Sept. 24, 2019.

#4, p. 765. $z = x^2+y^2$, $x^2+y^2=5$ (directed clockwise as viewed from the origin)

Find parametric and vector representations.

$$\begin{cases} z = x^2+y^2 \\ x^2+y^2=5 \end{cases} \iff \begin{cases} x^2+y^2=5 \\ z=5 \end{cases}$$

$(x+a)^2+(y+b)^2=R^2$

$\Uparrow$

$x+a = R\cos t$

$y+b = R\sin t$

$\Downarrow$

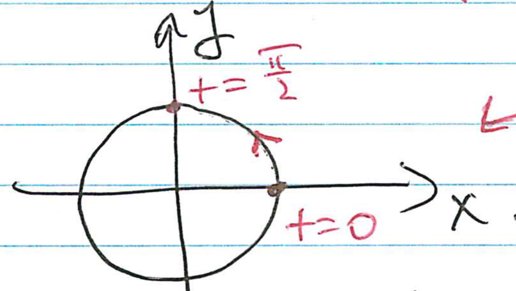
$x = -a + R\cos t$

$y = -b + R\sin t$, $t \in \mathbb{R}$.

$$\begin{cases} x = \sqrt{5}\cos t \\ y = \sqrt{5}\sin t \\ z = 5 \end{cases}, \ t \in \mathbb{R}$$

Orientation?

Answer.



orientation is correct.

#7, p. 765 $z = x+y$, $y = x^2$ directed so that $x \uparrow$ along the curve.

$$\begin{cases} x = t \\ y = t^2 \\ z = t+t^2, \ t \in \mathbb{R} \end{cases}$$

Answer.

Orientation?

Yes, correct.

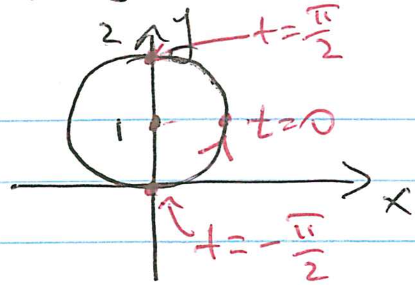
#8, p. 765 $z = \sqrt{4-x^2-y^2}$, $x^2+y^2-2y=0$ directed so that $z \downarrow$ when x is positive.

$$\begin{cases} z = \sqrt{4-x^2-y^2} \\ x^2+y^2-2y=0 \end{cases} \iff \begin{cases} x^2+(y-1)^2=1 \\ z = \sqrt{4-x^2-y^2} \end{cases}$$

$$\begin{cases} x = \cos t \\ y = 1+\sin t \\ z = \sqrt{4-\cos^2 t-(1+\sin t)^2} \end{cases} \iff \begin{cases} x = \cos t \\ y = 1+\sin t \\ z = \sqrt{2-2\sin t}, \ t \in \mathbb{R}. \end{cases}$$

Answer.

I. Projection onto xy-plane



$t=-\frac{\pi}{2}$: $(0, 0, 2)$

$t=0$: $(1, 1, \sqrt{2})$

$t=\frac{\pi}{2}$: $(0, 2, 0)$

$\therefore$ $z \downarrow$ as t goes from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

(for these t, x is positive).

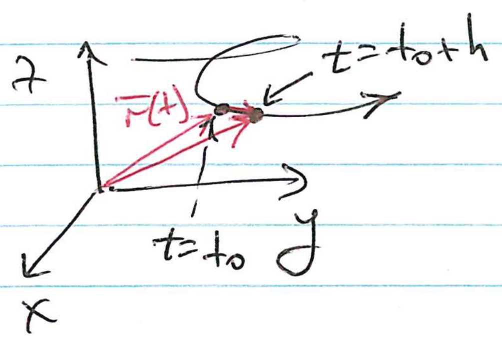
$\therefore$ Orientation was correct.

II. $x \geq 0$ if $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$

Find $\frac{dz}{dt}$ for these t. Is $\frac{dz}{dt} < 0$?

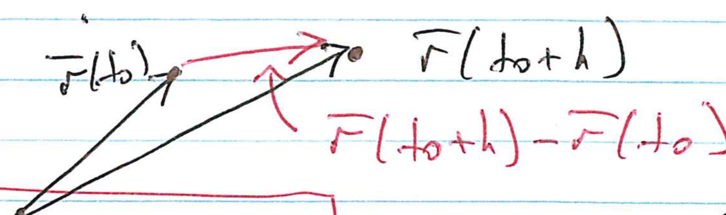
Hw: #9, 6 p. 765.

Section 11.11. Tangent vectors & Lengths of curves.



$\bar{r}(t)$ position vector at time t

$\bar{r}(t_0+h)$ = position vector at time t_0+h.



$$\boxed{\lim_{h \to 0} \frac{\bar{r}(t_0+h)-\bar{r}(t_0)}{h} = \bar{r}'(t_0)} \left\{ = \frac{d\bar{r}}{dt}(t_0) \right\}$$

tangent vector to the curve $\bar{r}=\bar{r}(t)$ at the pt. $t=t_0$.

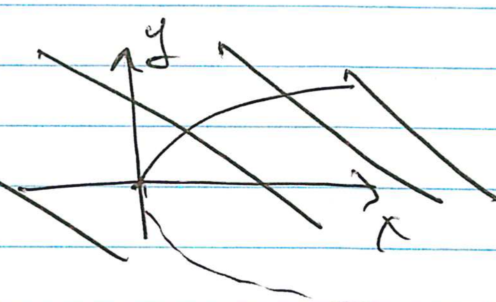
Note: 1) $\frac{d\bar{r}}{dt}$ points in the direction in which $t \uparrow$ along the curve.

2) ~~Also~~ If $\bar{r}(t) = (x(t), y(t), z(t))$, then
$$\bar{r}'(t) = (x'(t), y'(t), z'(t)).$$

Ex: **Q-n:** If x, y and z are all differentiable does it mean that the tangent exists?

Answer: No (we also need the condition: "x', y' and z' do not all vanish at that pt. t").

$$\cancel{\begin{cases} x = t^2 \\ y = t \\ z = 0, \quad t \in \mathbb{R}. \end{cases}}$$



Example to follow.

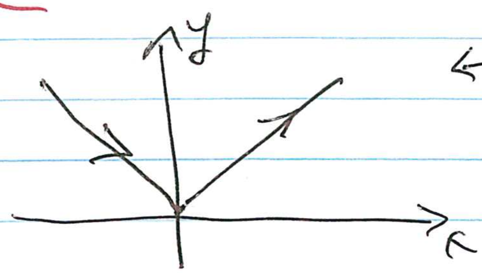
Fact: If $\bar{r}(t) = (x(t), y(t), z(t))$, $\alpha \le t \le \beta$, is the vector representation of a curve C, then at any point on C at which $x'(t), y'(t)$ and $z'(t)$ all exist and do not vanish simultaneously,
$$\frac{d\bar{r}}{dt} = (x'(t), y'(t), z'(t))$$
is a tangent vector to C.

Def-n: we say that a curve $C: \bar{r}(t) = (x(t), y(t), z(t))$, $\alpha \le t \le \beta$

is a <u>smooth curve</u> if

$x'(t)$, $y'(t)$ and $z'(t)$ are all continuous for $\alpha \le t \le \beta$ and do not vanish simultaneously for $\alpha < t < \beta$.

Def-n: A curve C is piecewise-smooth if it is <u>continuous</u> and can be divided into a finite number of smooth subcurves.

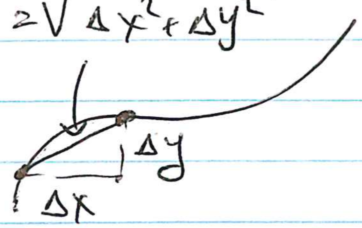


← piecewise-smooth curve.

$t=0$

<u>Unit tangent vector:</u> $\hat{T}(t) = \dfrac{\frac{d\bar{r}}{dt}}{\left|\frac{d\bar{r}}{dt}\right|}$

$\Delta s = \sqrt{\Delta x^2 + \Delta y^2}$



In 2D,
$$ds = \sqrt{dx^2 + dy^2}$$

In 3D,
$$ds = \sqrt{dx^2 + dy^2 + dz^2}$$

If $C: \bar{r}(t) = (x(t), y(t), z(t))$, then the length of this curve from the pt. corresp. to t_0 to a pt. corresp. to t is

$$L(t) = \int_{t_0}^{t} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du.$$

Note: $\bar{r}'(t) = \left(x'(t), y'(t), z'(t)\right)$

$$|\bar{r}'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2},$$

i.e., $L(t) = \int_{t_0}^{t} |\bar{r}'(u)|\, du$ and so

$$\frac{dL}{dt} = |\bar{r}'(t)|$$

If we set $s = s(t)$ to be $s(t) = \int_{t_0}^{t} |\bar{r}'(u)|\, du$,

then $\boxed{\dfrac{ds}{dt} = |\bar{r}'(t)|}$

[Note: ~~AB~~ you can rewrite $\bar{r} = \bar{r}(t)$ in terms of the new parameter s (so called arc-length parametrization).

$\bar{r} = \bar{r}(s)$, $s \in \mathbb{R}$.

$\dfrac{d\bar{r}}{ds} \leftarrow$ tangent.

$$\frac{d\bar{r}}{dt} = \frac{d\bar{r}}{ds} \cdot \frac{ds}{dt} \quad \text{(the Chain Rule)}.$$

$$\Updownarrow$$

$$\frac{d\bar{r}}{ds} = \frac{\frac{d\bar{r}}{dt}}{\frac{ds}{dt}} = \frac{\frac{d\bar{r}}{dt}}{\left|\frac{d\bar{r}}{dt}\right|}$$

$\therefore \dfrac{d\bar{r}}{ds}$ has unit length, i.e., $\boxed{\hat{T}(s) = \dfrac{d\bar{r}}{ds}}$.

[If ~~this~~ $\bar{r} = \bar{r}(s)$ is the arc-length parametrization, then $\dfrac{d\bar{r}}{ds}$ has unit length.]

#2, p. 770 Find $\hat{T}$ (unit tangent vector) at each point on the curve:

$$\begin{cases} x = t \\ y = t^2 \\ z = t^3, \quad t \geq 1 \end{cases} \qquad \begin{aligned} \bar{r}(t) &= (t, t^2, t^3) \\ \bar{r}'(t) &= (1, 2t, 3t^2) \end{aligned}$$

Unit tangent: $\hat{T}(t) = \dfrac{\bar{r}'(t)}{|\bar{r}'(t)|} = \cancel{\dfrac{(t, t^2, t^3)}{\sqrt{1+4t^2+9t^4}}}$

$$= \frac{(1, 2t, 3t^2)}{\sqrt{1+4t^2+9t^4}}.$$

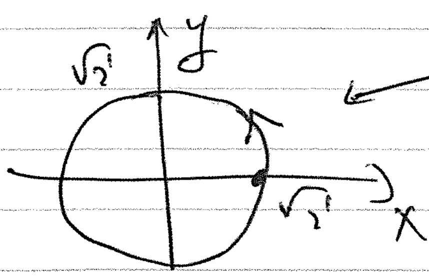
#8, p. 770 Find $\hat{T}$ at $(1, 1, \sqrt{2})$:

$x^2+y^2+z^2=4$, $z = \sqrt{x^2+y^2}$ directed so that $x \uparrow$ when y is positive

$$\begin{cases} x^2+y^2+z^2=4 \\ z=\sqrt{x^2+y^2} \end{cases} \Leftrightarrow \begin{cases} z^2 = x^2+y^2, \quad z \geq 0 \\ x^2+y^2+z^2=4 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} z^2 = 2, \quad z \geq 0 \\ z^2 = x^2+y^2 \end{cases} \Leftrightarrow \begin{cases} z = \sqrt{2} \\ x^2+y^2 = 2 \end{cases}$$

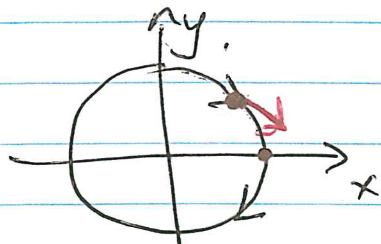
$$\Leftrightarrow \begin{cases} x = \sqrt{2}\cos t \\ y = \sqrt{2}\sin t \\ z = \sqrt{2}, \quad t \in \mathbb{R}. \end{cases}$$



x is $\downarrow$ when y is positive.

$\therefore$ Replace t by $-t$ (!).

$$\begin{cases} x = \sqrt{2}\cos t \\ y = -\sqrt{2}\sin t \\ z = \sqrt{2} \end{cases}, \quad t \in \mathbb{R}.$$ ← The right parametrization.



Now, $x \uparrow$ when $y > 0$.

$\bar{r}(t) = (\sqrt{2}\cos t, -\sqrt{2}\sin t, \sqrt{2}),\ t \in \mathbb{R}.$

$t = ?$ $\quad \bar{r}(t) = (1, 1, \sqrt{2})$

$t = -\frac{\pi}{4}$

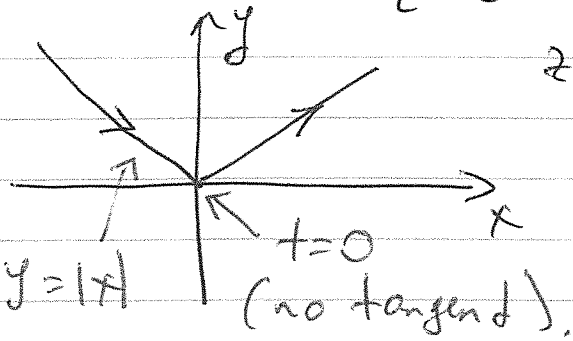
$$\therefore \hat{T}\left(-\frac{\pi}{4}\right) = \left.\frac{\bar{r}'(t)}{|\bar{r}'(t)|}\right|_{t=-\frac{\pi}{4}} =$$

$$= \left.\frac{(-\sqrt{2}\sin t, -\sqrt{2}\cos t, 0)}{\sqrt{2\sin^2 t + 2\cos^2 t}}\right|_{t=-\frac{\pi}{4}} =$$

$$= (-\sin t, -\cos t, 0)\Big|_{t=-\frac{\pi}{4}} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$$

$$\therefore \hat{T}\left(-\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right).$$

Example:
$$\begin{cases} x = t^3 \\ y = t^2|t| \\ z = 0 \end{cases}, \quad t \in \mathbb{R}.$$



$z=0$

x, y and z are all differentiable:
$$\begin{cases} x'(t) = 3t^2 \\ y'(t) = 3t|t| \quad (\text{HW}) \\ z'(t) = 0 \end{cases}$$

Also, $x'(0) = y'(0) = z'(0) = 0$