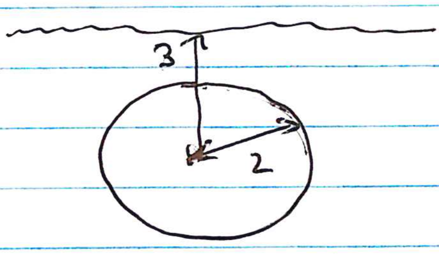
$$F = \iint_D P(x,y)\, dA = \iint_D \rho g(-y)\, dA =$$

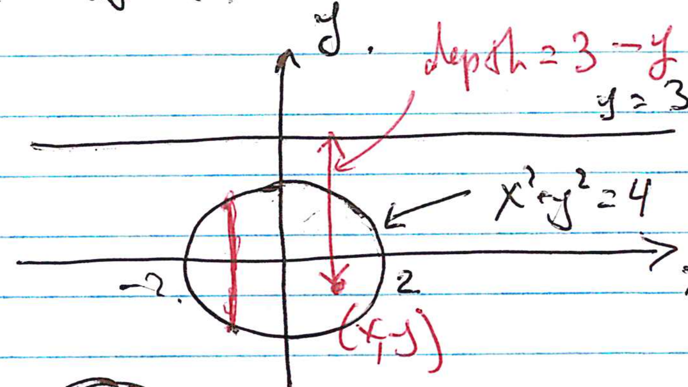
$$= \int_{-6}^{6} \int_{\frac{1}{9}x^2-4}^{0} \rho g(-y)\, dy\, dx =$$

$$= 2\int_0^6 \rho g \left(-\frac{1}{2}y^2\right)\Bigg|_{y=\frac{1}{9}x^2-4}^{y=0} dx =$$

$$= 2\rho g \int_0^6 \frac{1}{2}\left(\frac{1}{9}x^2-4\right)^2 dx = \dots$$

#11, p. 915 A circle of radius 2 with centre 3 units below the surface.





$$F = \iint_D \rho g(3-y)\, dA =$$

$$= \iint_D \rho g(3-y)\,dA, \text{ where } D=\{(x,y)\mid x^2+y^2\le 4\}.$$

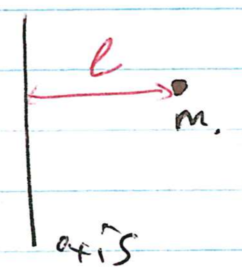
$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \rho g(3-y)\,dy\,dx =$$

$$= 2\int_0^2 \rho g\left[-\frac{(3-y)^2}{2}\Bigg|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\right]dx =$$

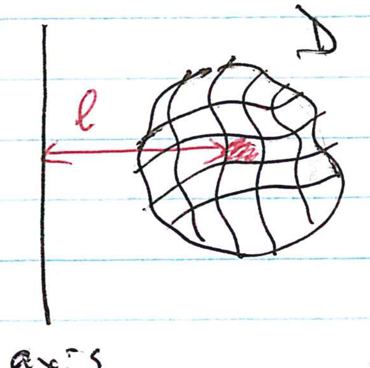
$$= 2\int_0^2 \frac{\rho g}{2}\left[-\left(3-\sqrt{4-x^2}\right)^2+\left(3+\sqrt{4-x^2}\right)^2\right]dx$$

$= \ldots$ (HW).

Sect. 13.5 Centers of Mass and Moments of Inertia.



1st moment about the axis $= m\ell$, where ℓ is the signed distance.



1st moment of small region =

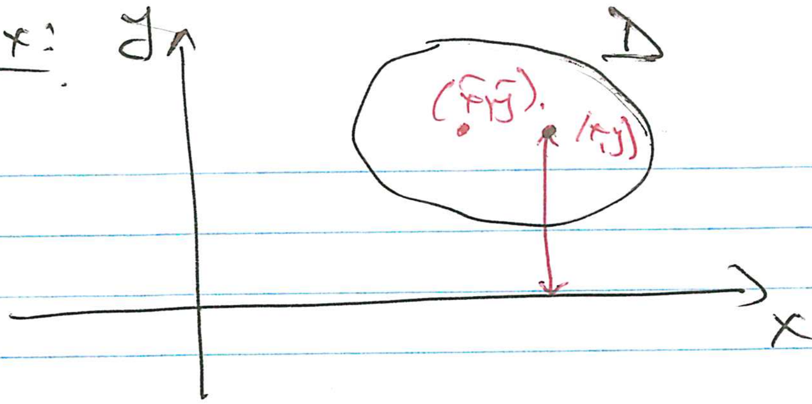
$$= (\underbrace{\rho}_{\text{density}}\Delta A)\cdot \ell$$

(mass)

1st moment of plate about this axis

$$= \iint_D \rho\, \underbrace{\ell(x,y)}_{\substack{\text{signed distance from } (x,y) \text{ to}\\ \text{the axis.}}}\,dA$$

Ex:



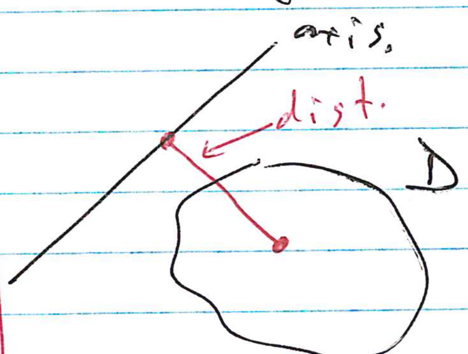
1st moment about the x-axis $= \iint_D \rho\, y \, dA$

1st moment about the y-axis $= \iint_D \rho\, x \, dA$

Mass of this plate $= \iint_D \rho \, dA = M$. (note: ρ may depend on x, y).

Centre of mass of the plate is $(\bar{x}, \bar{y})$, where
$$\bar{y} = \frac{1}{M} \iint_D \rho\, y \, dA, \quad \bar{x} = \frac{1}{M} \iint_D \rho\, x \, dA.$$
| centroid

Moment of inertia about an axis =



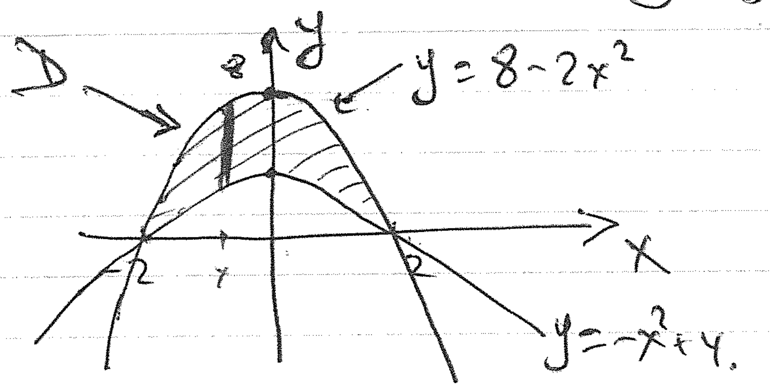
$$= \iint_D \rho \left[d(x,y)\right]^2 dA, \text{ where}$$

$d(x,y)$ = distance from (x,y) to the axis.

#2, p. 924 Find the centroid of the region bounded by $y = 8 - 2x^2$, $y + x^2 = 4$.

$$\Updownarrow$$

$$y = -x^2 + 4.$$



$$\bar{x} = \frac{1}{M} \iint_D \rho\, x\, dA$$

$$M = \iint_D \rho\, dA$$

$$\therefore \bar{x} = \frac{\iint_D \rho x\, dA}{\iint_D \rho\, dA} = \frac{\rho \iint_D x\, dA}{\rho \iint_D dA} = \frac{\iint_D x\, dA}{\iint_D dA},$$

i.e., we can assume that $\rho = 1$.

$$M = \iint_D dA = \int_{-2}^{2} \int_{-x^2+4}^{8-2x^2} 1 \; dy\, dx =$$

$$= \int_{-2}^{2} \left[8 - 2x^2 - (-x^2 + 4)\right] dx = \int_{-2}^{2} (8 - x^2 - 4)\, dx =$$

$$= 2 \int_0^2 (4 - x^2)\, dx = 2\left(4x - \tfrac{1}{3}x^3\right)\Big|_0^2 =$$

$$= 2\left(8 - \tfrac{1}{3} \cdot 8\right) = 2 \cdot \tfrac{2}{3} \cdot 8 = \tfrac{32}{3}.$$

Note: $\bar{x} = 0$ by symmetry.

If we had to set it up, then

$$\bar{x} = \frac{1}{M} \iint_D x\, dA = \frac{1}{M} \int_{-2}^{2} \int_{-x^2+4}^{8-2x^2} x \; dy\, dx =$$

$$= \frac{1}{M} \int_{-2}^{2} x(4 - x^2)\, dx = 0$$

odd fcn.

$$\bar{y} = \frac{1}{M}\iint_D y\,dA = \frac{1}{M}\int_{-2}^{2}\int_{-x^2+4}^{8-2x^2} y\,dy\,dx$$

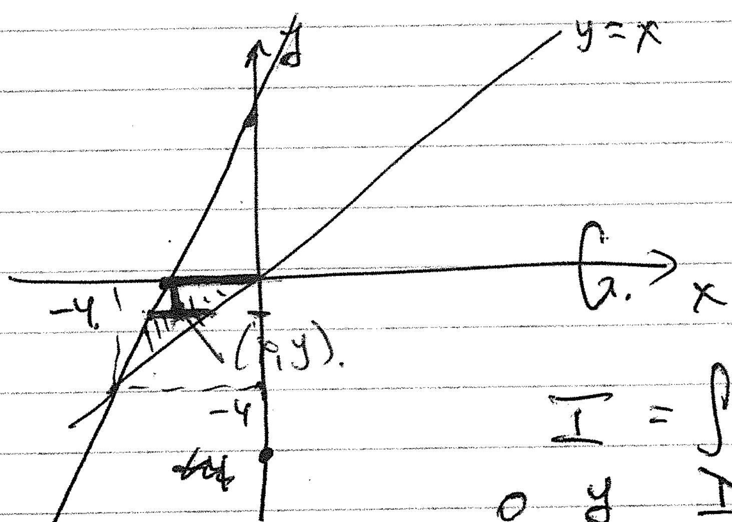
$$= \frac{1}{M}\int_{-2}^{2}\left(\frac{1}{2}y^2\right)\Bigg|_{y=-x^2+4}^{y=8-2x^2} dx =$$

$$= \frac{1}{M}\int_{-2}^{2}\frac{1}{2}\left[(8-2x^2)^2-(-x^2+4)^2\right]dx =$$

$$= \frac{1}{2M}\cdot 2\int_{0}^{2}\left[64-32x^2+4x^4-x^4-16+8x^2\right]dx$$

$$= \dots \text{(HW)} \quad \left(\text{recall } M = \frac{32}{3}\right).$$

#12, p. 924   Find the second moment of area of the region bounded by $y=x$, $y=2x+4$, $y=0$ ~~(below the x axis)~~ about the x-axis.



$$\begin{cases} y = x \\ y = 2x+4 \end{cases} \Leftrightarrow \begin{cases} x = 2x+4 \\ y = x \end{cases} \Leftrightarrow \begin{cases} x = -4 \\ y = -4. \end{cases}$$

$$I = \iint_D (-y)^2\,dA =$$

$$= \int_{-4}^{0}\int_{\frac{y-4}{2}}^{y} y^2 \, dx\,dy =$$

$y = 2x+4.$

$\Leftrightarrow$ $x = \frac{y-4}{2}$

$$= \int_{-4}^{0} y^2\left[y - \frac{y-4}{2}\right]dy = \int_{-4}^{0} y\cdot\left[\frac{y}{2}+2\right]dy = \dots$$