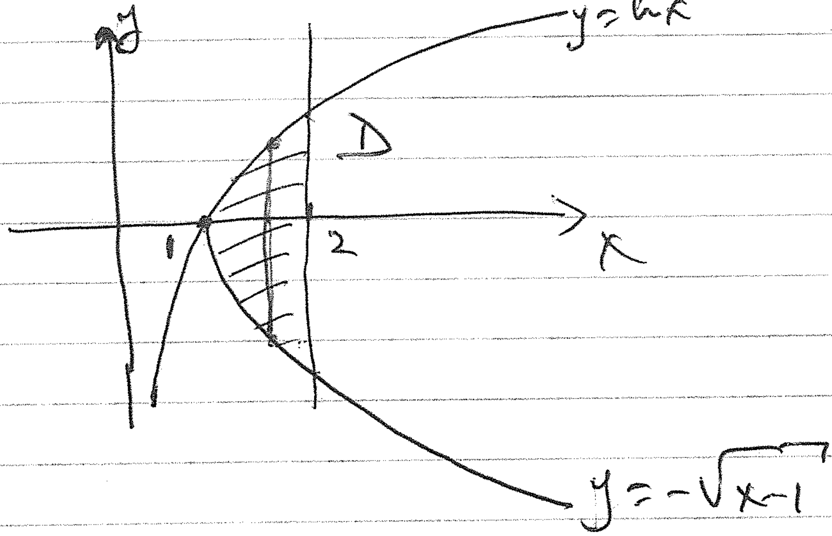
Nov. 7, 2019

#25, p. 924 Find the centroid of the region bounded by

$y = \ln x$, $y + \sqrt{x-1} = 0$, $x = 2$.



$\{ y = -\sqrt{x-1}$

$M = \iint_D dA$

$M\bar{x} = \iint_D x\, dA$

$M\bar{y} = \iint_D y\, dA$

$$M = \int_1^2 \int_{-\sqrt{x-1}}^{\ln x} 1 \, dy\, dx =$$

$$= \int_1^2 \left(\ln x + \sqrt{x-1}\right) dx =$$

$$\int_1^2 \sqrt{x-1}\, dx = \frac{2}{3}(x-1)^{\frac{3}{2}}\Big|_1^2 = \frac{2}{3}\left[1^{3/2} - 0^{3/2}\right] = \frac{2}{3}$$

$$\int_1^2 \ln x\, dx = \int_1^2 x' \cdot \ln x\, dx = x \ln x \Big|_1^2 - \int_1^2 x \cdot \frac{1}{x} dx$$

$$= 2\ln 2 - \cancel{1 \ln 1}_{0} - (2-1) = 2\ln 2 - 1$$

$$= 2\ln 2 - 1 + \frac{2}{3} = 2\ln 2 - \frac{1}{3}.$$

$$m\bar{x} = \iint_D \rho x \, dA = \int_1^2 \int_{-\sqrt{x-1}}^{\ln x} x \, dy \, dx =$$

$$= \int_1^2 x\left[\ln x + \sqrt{x-1}\right] dx \; \textcircled{E} \; \ldots \; (HW).$$

$$\int_1^2 x\sqrt{x-1}\, dx = \begin{cases} u = x-1 \\ du = dx \\ x=1 \to u=0 \\ x=2 \to u=1 \end{cases} = \int_0^1 (u+1)\sqrt{u}\, du = \ldots$$

$$\int_1^2 x \ln x \, dx = \int_1^2 \left(\frac{x^2}{2}\right)' \ln x \, dx = \frac{1}{2}x^2 \ln x \Big|_1^2 - \int_1^2 \frac{1}{2}x^2 \cdot \frac{1}{x} d$$

$$= \ldots$$

$$m\bar{y} = \iint_D y \, dA = \int_1^2 \int_{-\sqrt{x-1}}^{\ln x} y \, dy \, dx = \int_1^2 \left(\frac{1}{2}y^2\right)\Big|_{-\sqrt{x-1}}^{\ln x} dx$$

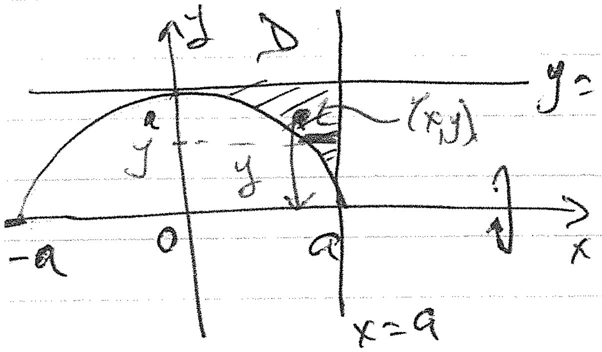
$$= \frac{1}{2} \int_1^2 \left[(\ln x)^2 - (x-1)\right] dx = \ldots \; (HW).$$

$$\int_1^2 (\ln x)^2 dx = \int_1^2 x'(\ln x)^2 dx = x(\ln x)^2 \Big|_1^2 - \int_1^2 x \cdot (2\ln x)\frac{1}{x} dx$$

$$= 2(\ln 2)^2 - 2\int_1^2 \ln x \, dx = \ldots \; (HW)$$

#31, p.924. Find the second moment :

$y = \sqrt{a^2 - x^2}$, $y = a$, $x = a$ $(a > 0)$ about the x-axis.



① $I = \iint_D y^2 \, dA =$

$= \int_0^a \int_{\sqrt{a^2-x^2}}^{a} y^2 \, dy \, dx$

$= \int_0^a \left[\frac{1}{3} y^3 \Big|_{\sqrt{a^2-x^2}}^{a} \right] dx = \frac{1}{3} \int_0^a \left[a^3 - (a^2 - x^2)\sqrt{a^2-x^2} \right] dx$

$= \dots$ (HW) (use $x = a \cos t$).

$\left\{ \int \sin^4 t \, dt = \left\{ \begin{array}{l} \cos 2t = 1 - 2\sin^2 t \\ \Leftrightarrow \sin^2 t = \frac{1 - \cos 2t}{2} \end{array} \right\} = \dots \right\}$

② Switch the order of integration.

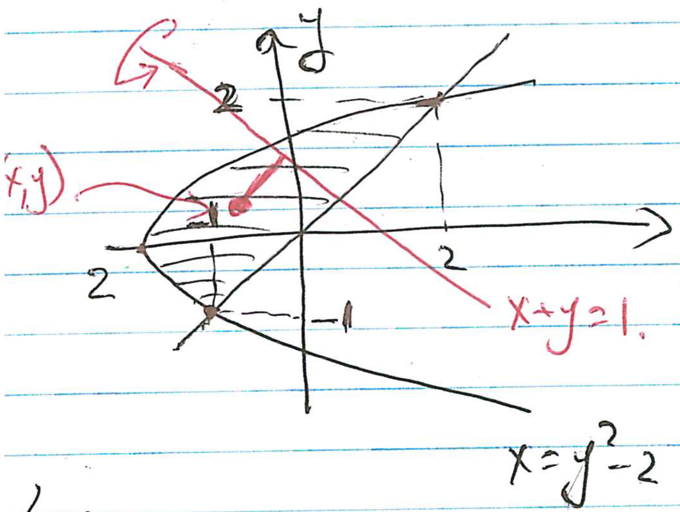
$I = \int_0^a \int_{\sqrt{a^2-y^2}}^{a} y^2 \, dx \, dy$

$= \int_0^a y^2 \left(a - \sqrt{a^2 - y^2} \right) dy = \dots$ (HW).

$\left\{ \int y^2 \sqrt{a^2 - y^2} \, dy = \left\{ y = a \cos t \right\} = \dots \right\}$

#32, p. 927 Find the 1st and 2nd moments:

$x = y^2 - 2$, $y = x$ about $x + y = 1$.



Find pts of intersection.

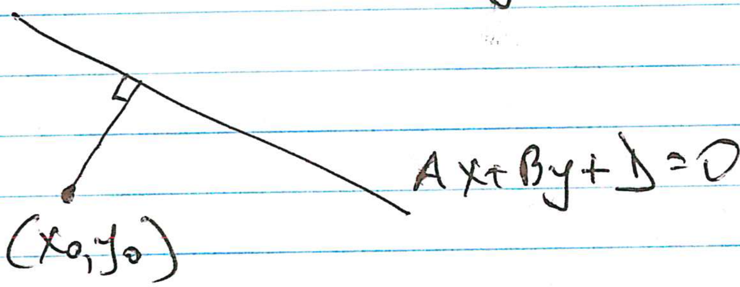
$$\begin{cases} y = x \\ x = y^2 - 2 \end{cases}$$

$x^2 - x - 2 = 0$

$x = \dfrac{1 \pm \sqrt{1+8}}{2}$

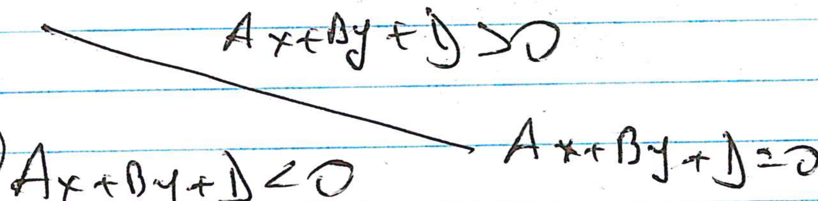
$= \dfrac{1 \pm 3}{2}$

$\therefore$ $x = -1$ or $x = 2$



$$\text{Dist}\{(x_0, y_0), Ax + By + D = 0\} = \frac{|Ax_0 + By_0 + D|}{\sqrt{A^2 + B^2}}$$

~~Angle moment~~ $\angle$ = Signed distance $= \dfrac{Ax_0 + By_0 + D}{\sqrt{A^2 + B^2}}$.



$$\text{1st moment } \angle = \iint_{\text{region}} \frac{Ax + By + D}{\sqrt{A^2 + B^2}}\, dA$$

$$\therefore \angle = \iint_{\text{region}} \frac{x + y - 1}{\sqrt{2}}\, dA$$

Region $= \{(x,y) \mid -1 \le y \le 2 \text{ and } y^2-2 \le x \le y\}$

$$\therefore L = \int_{-1}^{2} \int_{y^2-2}^{y} \frac{x+y-1}{\sqrt{2}} \, dx \; dy =$$

$$= \frac{1}{\sqrt{2}} \int_{-1}^{2} \left[\left(\frac{1}{2}x^2 + xy - x \right) \Big|_{x=y^2-2}^{x=y} \right] dy =$$

$$= \frac{1}{\sqrt{2}} \int_{-1}^{2} \left\{ \left(\frac{1}{2}y^2 + y^2 - y \right) - \left[\frac{(y^2-2)^2}{2} + (y^2-2)y - (y^2-2) \right] \right\} dy$$

$= \ldots$

$$\text{2nd moment } I = \iint_{\text{region}} \left[\frac{x+y-1}{\sqrt{2}} \right]^2 dA =$$

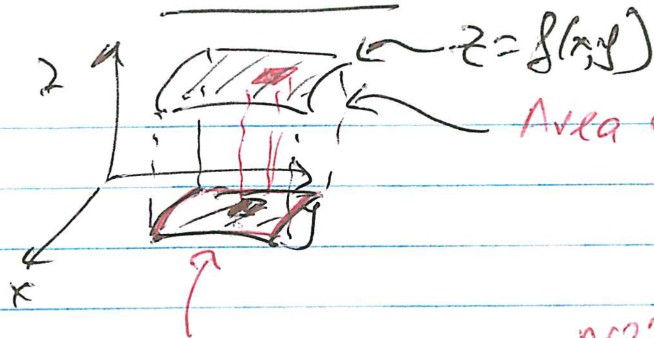
$$= \int_{-1}^{2} \int_{y^2-2}^{y} \frac{1}{2} (x+y-1)^2 \, dx \, dy =$$

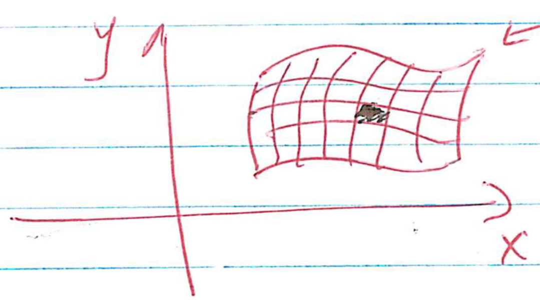
$$= \left\{ \begin{aligned} \int (x+\underbrace{(y-1)}_{a})^2 dx &= \frac{(x+a)^3}{3} + C = \\ &= \frac{(x+y-1)^3}{3} + C \end{aligned} \right\}$$

$$= \frac{1}{6} \int_{-1}^{2} \left\{ (x+y-1)^3 \Big|_{x=y^2-2}^{x=y} \right\} dy =$$

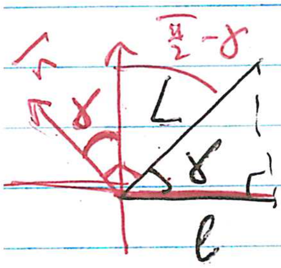
$$= \frac{1}{6} \int_{-1}^{2} \left[(2y-1)^3 - (y^2+y-3)^3 \right] dy = \ldots \text{ HW.}$$

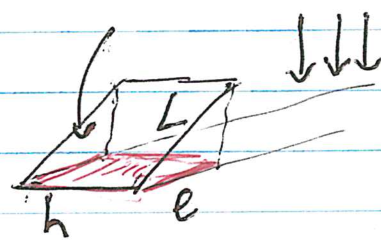
Sect. 13.6 Surface Areas.





Cross section of





Q-n: Comparison of the area of the slanted rectangle and its projection onto the xy-plane.

$$\frac{l}{L} = \cos\gamma$$

Area of slanted rect. $= hL = h \cdot \frac{l}{\cos\gamma}$

$$= \frac{\text{Area of projection}}{\cos\gamma}$$

i.e., $\underbrace{\Delta S_{ij}}_{\text{area on the surface.}} \approx \frac{1}{\cos\gamma} \underbrace{\Delta A_{ij}}_{\text{projection.}}$

$\bar{n} \cdot \bar{k} = |\bar{n}| \cdot |\bar{k}| \cos\gamma$

normal to surface

normal to the xy-plane.

~~So~~ $\bar{k} = (0,0,1)$

$\bar{n}$? $\quad z = f(x,y) \Leftrightarrow \underbrace{z - f(x,y)}_{F(x,y,z)} = 0$

$$\bar{n} = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right)$$

$$\therefore \underbrace{\bar{n} \cdot \bar{k}}_{1} = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \cdot 1 \cdot \cos\gamma$$

$$\therefore \cos\gamma = \frac{1}{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}}$$

$$\boxed{\therefore \Delta S_{ij} = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, \Delta A_{ij}}$$

$$\text{Surface area} = \iint_{\substack{\text{projection} \\ \text{onto xy-plane.}}} \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dA$$