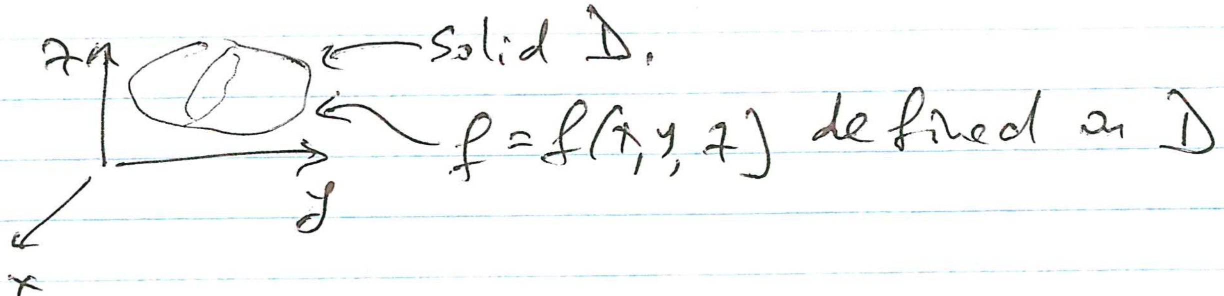
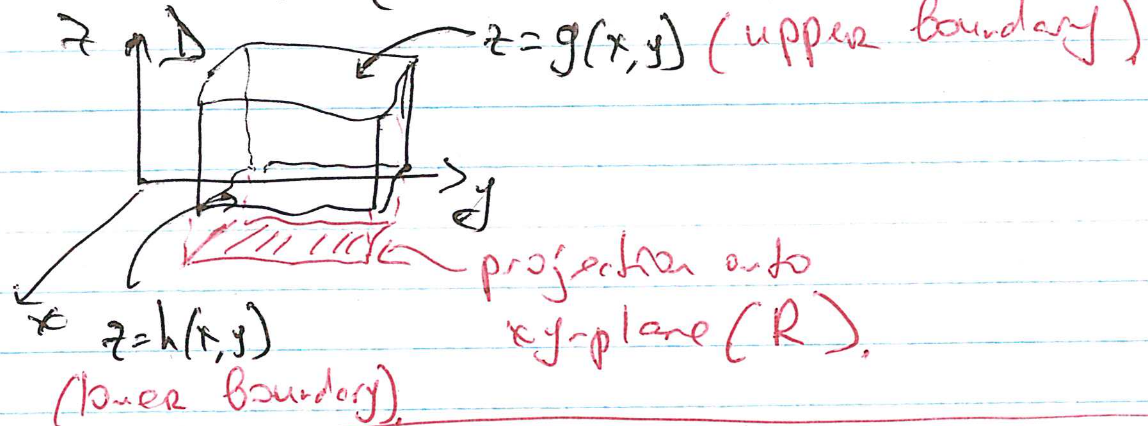
NOV. 26, 2019.

Sect. 13.8 Triple Integrals.



$$\iiint_D f(x,y,z)\,dV = \lim_{\substack{n\to\infty \\ \|P\|\to 0}} \sum_i f(x_i^*, y_i^*, z_i^*)\,\Delta V_i .$$

Note: If f is piecewise continuous and the boundary of D is piecewise smooth, then $\iiint_D f(x,y,z)\,dV$ exists (is well defined).



$$\boxed{D = \{(x,y,z) \mid h(x,y) \le z \le g(x,y),\ (x,y) \in R\}.}$$

Then $$\boxed{\iiint_D f(x,y,z)\,dV = \iint_R \left[\int_{h(x,y)}^{g(x,y)} f(x,y,z)\,dz\right] dA}$$

#1, p. 945. $\iiint\limits_V (x^2 z + y e^x)\, dV$, where

V is bounded by $x=0$, $x=1$, $y=1$, $y=2$, $z=0$, $z=1$.

$V = \{(x,y,z) \mid 0 \le z \le 1 \text{ and } (x,y) \in R\}$, where

$R = \{(x,y) \mid 0 \le x \le 1,\ 1 \le y \le 2\}$.

You want to present the solid V over which the triple integral is taken in the form:

$V = \{(x,y,z) \mid h(x,y) \le z \le g(x,y),\ h_1(x) \le y \le g_1(x),\ a \le x \le b\}$

Then $\iiint\limits_V f(x,y,z)\, dV = \int_a^b \int_{h_1(x)}^{g_1(x)} \int_{h(x,y)}^{g(x,y)} f(x,y,z)\, dz\, dy\, dx$

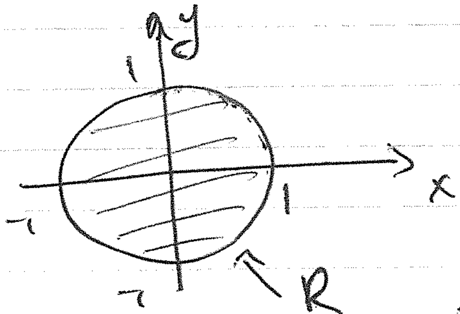
$\therefore \iiint\limits_V (x^2 z + y e^x)\, dV = \int_0^1 \int_1^2 \int_0^1 (x^2 z + y e^x)\, dz\, dy\, dx =$

$= \int_0^1 \int_1^2 \left\{ \left(\frac{1}{2} x^2 z^2 + y e^x z \right) \Big|_{z=0}^{z=1} \right\} dy\, dx =$

$= \int_0^1 \int_1^2 \left(\frac{1}{2} x^2 + y e^x \right) dy\, dx = \int_0^1 \left[\left(\frac{1}{2} x^2 y + \frac{1}{2} y^2 e^x \right) \Big|_{y=1}^{y=2} \right]$

$= \int_0^1 \left[x^2 + 2e^x - \left(\frac{1}{2} x^2 + \frac{1}{2} e^x \right) \right] dx = \ldots$

#4, p. 945 $\iiint_V xy\,dV$, where V is enclosed by $z=\sqrt{1-x^2-y^2}$, $z=0$.



$R=\{(x,y) \mid$ "enclosed" by $\sqrt{1-x^2-y^2}=0\}$

$=\{(x,y) \mid x^2+y^2\le 1\}$.

$$\therefore \iiint_V xy\,dV = \iint_R \left(\int_0^{\sqrt{1-x^2-y^2}} xy\,dz\right) dA$$

$$= \iint_R xy\sqrt{1-x^2-y^2}\,dA = \left\{\begin{array}{l}\text{Polar coordinates:}\\ x=r\cos\theta\\ y=r\sin\theta\\ R=\{(r,\theta) \mid 0\le\theta\le 2\pi,\ 0\le r\le 1\}\end{array}\right\}$$

$$= \int_0^{2\pi}\int_0^1 r^2\cos\theta\sin\theta\sqrt{1-r^2}\cdot r\,dr\,d\theta =$$

$$= \int_0^{2\pi}\int_0^1 \left(\tfrac{1}{2}\sin 2\theta\right) r^3\sqrt{1-r^2}\,dr\,d\theta =$$

$$= \left[\int_0^{2\pi}\tfrac{1}{2}\sin 2\theta\,d\theta\right]\cdot\left[\int_0^1 r^3\sqrt{1-r^2}\,dr\right] = 0.$$

$$\overset{\shortparallel}{\frac{-\cos 2\theta}{4}}\Bigg|_0^{2\pi} \qquad\qquad " \left\{ u = 1-r^2 \right\}$$

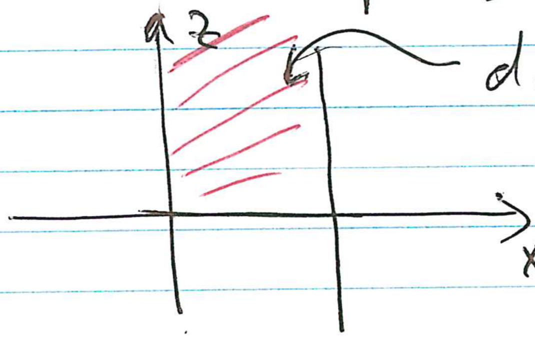
$$\overset{\shortparallel}{-\tfrac{1}{4}}(\cos 4\pi - \cos 0)$$

$$\overset{\shortparallel}{0}$$

#7, p. 945. $\iiint\limits_V x^2y^2z^2\,dV$, where V is bounded by $z=1+y$, $y+z=1$, $x=1$, $x=0$, $z=0$.

I. The inner variable is y:

$\underbrace{y=z-1,\ y=1-z}_{y \text{ is between these planes}}$ and $(x,z)\in R$ where R is bounded by $x=1$, $x=0$, $z=0$.

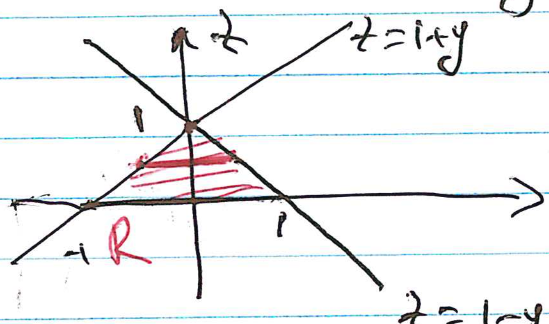


did not get a bounded region.

$\therefore$ Try the other order

II. The inner variable of integration is x:

$0\le x\le 1$ and $(y,z)\in R$, where R is bounded by $z=1+y$, $y+z=1$, $z=0$.



$\therefore V=\{(x,y,z)\mid 0\le x\le 1, (y,z)\in R\}$ with

$R=\{(y,z)\mid 0\le z\le 1,\ z-1\le y\le 1-z\}$

$$\therefore \iiint\limits_V x^2y^2z^2\,dV=\iint\limits_R\left(\int_0^1 x^2y^2z^2\,dx\right)dA=$$

$$=\int_0^1\int_{z-1}^{1-z}\int_0^1 x^2y^2z^2\,dx\;dy\;dz=$$

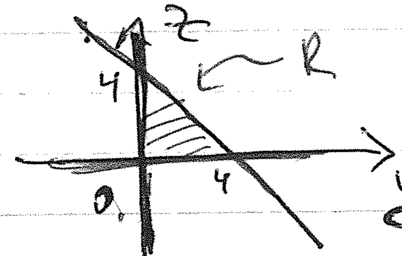
$$= \int_0^1 \int_{z-1}^{1-z} \left(y^2 z^2 \tfrac{1}{3}\right) dy\, dz = \int_0^1 \left(\tfrac{1}{3} z^2 \cdot \tfrac{1}{3} y^3 \Big|_{y=z-1}^{y=1-z} \right) dz =$$

$$= \int_0^1 \tfrac{1}{9} z^2 \left[(1-z)^3 - (z-1)^3 \right] dz = \dots \text{ HW.}$$

#9, p. 945 $\iiint_V dV$, where V is bounded by

$z = x^2$, $y + z = 4$, $y = 0$.

Try projection onto the yz-plane.



$x^2 = z \Rightarrow z \geq 0$.

$V = \{(x,y,z) \mid -\sqrt{z} \leq x \leq \sqrt{z},\ (y,z) \in R\}$,

where $R = \{(y,z) \mid 0 \leq y \leq 4,\ 0 \leq z \leq 4 - y\}$

$$\iiint_V dV = \int_0^4 \int_0^{4-y} \int_{-\sqrt{z}}^{\sqrt{z}} 1 \cdot dx\, dz\, dy = \dots \text{ HW}$$

#16, p. 945. $\iiint_V xyz\, dV$, where V is bounded by

$z = x^2 + 4y^2$, $2x + 8y + z = 4$.

Projection onto the xy-plane?

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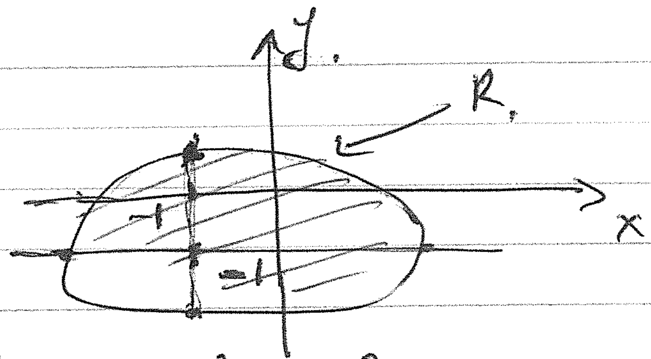
$$x^2+4y^2+(2x+8y)=4$$

$$\iff x^2+2x+4(y^2+2y)=4$$

$$\iff (x+1)^2+4(y+1)^2=9$$



$$R=\{(x,y)\mid (x+1)^2+4(y+1)^2\le 9\}$$

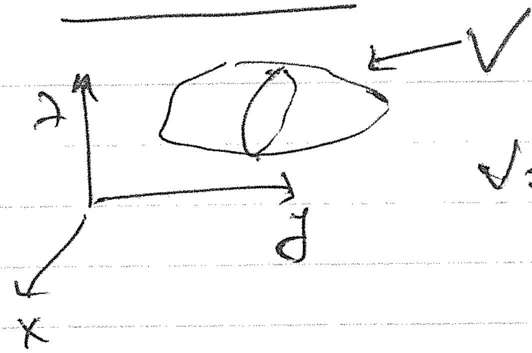
$$V=\{(x,y,z)\mid x^2+4y^2\le z\le 4-2x-8y,\ (x,y)\in R\}.$$

$$\iiint_V xyz\,dV=\iint_R\left(\int_{x^2+4y^2}^{4-2x-8y} xyz\,dz\right)dA =$$

$$= \left\{ \begin{array}{l} R=\{(x,y)\mid -\frac{5}{2}\le y\le\frac{1}{2},\quad |x+1|\le\sqrt{9-4(y+1)^2}\} = \\ x=-1:\ (y+1)^2\le\frac{9}{4} \iff |y+1|\le\frac{3}{2} \iff \\ \iff -\frac{3}{2}\le y+1\le\frac{3}{2} \iff -\frac{5}{2}\le y\le\frac{1}{2} \\ = \{(x,y)\mid -\frac{5}{2}\le y\le\frac{1}{2},\ -1-\sqrt{9-4(y+1)^2}\le x\le -1+\sqrt{9-4(y+1)^2}\} \end{array} \right.$$

$$= \int_{-\frac{5}{2}}^{\frac{1}{2}}\int_{-1-\sqrt{9-4(y+1)^2}}^{-1+\sqrt{9-4(y+1)^2}}\int_{x^2+4y^2}^{4-2x-8y} xyz\,dz\,dx\,dy.$$

Section 13.9. Volumes.

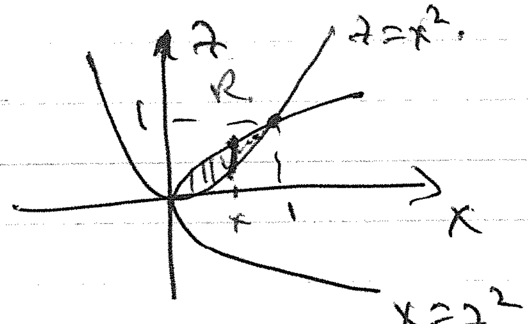


Volume of $V = \iiint_V 1 \, dV$

#2, p. 949. V is bounded by $x = z^2$, $z = x^2$, $y=0$, $y=2$. Find the volume of V.

$V = \{(x,y,z) \mid 0 \le y \le 2 \text{ and } (x,z) \in R\}$, where

$R = \{(x,z) \mid \text{"bounded" by } x = z^2 \text{ and } z = x^2\}$.



Pts of intersection:

$$\begin{cases} x = z^2 \\ z = x^2 \end{cases} \Rightarrow \begin{cases} x = x^4 \\ z = x^2 \end{cases} \Leftrightarrow \begin{cases} x = 0, 1 \\ z = x^2 \end{cases}$$

$\therefore$ $(0,0)$ and $(1,1)$ are pts. of inter.

$$\text{Volume} = \iint_R \left(\int_0^2 1 \, dy \right) dA = \iint_R 2 \, dA \; =$$

$$R = \{(x,z) \mid 0 \le x \le 1, \quad x^2 \le z \le \sqrt{x}\}$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} 2 \, dz \, dx =$$

$$= \int_0^1 2(\sqrt{x} - x^2) \, dx = \ldots \text{HW.}$$