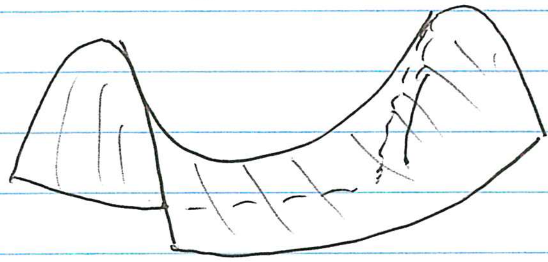
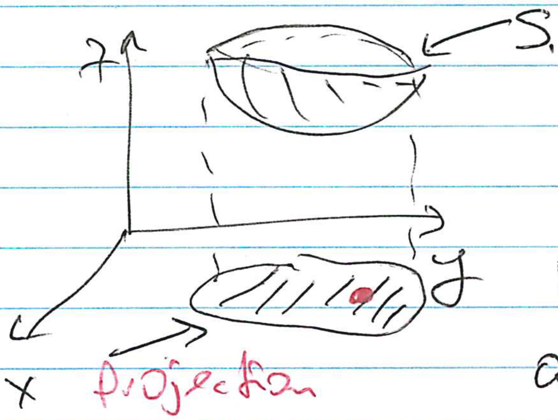
Sept. 12, 2019.

Ex: Hyperbolic paraboloid.

$$z = x^2 - y^2$$



Projections onto planes.



Object $S = \{ (x,y,z) \in \mathbb{R}^3 \mid Q(x,y,z) = 0 \}$

Projection of this object S onto the xy-plane?

$$\text{Projection onto xy-plane} = \{ (x,y) \in \mathbb{R}^2 \mid \text{there is at least one } z \text{ satisfying } Q(x,y,z) = 0 \}$$

Ex (Ex #14 from Lab Section 11.2):

$$\begin{cases} z = 2x^2 + 4y^2 & (*) \\ y + z = 1 \end{cases}$$

Projection onto the xy-plane =

$= \{ (x,y) \in \mathbb{R}^2 \mid (*) \text{ has at least one sol-n for } z \}$

$= P$

Eliminate z; resulting eq-n has to be satisfied (!).

$$\begin{cases} z = 1-y \\ 1-y = 2x^2+4y^2 \end{cases} \quad (*)$$

$\therefore$ If $(x,y) \in P$, then
$2x^2+4y^2+y=1$ ← ellipse
has to be satisfied !

[Warning (!): this does not (yet) mean that every point on this ellipse is in P]

So we have to see if, for every (x,y) s.t. $2x^2+4y^2+y=1$ one can find z so that $(*)$ holds.

Yes (!) : $z = 1-y$

$\therefore$ P = projection onto xy-plane =
$= \{(x,y) \in \mathbb{R}^2 \mid 2x^2+4y^2+y=1\}$.

II. Projection onto xz-plane = Q =
$= \{(x,z) \in \mathbb{R}^2 \mid (*)$ has at least one soln for $y\}$.

1) Eliminate y: $\begin{cases} y = 1-z \\ z = 2x^2+4(1-z)^2 \end{cases}$

What is the curve $z = 2x^2+4(1-z)^2$ in the xz-plane?

$z = 2x^2+4(z^2-2z+1)$

$\Updownarrow$

$2x^2+4z^2-8z+4-z=0$

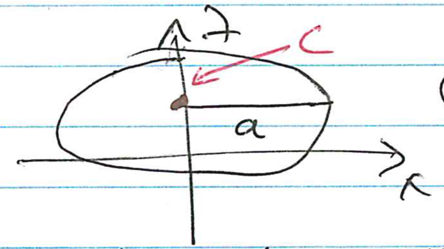
$\Updownarrow$

$2x^2+4z^2-9z+4=0$

$$4z^2 - 9z = 4\left(z^2 - \frac{9z}{4}\right) = 4\left[\left(z - \frac{9}{8}\right)^2 - \left(\frac{9}{8}\right)^2\right]$$
$$= 4\left(z - \frac{9}{8}\right)^2 - \frac{81}{16}$$

$$2x^2 + 4\left(z - \frac{9}{8}\right)^2 - \frac{81}{16} + 4 = 0$$
$$\Leftrightarrow$$
$$2x^2 + 4\left(z - \frac{9}{8}\right)^2 = \frac{17}{16}$$
$$\Leftrightarrow$$
$$\frac{32}{17}x^2 + \frac{64}{17}\left(z - \frac{9}{8}\right)^2 = 1 \quad \leftarrow \text{ellipse}$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{z-b}{c}\right)^2 = 1$$



2) Note: for every (x, z) on this ellipse, there is y s.t. $(*)$ is satisfied $(y = 1 - z)$

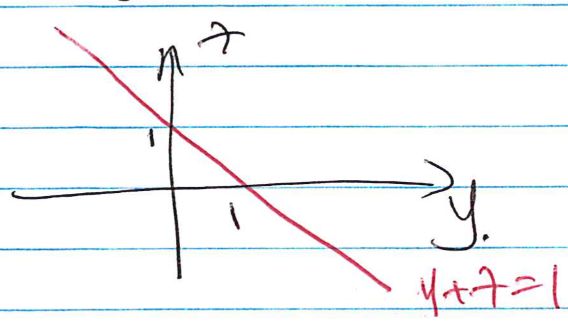
$\therefore Q = \{(x, z) \in \mathbb{R}^2 \mid z = 2x^2 + 4(1-z)^2\}$

III. Projection onto the yz-plane

$$(*): \begin{cases} z = 2x^2 + 4y^2 \\ y + z = 1 \end{cases}$$

Projection onto yz-plane $= T = \{(y, z) \in \mathbb{R}^2 \mid$ there is at least one x s.t. $(*)$ is satisfied$\}$

1) $y + z = 1$



2) Q-n: find all (y, z) on this line s.t. there is x s.t. $(*)$ is satisfied.

Try to solve $z = 2x^2 + 4y^2$ for x:

$$2x^2 = z - 4y^2 \iff x^2 = \frac{z-4y^2}{2} \iff x = \pm\sqrt{\frac{z-4y^2}{2}}$$

$$\therefore \boxed{z - 4y^2 \geq 0}$$

$$\boxed{\therefore T = \{(y,z) \in \mathbb{R}^2 \mid y + z = 1 \text{ and } 4y^2 \leq z\}.}$$

$$z = 1 - y \implies 4y^2 \leq 1 - y \iff 4y^2 + y \leq 1$$

$$\iff \boxed{4y^2 + y - 1 \leq 0.}$$

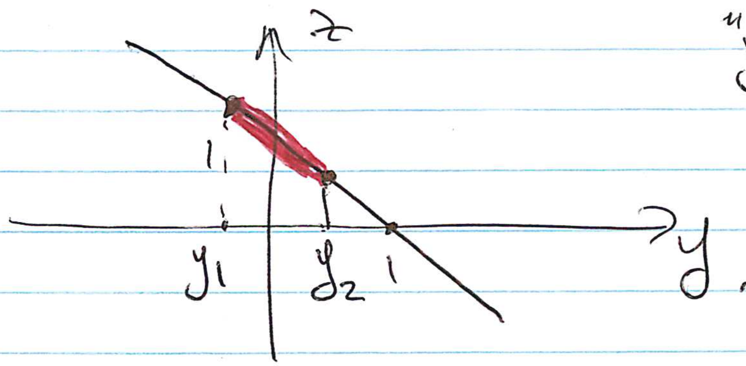
$$\boxed{\therefore T = \{(y,z) \in \mathbb{R}^2 \mid z = 1 - y \text{ and } 4y^2 + y - 1 \leq 0\}}$$

Solve $4y^2 + y - 1 \leq 0$

$$4y^2 + y - 1 = 0 \iff y = \frac{-1 \pm \sqrt{1 + 4 \cdot 4}}{2 \cdot 4} = \frac{-1 \pm \sqrt{17}}{8}$$

$$\therefore 4y^2 + y - 1 \leq 0 \iff \frac{-1-\sqrt{17}}{8} \leq y \leq \frac{-1+\sqrt{17}}{8}$$

$$\therefore T = \{(y,z) \in \mathbb{R}^2 \mid z = 1 - y,\ \underbrace{\frac{-1-\sqrt{17}}{8}}_{y_1} \leq y \leq \underbrace{\frac{-1+\sqrt{17}}{8}}_{y_2}\}.$$



Ex: $z = \sqrt{1-x^2-2y^2}$ $(*)$

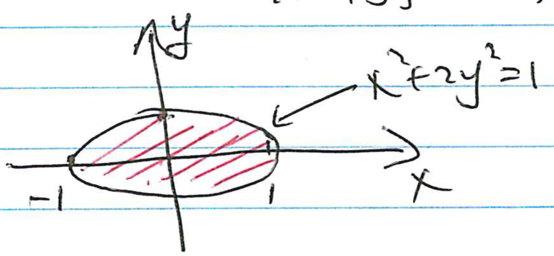
Find the projection of this surface onto 1) the xy-plane and 2) xz-plane.

1) P_1 = Projection onto xy-plane =

$= \{(x,y) \in \mathbb{R}^2 \mid \text{there is } z \text{ s.t. } (*) \text{ is satisfied}\}$

$= \{(x,y) \in \mathbb{R}^2 \mid 1-x^2-2y^2 \geq 0\}$

$= \{(x,y) \in \mathbb{R}^2 \mid x^2+2y^2 \leq 1\}$.



2) P_2 = Projection onto xz-plane

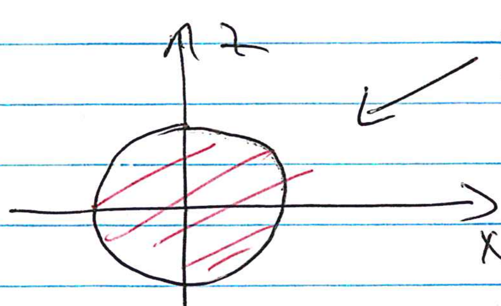
$= \{(x,z) \in \mathbb{R}^2 \mid \text{there is } y \text{ s.t. } (*) \text{ is satisfied}\}$.

Try to solve $(*)$ for y: $z \geq 0$

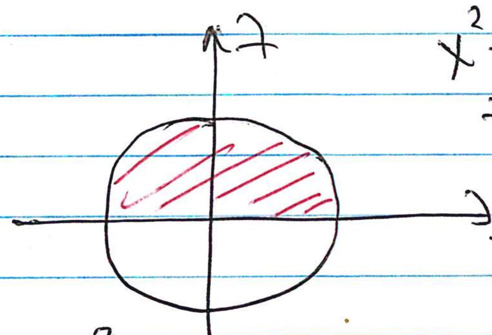
$z = \sqrt{1-x^2-2y^2} \Rightarrow z^2 = 1-x^2-2y^2 \Leftrightarrow$

$\Leftrightarrow 2y^2 = 1-x^2-z^2 \Leftrightarrow y = \pm\sqrt{\frac{1-x^2-z^2}{2}}$

$\therefore\ 1-x^2-z^2 \geq 0 \Leftrightarrow x^2+z^2 \leq 1.$



Is NOT the projection



$x^2+z^2 \leq 1$
$z \geq 0$

$P_2 = \{(x,z) \in \mathbb{R}^2 \mid x^2+z^2 \leq 1,\ z \geq 0\}$.

#48, p. 707 $\begin{cases} x^2+y^2=4 \\ z=4 \end{cases}$ (*) Qn: find projections onto xy; xz-, yz-plane

1) Proj. onto xy-plane $= P_1$.

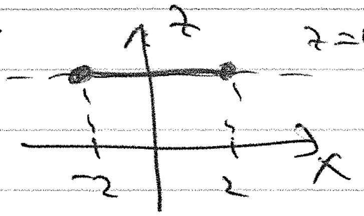
$x^2+y^2=4$ $\therefore$ $P_1=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2=4\}$.

2) Proj. onto xz-plane $= P_2$.

$y^2=4-x^2 \iff y=\pm\sqrt{4-x^2}$

$\therefore$ $z=4$ and $4-x^2\geq 0 \iff z=4$ and $-2\leq x\leq 2$

$P_2=\{(x,z)\in\mathbb{R}^2 \mid z=4,\ -2\leq x\leq 2\}$



3) Proj. onto yz-plane $= P_3$.

$x^2+y^2=4 \iff x=\pm\sqrt{4-y^2}$

$\therefore$ $z=4$ and $4-y^2\geq 0 \iff z=4$ and $-2\leq y\leq 2$.

$P_3=\{(y,z)\in\mathbb{R}^2 \mid z=4 \text{ and } -2\leq y\leq 2\}$

#61, p. 707 $\begin{cases} x^2+y^2-2y=0 \\ z^2=x^2+y^2 \end{cases}$ (*) Find proj. onto xz-plane

$P=\{(x,z)\in\mathbb{R}^2 \mid \text{there is } y \text{ s.t. } (*) \text{ is satisfied}\}$

$\begin{cases} \cancel{(y-1)^2+x^2=1} \\ \cancel{y^2=z^2-x^2} \end{cases}$ (*) $\begin{cases} x^2+(z^2-x^2)-2y=0 \\ y^2=z^2-x^2 \end{cases} \iff$

$\iff \begin{cases} z^2-2y=0 \\ y^2=z^2-x^2 \end{cases} \iff \begin{cases} y=\frac{1}{2}z^2 \\ y^2=z^2-x^2 \end{cases} \iff$

$$\Leftrightarrow \begin{cases} y = \frac{1}{2}z^2 \\ \frac{1}{4}z^4 = z^2 - x^2 \end{cases}$$

$$\boxed{P = \{(x,z) \in \mathbb{R}^2 \mid z^4 = 4z^2 - 4x^2\}}$$

$$(z^2-2)^2 + 4x^2 = 4.$$

$$(z^2-2)^2 \overset{?}{=} 4 - 4x^2 \Leftrightarrow z^2 - 2 = \pm\sqrt{4-4x^2}, \quad x^2 \le 1.$$

i.e., $P = \{(x,z) \mid z = 2 + \sqrt{4-4x^2}, \ -1 \le x \le 1\}$

$\cup \{(x,z) \mid z = 2 - \sqrt{4-4x^2}, \ -1 \le x \le 1\}.$

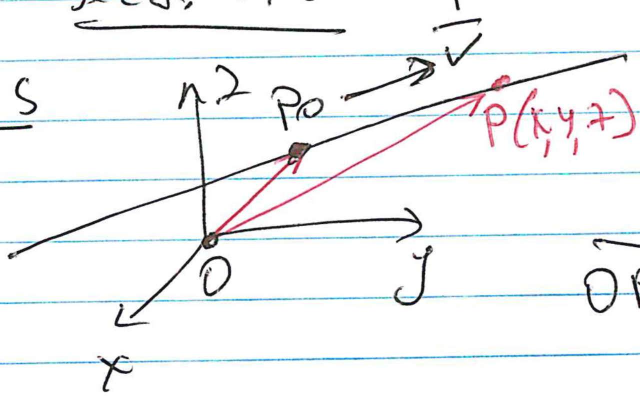
Solve for x: $4x^2 = 4z^2 - z^4$

$$x^2 \overset{?}{=} z^2 - \frac{z^4}{4} \Leftrightarrow x = \pm\sqrt{z^2 - \frac{z^4}{4}}.$$

Another way to write $\vec{P}$.

Sect. 11.5: planes and lines.

Lines



$P_0(a,b,c)$

$\vec{v} = (u, w, p)$

$\overrightarrow{OP} = \overrightarrow{OP_0} + t\vec{v}, \quad t \in \mathbb{R}$

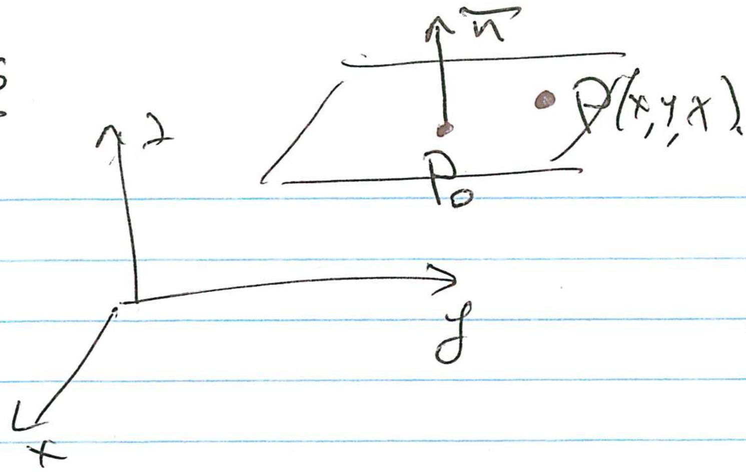
I. $(x,y,z) = (a,b,c) + t(u,w,p), \quad t \in \mathbb{R}$

II. $\begin{cases} x \overset{?}{=} a + tu \\ y = b + tw \\ z = c + tp, \quad t \in \mathbb{R} \end{cases}$

III. $\dfrac{x-a}{u} = \dfrac{y-b}{w} = \dfrac{z-c}{p}$

3 ways to write eq-s for this line.

Planes



$P_0(a,b,c)$

$\bar{n} \perp$ plane
(normal vector)

Find all pts $P(x,y,z)$ on this plane.

P has to be s.t. $\overline{P_0P} \perp \bar{n}$

$\overline{P_0P} = (x-a, y-b, z-c)$

If $\bar{n} = (A, B, C)$

$\bar{n} \perp \overline{P_0P}$ iff $\bar{n} \cdot \overline{P_0P} = 0$

$$\therefore \boxed{A(x-a) + B(y-b) + C(z-c) = 0}$$