Sept. 10, 2019

Section 11.2 : Surfaces,

3D : 3-dimensional space

Quadric Surfaces:
(polynomial of total degree 2 in 3 variables) :

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J$$

"$Q(x,y,z)$"

~~Graphs~~ Graphs (of functions) / surfaces:

$$\{(x,y,z) \in \mathbb{R}^3 \mid Q(x,y,z) = 0\}$$

Ex: $S = \{(x,y) \in \mathbb{R}^2 \mid x^2 - x + \sqrt{y} + 5 = 0\}$

Q-n: Is $(1,1)$ in the set S? (Does this curve pass through the pt. $(1,1)$?)

LHS of eq-n $x^2 - x + \sqrt{y} + 5 = 0$ at $(1,1)$: $6 \neq 0 = \text{RHS}$

A: $(1,1) \notin S$

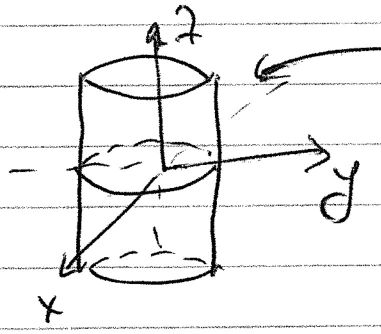
$\notin$ is not an element of

Ex: Q-n: Does $Q(x,y,z) = 0$ always represent a surface?

1) $x^2 + y^2 + z^2 = 1$ ← sphere of radius 1.

2) $x^2 + y^2 = 1$

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}, \quad z \in \mathbb{R}$$



← Cylinder

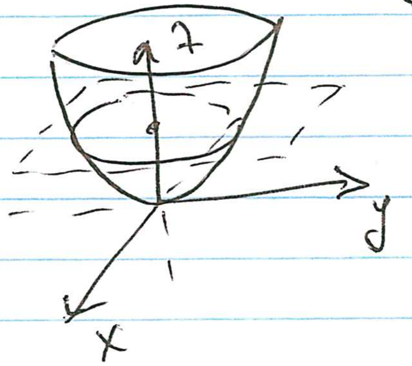
5

3) $z - x^2 - y^2 = 0 \iff z = x^2 + y^2$

xy-plane : $\{(x,y,z) \in \mathbb{R}^3 \mid z = 0\}$

$z=0$: Solve $x^2+y^2=0 \iff (x,y)=(0,0)$

$z=1$; $x^2+y^2=1$ ← unit circle in the plane $z=1$.



(Circular) paraboloid.

Consider intersection of $z=x^2+y^2$ with $y=kx$, $k \in \mathbb{R}$

Curve of intersection is

$$\{(x,y,z) \in \mathbb{R}^3 \mid z = x^2+y^2,\ y = kx\}$$

$$= \{(x,y,z) \in \mathbb{R}^3 \mid z = (1+k^2)x^2,\ y = kx\}.$$

↑ parabola in the plane $y=kx$

$\therefore$ Intersection of $z=x^2+y^2$ with any plane containing the z-axis is a parabola (hence, the name paraboloid)

4) $x^2+y^2+z^2=0$

$S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=0\} = \{(0,0,0)\}$

↑ a single point

5) $x^2+y^2+z^2+1=0$

$S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2+1=0\} = \emptyset$ ← empty set

(no points).

6) $0 \cdot x^2 + 0 \cdot y^2 + 0 \cdot z^2 = 0$

$S = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \cdot x^2 + 0 y^2 + 0 z^2 = 0\} =$

$= \{(x,y,z) \in \mathbb{R}^3 \mid 0 = 0\} = \{(x,y,z) \in \mathbb{R}^3 \mid x,y,z \in \mathbb{R}\}$

$= \mathbb{R}^3$.

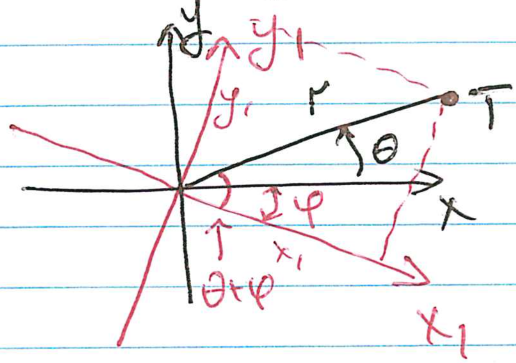
Review 2D:

Q-n: what are all possible curves (in 2D) represented by $Ax^2+By^2+Cxy+Dx+Ey+F=0$.

$$\text{Curve} = \{(x,y)\in\mathbb{R}^2 \mid P(x,y)=0\}.$$

$P(x,y)$

Idea: Change variables so that the term Cxy disappears (rotate the object OR rotate the coordinate system).



Point T has coordinates (x,y) in the xy-plane and coordinates of T in the x_1,y_1-coordinate plane are x_1 and y_1.

How do x_1 and y_1 depend on x and y?

Polar coordinates: (r,θ)
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\begin{cases} x_1 = r\cos(\theta+\varphi) \\ y_1 = r\sin(\theta+\varphi) \end{cases}$$

$$\begin{cases} \cos(A+B) - ? \\ \sin(A+B) - ? \end{cases}$$

$$\boxed{\text{Euler's identity: } e^{ix} = \cos x + i\sin x}$$

$$e^{i(x+y)} = e^{ix}\cdot e^{iy} = (\cos x + i\sin x)(\cos y + i\sin y)$$
$$= \cos x\cos y + i^2\sin x\sin y + i(\sin x\cos y + \cos x\sin y)$$
$$= \cos x\cos y - \sin x\sin y + i(\sin x\cos y + \cos x\sin y)$$

$$e^{i(x+y)} = \cos(x+y) + i\sin(x+y)$$

$$\boxed{\begin{aligned}\cos(x+y) &= \cos x\cos y - \sin x\sin y \\ \sin(x+y) &= \sin x\cos y + \cos x\sin y.\end{aligned}}$$

We have:

$$x_1 = r\cos(\theta+\varphi) = r\cos\theta\cos\varphi - r\sin\theta\sin\varphi$$
$$= x\cos\varphi - y\sin\varphi$$

$$y_1 = r\sin(\theta+\varphi) = r\sin\varphi\cos\theta + r\cos\varphi\sin\theta$$
$$= x\sin\varphi + y\cos\varphi$$

$$\begin{cases} x_1 = x\cos\varphi - y\sin\varphi \\ y_1 = x\sin\varphi + y\cos\varphi \end{cases} \iff \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Conclusion (HW: check details): By rotating the coordinate system you can get rid of Cxy in the eq-n $P(x,y) = 0$.

$\therefore$ Consider $Ax^2 + By^2 + Cx + Dy + E = 0$.

If $A = B = 0$, then $Cx + Dy + E = 0$ is a line in the the xy-plane.

If $B = 0$ and $A \neq 0$, then we get a <u>parabola</u>.

Idea: If $A \neq 0$, then rewrite $Ax^2 + Cx + E$ by completing the square.

<u>Recall:</u> $\underline{ax^2 + bx + c} = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) =$

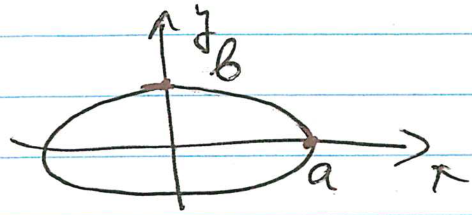
$$= a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right) =$$

$$= \underline{a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}}.$$

After simplifications:

$$Ax^2 + By^2 + C = 0 \quad , \; A, B, C \in \mathbb{R} \; (\text{assume } A \neq 0, B \neq 0).$$

1) $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$



Ellipse.

2) $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = -1$; Nothing

3) $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$: hyperbola.

<u>Ex:</u> Sketch $x^2 - 2y^2 = 1$

"Solve" for y? Use symmetry: if (a, b) is on this curve, then $(\pm a, \pm b)$ is also on this curve.

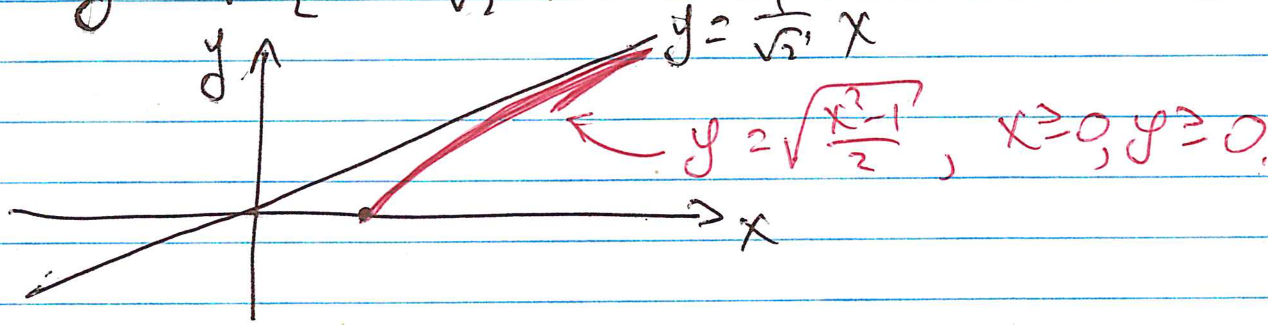
$\therefore$ The curve is symmetric about the x-axis and the y-axis.

Assume $x \geq 0, y \geq 0$.

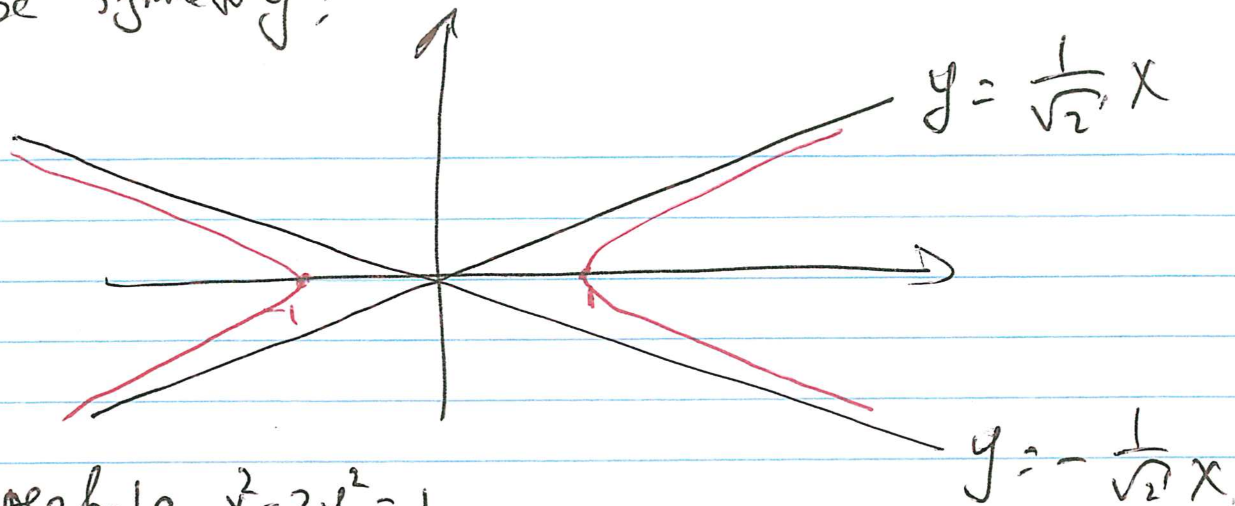
Solve for y: $2y^2 = x^2 - 1 \iff y^2 = \frac{x^2-1}{2} \iff y = \sqrt{\frac{x^2-1}{2}}$

<u>Q-n:</u> what happens when $x \to \infty$?

$$y \approx \sqrt{\frac{x^2}{2}} = \frac{1}{\sqrt{2}} x$$



Use symmetry:



Hyperbola $x^2 - 2y^2 = 1$

In general, $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$ is a hyperbola with asymptotes $\frac{x}{a} = \pm \frac{y}{b} \Leftrightarrow y = \pm \frac{b}{a} x$.

Note: Formally, we need to show

$$\lim_{x \to \infty} \left(\frac{1}{\sqrt{2}} x - \sqrt{\frac{x^2 - 1}{2}} \right) = 0 \quad (?)$$

$$LHS = \lim_{x \to \infty} \frac{1}{\sqrt{2}} \left(x - \sqrt{x^2 - 1} \right) = \left\{ \begin{matrix} \infty - \infty \\ \text{no information} \end{matrix} \right\} =$$

$$= \lim_{x \to \infty} \frac{1}{\sqrt{2}} \cdot \frac{(x - \sqrt{x^2 - 1})}{1} \cdot \frac{x + \sqrt{x^2 - 1}}{x + \sqrt{x^2 - 1}} =$$

$$= \lim_{x \to \infty} \frac{1}{\sqrt{2}} \cdot \frac{x^2 - (x^2 - 1)}{x + \sqrt{x^2 - 1}} = \lim_{x \to \infty} \frac{1}{\sqrt{2}} \cdot \frac{1}{x + \sqrt{x^2 - 1}} = \left\{ \frac{1}{\infty} \right\}$$

$$= 0.$$

Conclusion: In 3D, it is sufficient to consider

1) $Ax^2 + By^2 + Cz^2 + D = 0$
2) $z = Ax^2 + By^2$
3) $F(x, y) = 0$ (i.e., z is missing)
4) Degenerate situation
   $Ax + By + Cz = 0$ (plane)
   $0 \cdot x^2 = 0$ (the whole space).