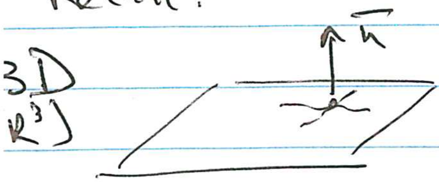
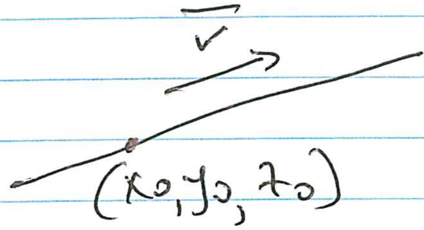
Sept. 17, 2019

Recall:



$Ax + By + Cz = D$

$\bar{n} = (A, B, C)$ $\{\bar{n}(A, B, C)\}$



$(x, y, z) = (x_0, y_0, z_0) + t\bar{v}$, $t \in \mathbb{R}$ ← vector eq-n

If $\bar{v} = (a, b, c)$, then

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct, \quad t \in \mathbb{R} \end{cases}$$ ← parametric eq-s

If $abc \neq 0$, then solving for t we get

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$ ← symmetric eq-s.

Ex: $\dfrac{2x-1}{-1} = \dfrac{y+1}{2} = \dfrac{3z+2}{4}$

Find a pt. on this line and a vector $\parallel$ line.

Point: $\begin{cases} 2x - 1 = 0 \\ y + 1 = 0 \\ 3z + 2 = 0 \end{cases} \Rightarrow P\left(\frac{1}{2}, -1, -\frac{2}{3}\right)$

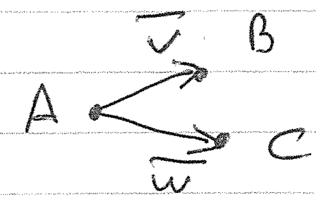
Vector: $\boxed{\bar{v} = (-1, 2, 4)}$ ← WRONG!

Rewrite in the symmetric form:

$$\frac{x - \frac{1}{2}}{-\frac{1}{2}} = \frac{y+1}{2} = \frac{z + \frac{2}{3}}{\frac{4}{3}}$$

Vector $\bar{v}$ which is $\parallel$ line: $\bar{v} = \left(-\frac{1}{2}, 2, \frac{4}{3}\right)$.

Ex: Find eqn of the plane through
$A(0,1,0)$, $B(2,1,-1)$, $C(3,0,-2)$.



$\bar{v} \times \bar{w} \perp$ plane

$\bar{v} = \overline{AB} = (2, 0, -1)$

$\bar{w} = \overline{AC} = (3, -1, -2)$

$$\bar{v} \times \bar{w} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 0 & -1 \\ 3 & -1 & -2 \end{vmatrix} = \bar{i}\begin{vmatrix} 0 & -1 \\ -1 & -2 \end{vmatrix} - \bar{j}\begin{vmatrix} 2 & -1 \\ 3 & -2 \end{vmatrix} + \bar{k}\begin{vmatrix} 2 & 0 \\ 3 & -1 \end{vmatrix}$$

$$= \bar{i}\left(0\cdot(-2) - (-1)(-1)\right) - \bar{j}\left(2\cdot(-2) - 3\cdot(-1)\right) +$$

$$+ \bar{k}\left(2\cdot(-1) - 0\cdot 3\right) =$$

$$= -\bar{i} + \bar{j} - 2\bar{k} = (-1, 1, -2)$$

$$\left\{ \text{Recall: } \bar{i} = (1,0,0),\ \bar{j} = (0,1,0),\ \bar{k} = (0,0,1) \right\}$$

$\therefore\ \bar{n} = (-1, 1, -2) \perp$ plane

Eqn of plane: $-1(x-0) + 1\cdot(y-1) - 2(z-0) = 0$

$$-x + y - 1 - 2z = 0$$

$$\boxed{x - y + 2z + 1 = 0}$$

Ex: Find eq-n of the line of intersection of planes $x+y+z=1$ and $-x+2y+3z=4$.

Observation: line of intersection of two planes with normals $\bar{n}_1$ and $\bar{n}_2$ is parallel to $n_1 \times n_2$.

~~Let~~ $\bar{n}_1 = (1,1,1)$, $\bar{n}_2 = (-1,2,3)$

$$\bar{n}_1 \times \bar{n}_2 = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 1 & 1 \\ -1 & 2 & 3 \end{vmatrix} = (1,-4,3) \leftarrow \parallel \text{ line.}$$

Find a pt.:

let $z=0$: $\begin{cases} x+y=1 \\ -x+2y=4 \end{cases} \Leftrightarrow \begin{cases} 3y=5 \\ x=1-y \end{cases} \Leftrightarrow$

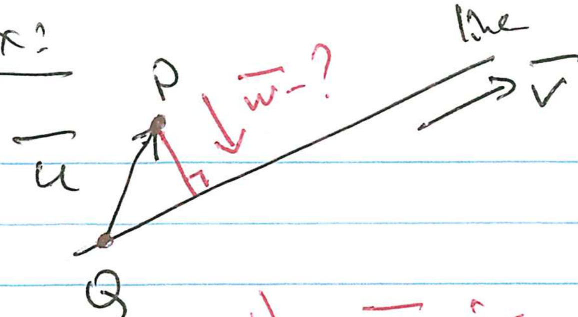
$\Leftrightarrow \begin{cases} y=\frac{5}{3} \\ x=-\frac{2}{3} \end{cases}$ $\therefore \left(-\frac{2}{3}, \frac{5}{3}, 0\right)$ is a point on line.

Vector eq-n: $(x,y,z) = \left(-\frac{2}{3}, \frac{5}{3}, 0\right) + t(1,-4,3)$, $t \in \mathbb{R}$

Param. eq-ns: $\begin{cases} x = -\frac{2}{3} + t \\ y = \frac{5}{3} - 4t \\ z = 3t, \quad t \in \mathbb{R}. \end{cases}$

Symmetric: $x + \frac{2}{3} = \frac{y - \frac{5}{3}}{4} = \frac{z}{3}$

Ex:



$P(1,0,1)$

$Q(-1,1,0)$

$\overline{v} = (-2,1,1)$

1) $\overline{w}$ is $\perp$ line

2) Initial point of $\overline{w}$ is P, i.e., $\overline{w}$ is $\parallel$ plane containing line and P.

Let $\overline{u}$ be $\overline{QP}$.

$\overline{w}$ is in plane through P and line: $\overline{w} \perp \overline{n}$, $\overline{n}$ - normal

$\overline{w}$ is $\perp \overline{v}$ $\therefore$ $\boxed{\overline{w} \parallel \overline{n} \times \overline{v}}$

$\boxed{\overline{n} \parallel \overline{u} \times \overline{v}}$

$\therefore \overline{w} \parallel (\overline{u} \times \overline{v}) \times \overline{v}$

$\overline{u} = \overline{QP} = (2,-1,1)$

$\overline{v} = (-2,1,1)$

$$\overline{u} \times \overline{v} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ 2 & -1 & 1 \\ -2 & 1 & 1 \end{vmatrix} = (-2,-4,0) \quad \{= -2(1,2,0)\}$$

$$(\overline{u} \times \overline{v}) \times \overline{v} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ -2 & -4 & 0 \\ -2 & 1 & 1 \end{vmatrix} = (-4,2,-10)$$

Note: could have taken any vector $\parallel$ to $\overline{u} \times \overline{v}$, e.g., $(1,2,0)$

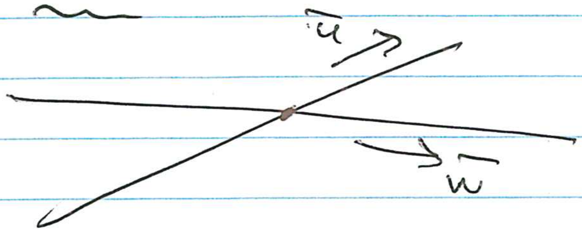
$\therefore \overline{w} \parallel (-4,2,-10)$ $\quad \{-2(2,-1,5)\}$

$\boxed{\overline{w} \parallel (2,-1,5).}$

#5, p. 734 Find Eq-n of plane containing the lines

$x = 2y = \frac{z+1}{4}$ (line 1) and $x = t,\ y = 2t,\ z = 6t-1$ (line 2).

HW: Check that these lines intersect each other!



Normal $\bar n = \bar u \times \bar w$

Line 1: $\bar u \parallel$ line 1

$$\frac{x}{1} = \frac{y}{\frac{1}{2}} = \frac{z+1}{4}$$

$$\bar u_1 = \left(1, \tfrac{1}{2}, 4\right) \longrightarrow \bar u_1 \parallel (2, 1, 8) = 2\bar u_1$$

$\therefore$ Take $\boxed{\bar u = (2, 1, 8)}$

Line 2: $\boxed{\bar w = (1, 2, 6)}$

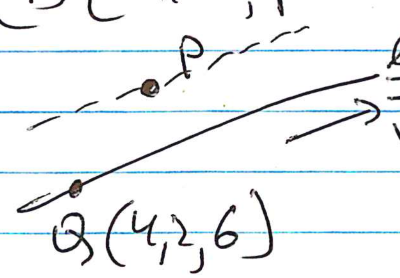
$$\bar n = \begin{vmatrix} \bar i & \bar j & \bar k \\ 2 & 1 & 8 \\ 1 & 2 & 6 \end{vmatrix} = (-10, -4, 3) \leftarrow \text{normal.}$$

$P(0, 0, -1)$ is on line 1 and so is on the plane.

Eq-n: $-10(x-0) - 4(y-0) + 3(z+1) = 0.$

Ex: Find eq-n of the line through $(2, 0, 3)$ and (i) parallel (ii) perpendicular (and crossing) to
$$\begin{cases} x = 4 + t \\ y = 2 \\ z = 6 - 2t \end{cases}$$

(i) (#18, p. 735).



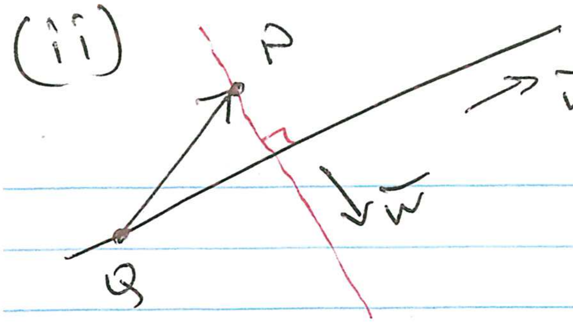
line 1. $P(2, 0, 3)$, $Q(4, 2, 6)$

$\bar v = (1, 0, -2)$.

Eq-n of line $\parallel$ line 1 is:

$$(x, y, z) = (2, 0, 3) + t(1, 0, -2),\ t \in \mathbb{R}.$$

(ii)



$\bar n = \overline{QP} \times \bar v \perp$ plane

$\bar n \times \bar v \perp$ "both $\bar v$ and $\bar n$"

$\therefore \bar n \times \bar v \parallel \bar w$

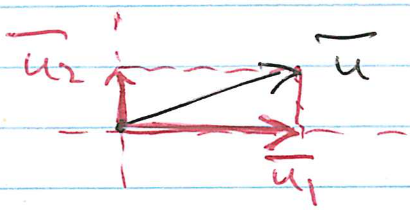
$\therefore$ ~~$(\bar n \times \bar v) \times \bar v$~~

$\bar n = \overline{QP} \times \bar v \quad \therefore \bar w \parallel (\overline{QP} \times \bar v) \times \bar v$.

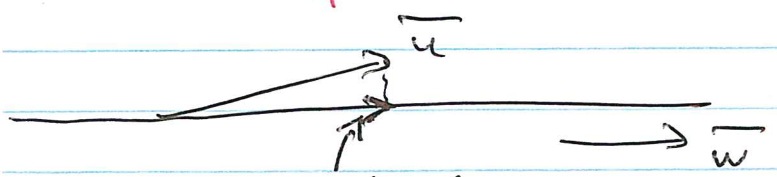
$\overline{QP} = (-2, -2, -3)$

$\bar v = (1, 0, -2)$. ... (HW).

Sect. 11.6 Applications.

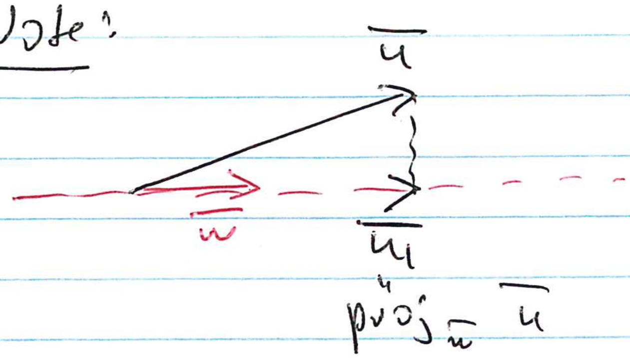


$\bar u = \bar u_1 + \bar u_2$



component of $\bar u$ in the direction $\bar w$

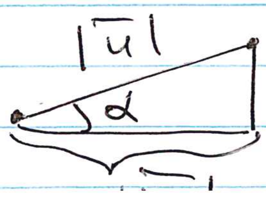
Note:



Assume that $|\bar w| = 1$ (i.e., $\bar w$ is a unit vector)

$\left\{ \text{proj}_{\bar w} \bar u \right.$ projection of $\bar u$ onto $\left. \bar w \right\}$

Find the length of $\bar u_1$?

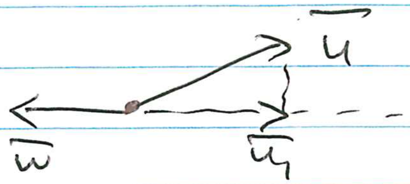


$|\bar u_1| = |\bar u| \cos\alpha$

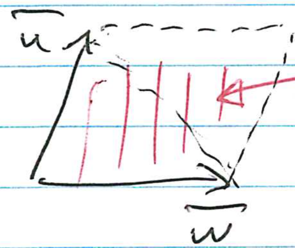
$\bar u \cdot \bar w = |\bar u| \cdot \underbrace{|\bar w|}_{1} \cos\alpha = |\bar u_1|$.

$$\boxed{\text{Length of projection of } \overline{u} \text{ onto a unit vector } \overline{w} \text{ is } |\overline{u} \cdot \overline{w}|.}$$

Q-n: how can $\overline{u} \cdot \overline{w}$ be negative?

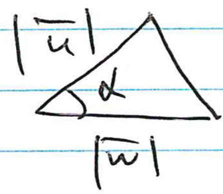


Recall:

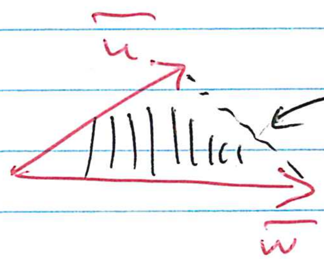


Area $= |\overline{u} \times \overline{w}|$

$|\overline{u} \times \overline{w}| = |\overline{u}| \cdot |\overline{w}| \sin \alpha$

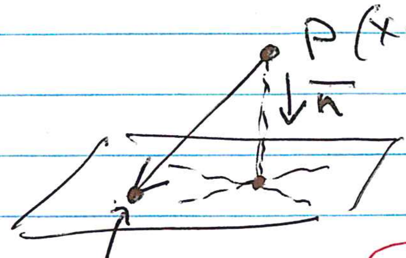


Area of triangle $= \frac{1}{2} |\overline{u}| \cdot |\overline{w}| \sin \alpha$

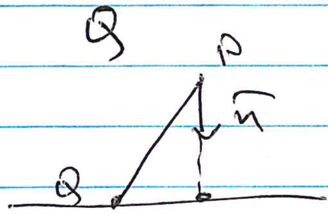


Area of triangle $=$
$= \frac{1}{2} |\overline{u} \times \overline{w}|.$

Distance from a point to a plane.



$Ax + By + Cz + D = 0.$



$\boxed{\text{Distance} = |\overline{PQ} \cdot \overline{n}|, \text{ where } \overline{n} \text{ is a unit vector normal to the plane.}}$

Let $Q\,(x_1, y_1, z_1)$.

$\overline{PQ} = (x_1 - x_0, y_1 - y_0, z_1 - z_0)$

Normal $\parallel (A, B, C)$

$\therefore$ Unit normal $= \dfrac{(A, B, C)}{\sqrt{A^2+B^2+C^2}} = \overline{n}$

Distance $= \left| \overline{PQ} \cdot \overline{n} \right| =$

$$= \frac{\left| A(x_1 - x_0) + B(y_1 - y_0) + C(z_1 - z_0) \right|}{\sqrt{A^2+B^2+C^2}}$$

Since Q is on the plane, $Ax_1 + By_1 + Cz_1 + D = 0$

$$\therefore \text{Distance} = \frac{\left| Ax_1 + By_1 + Cz_1 - Ax_0 - By_0 - Cz_0 \right|}{\sqrt{A^2+B^2+C^2}} =$$

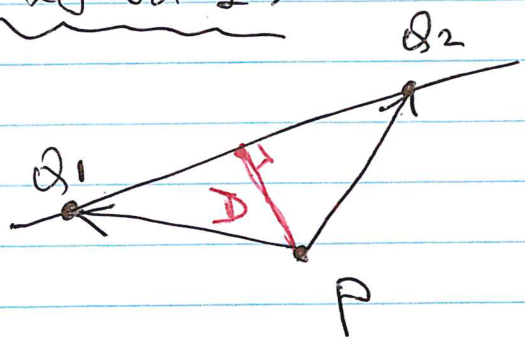
$$= \frac{\left| -D - Ax_0 - By_0 - Cz_0 \right|}{\sqrt{A^2+B^2+C^2}}$$

$$\boxed{= \frac{\left| Ax_0 + By_0 + Cz_0 + D \right|}{\sqrt{A^2+B^2+C^2}}}$$

Distance from (x_0, y_0, z_0) to plane $Ax+By+Cz+D=$

Distance from a point to a line

Method 1:



Area of triangle $Q_1 Q_2 P$:

1) $A = \frac{1}{2} \left| \overline{Q_1 Q_2} \right| \cdot D$

2) $A = \frac{1}{2} \left| \overline{PQ_1} \times \overline{PQ_2} \right|$

3C

$$\therefore \boxed{D = \frac{|\overline{PQ_1} \times \overline{PQ_2}|}{|\overline{Q_1Q_2}|}}$$

If $P(x_0, y_0, z_0)$

$Q_1(x_1, y_1, z_1)$

$Q_2(x_2, y_2, z_2)$, then

$$\boxed{\text{Distance } D = \frac{|(x_1-x_0, y_1-y_0, z_1-z_0) \times (x_2-x_0, y_2-y_0, z_2-z_0)|}{|(x_2-x_1, y_2-y_1, z_2-z_1)|}}$$