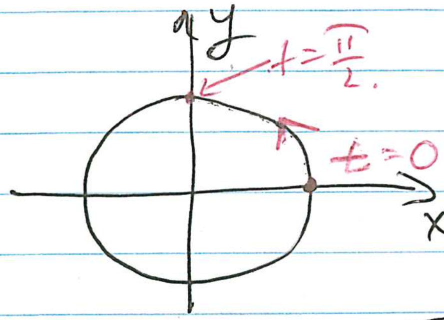
4+

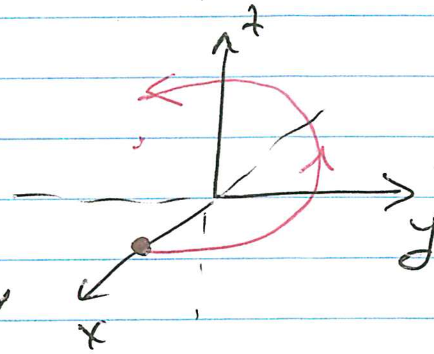
# Sept. 26, 2019.

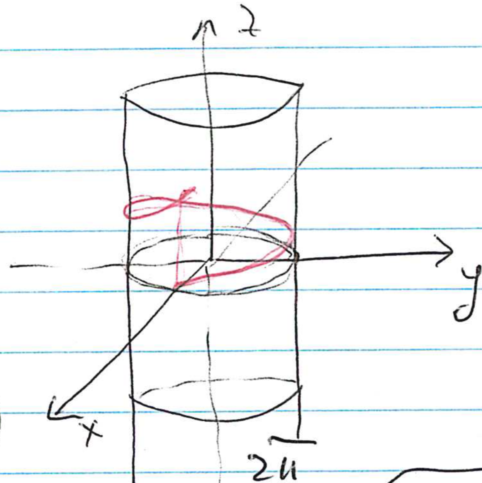
**#11, p.770** Find the length of the curve (+ draw the curve):

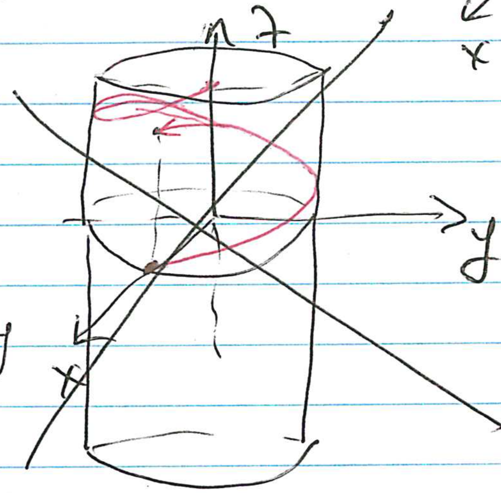
$x = 2\cos t,\ y = 2\sin t,\ z = 3t,\ 0 \le t \le 2\pi.$

$x^2 + y^2 = 4$



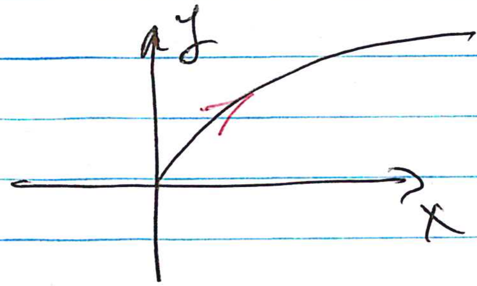



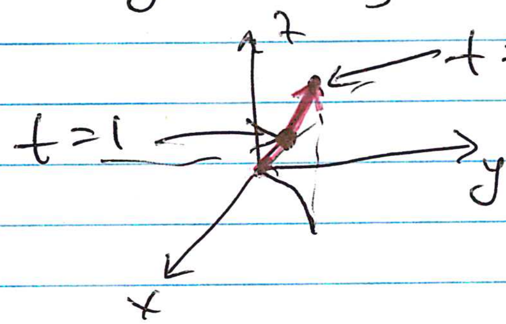

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt =$$

$$= \int_0^{2\pi} \sqrt{(-2\sin t)^2 + (2\cos t)^2 + 3^2}\, dt = 2\pi\sqrt{13}.$$

**#14, p.770** $x = t,\ y = t^{3/2},\ z = 4t^{3/2},\ 1 \le t \le 4.$

$y = x^{3/2},\quad z = 4y.$



$$L = \int_1^4 \sqrt{1 + \left(\tfrac{3}{2} t^{\frac{1}{2}}\right)^2 + \left(4 \cdot \tfrac{3}{2} t^{\frac{1}{2}}\right)^2}\, dt =$$

$$= \int_1^4 \sqrt{1 + \tfrac{9}{4} t + 36 t}\, dt = \dots \text{ HW.}$$

## Sect. 12.1 Multivariable F-s.

Curve: $\overline{r}(t) = (\underbrace{x(t), y(t), z(t)}_{\text{3 values.}})$

$\overline{r}: R \to R^3$

One variable

$$f(x_1, x_2, x_3) = y$$

$f: R^3 \to R$    1 value.

3 variables

**Ex:** $f(x, y, z) = x^2 + yz$

$f(1, 2, 3) = 1^2 + 2 \cdot 3 = 7$

In general, $f(\underbrace{x_1, \dots, x_n}_{n \text{ variables}}) = \underbrace{V}_{\in R}$.

$$f: R^n \to R$$

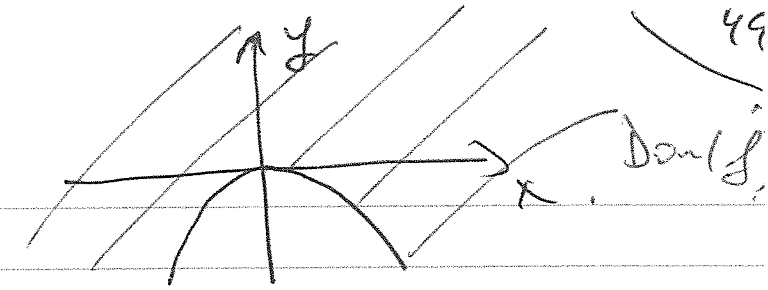
Domain of $f = f(x_1, \dots, x_n)$ is the set

$$S = \{(x_1, \dots, x_n) \mid f(x_1, \dots, x_n) \text{ is defined}\}.$$

**Ex:** $f(x, y) = \sqrt{y + x^2}$. Find Dom($f$).

$$\text{Dom}(f) = \{(x, y) \mid y + x^2 \geq 0\}$$

$y + x^2 = 0 \Leftrightarrow y = -x^2$




**Note:** 1) $f = f(x,y)$ defines the surface $z = f(x,y)$.

2) Graph of $f = f(x,y)$ is the graph of the surface $z = f(x,y)$

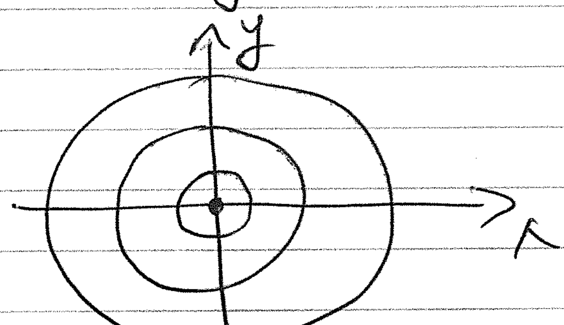
which is $\{(x,y,z) \in \mathbb{R}^3 \mid z = f(x,y),\ (x,y) \in Dom(f)\}$

3) 1) Given $f = f(x,y,z)$, $f(x,y,z) = K$, $K \in \mathbb{R}$, represents a level surface.

2) Given $f = f(x,y)$, $f(x,y) = K$, $K \in \mathbb{R}$, represents a level curve (for each $K$).

**Ex:** $f(x,y) = x^2 + y^2$. Level curves?

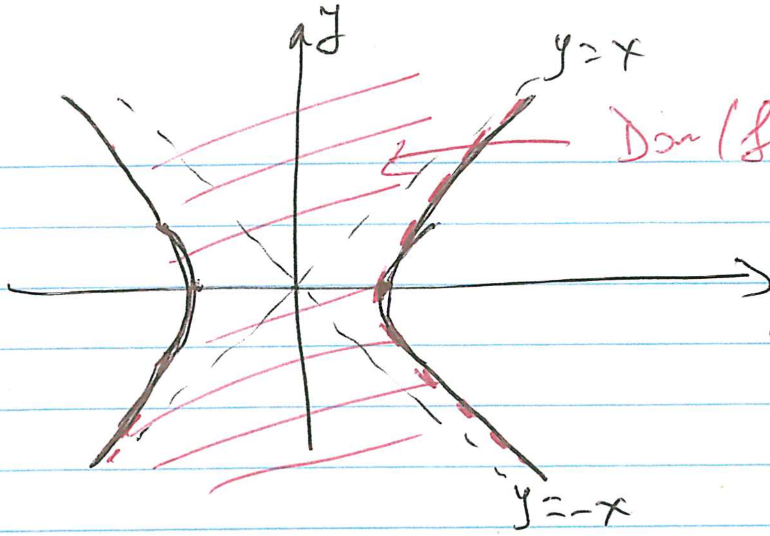
$x^2 + y^2 = K \leftarrow$ circle of radius $\sqrt{K}$, $K \geq 0$.




**# 4, p. 802** $f(x,y) = \ln(1 - x^2 + y^2)$

$Dom(f) = \{(x,y) \mid 1 - x^2 + y^2 > 0\}$

$1 - x^2 + y^2 = 0 \Leftrightarrow y^2 = x^2 - 1$



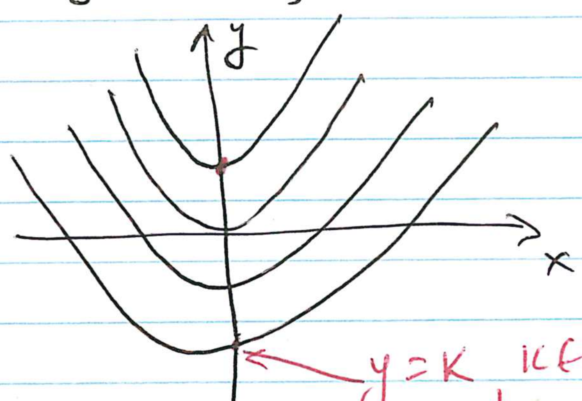

Dom(f) (note: the boundary is not included).

$$\left\{ \begin{array}{l} y=\sqrt{x^2-1} \\ \lim_{x\to\infty}\left(\sqrt{x^2-1}-x\right)=0 \quad \text{HW} \end{array} \right\}$$

#5, p. 802 HW.

#23, p. 802. Level curves? $f(x,y)=y-x^2$.

$y-x^2=K,\ K\in\mathbb{R} \iff y=x^2+K,\ K\in\mathbb{R}$.




$y=K,\ K\in\mathbb{R}$.

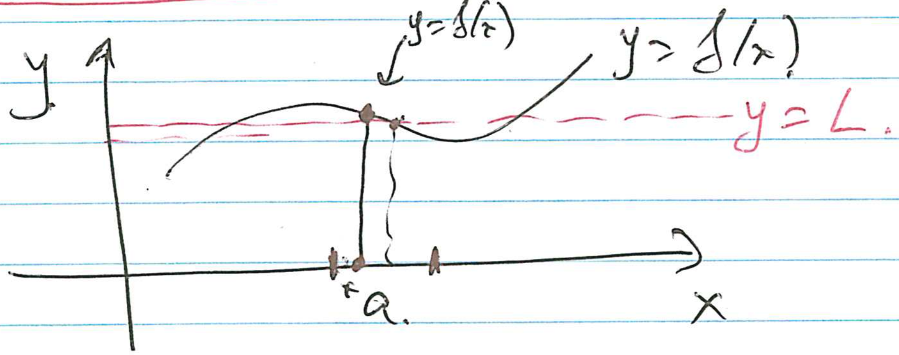
## Section 12.2 Limits and Continuity.

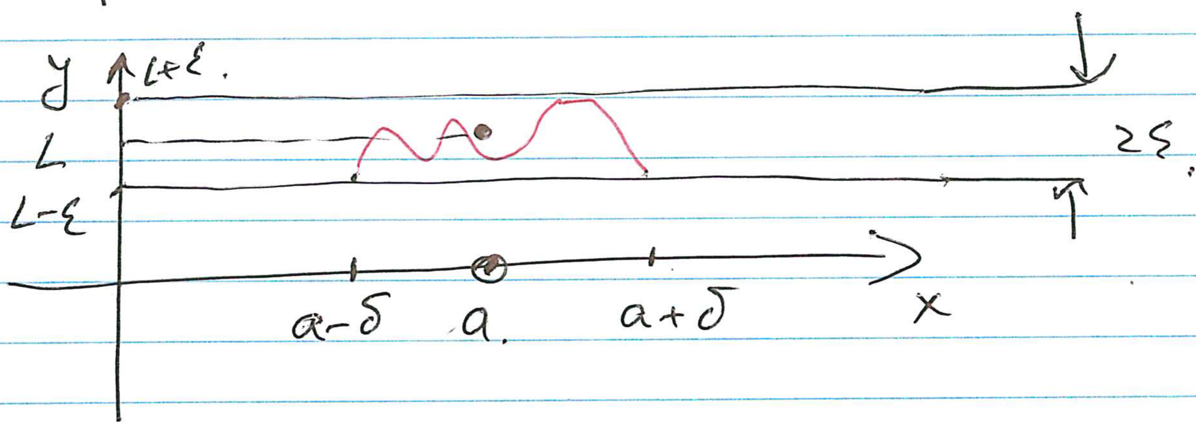
Recall the def-n of the limit of univariate f-s: $y=f(x)$

$\lim_{x\to a} f(x)=L \iff$

1st way For every $\varepsilon>0$, there is $\delta>0$ such that $|f(x)-L|<\varepsilon$ whenever $|x-a|<\delta$, $x\neq a$.

2nd way $\forall \varepsilon > 0 \; \exists \delta > 0$ s.t. $\forall x : |x-a| < \delta, x \neq a$
$$|f(x) - L| < \varepsilon.$$




Note: $\forall$ for every, for any

$\exists$ there is, there exist(s)

Def-n (Limit of $f = f(x,y)$).

A f-n $f = f(x,y)$ has a limit $L$ as $(x,y)$ approaches $(x_0, y_0)$, written $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ if, for every $\varepsilon > 0$, we can find $\delta > 0$ such that, for all $(x,y)$ satisfying $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$,

$$|f(x,y) - L| < \varepsilon.$$

<u>Remarks</u>: 1) $(x,y) \in \mathrm{Dom}(f)$ whenever $(x,y)$ is used in this def-n.

2) $0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$.


$$0 \overset{\shortparallel}{<} \mathrm{dist}\{(x,y),(x_0,y_0)\} < \delta$$

<u>Ex</u>: $f(x,y) = x^2+xy$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \left\{\begin{array}{l} x\to 0 : x^2 \to 0 \\ x\to 0, y\to 0 : xy \to 0 \\ \therefore x\to 0, y\to 0 : x^2+xy\to 0 \end{array}\right\} = 0.$$

<u>Formal proof</u>: $\lim_{(x,y)\to(0,0)} (x^2+xy) = 0$

Let $\varepsilon > 0$ be given. We need to find $\delta = \delta(\varepsilon) > 0$ s.t., if $0 < x^2+y^2 < \delta^2$, then $|x^2+xy| < \varepsilon$.

Suppose that $x^2+y^2<\delta^2 \Rightarrow \boxed{x^2<\delta^2}$
(since $x^2 < \delta^2 - y^2 < \delta^2$).

Q-n: what about $xy$?

$\boxed{|xy| \le \frac{x^2+y^2}{2}} \Leftrightarrow 2|xy| \le x^2+y^2 \Leftrightarrow x^2 - 2|x|\cdot|y| + y^2 \ge 0$
$\Leftrightarrow (|x|-|y|)^2 \ge 0$.

$\therefore$ If $x^2+y^2<\delta^2$ then $\boxed{|xy| < \frac{\delta^2}{2}}$

$\therefore$ If $x^2+y^2<\delta^2$, then

$$|x^2+xy| \underset{\substack{\uparrow \\ \text{triangle} \\ \text{inequality}}}{\le} |x^2| + |xy| < \delta^2 + \frac{\delta^2}{2} = \boxed{\frac{3}{2}\delta^2 < \varepsilon}$$

Is there $\delta$ s.t. this holds?

A: Yes. For example, pick $\delta = \sqrt{\frac{2\varepsilon}{3}} \cdot \frac{1}{100}$ 

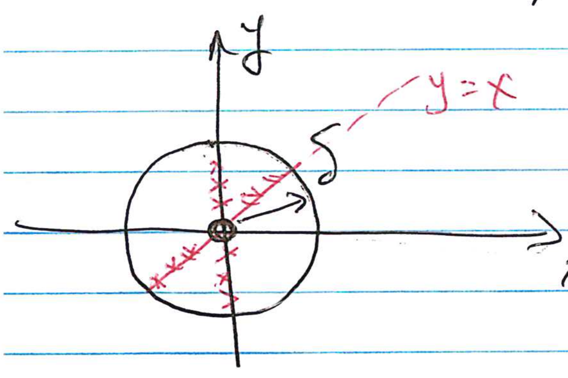
$$\left\{ \frac{3}{2}\delta^2 = \frac{3}{2} \cdot \frac{2\varepsilon}{3} \cdot \frac{1}{10{,}000} = \frac{\varepsilon}{10{,}000} < \varepsilon \right\}$$

Hence, for any $\varepsilon > 0$, there is $\delta = \delta(\varepsilon)$ (we found that $\delta = \sqrt{\frac{2\varepsilon}{3}} \cdot \frac{1}{100}$ works) s.t.
$|f(x,y) - 0| < \varepsilon$ whenever $0 < \sqrt{x^2+y^2} < \delta$.

$\therefore$ By the def-n, $\lim\limits_{(x,y)\to(0,0)} (x^2+xy) = 0$.

**Def-n** (continuity of $f = f(x,y)$ at a pt. $(x_0, y_0)$).
we say that $f = f(x,y)$ is continuous at $(x_0,y_0)$ **iff** $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists and $= f(x_0,y_0)$.

**Ex:** $f(x,y) = \dfrac{x^2}{x^2+y^2}$ $\qquad \mathrm{Dom}(f) = \{(x,y) \mid (x,y) \neq (0,0)\}$




$\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{x^2+y^2}$ — ?

If $y = x$, then $f(x,y) = f(x,x) =$
$$= \frac{x^2}{x^2+x^2} = \frac{x^2}{2x^2} = \frac{1}{2}.$$

Approach $(0,0)$ along $y$-axis:

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{and } x=0}} f(x,y) = \lim_{\substack{x=0\\ y\to 0}} 0 = 0$$

Approach $(0,0)$ along $y=x$:

$$\lim_{\substack{(x,y)\to(0,0)\\ x=y}} f(x,y) = \lim_{x\to 0} \frac{x^2}{x^2+x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}.$$

i.e. we get different "limits" as we approach $(0,0)$ along 2 different curves.

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ d.n.e.

Q-n: If limits along all lines $y=ax$ are the same, can we say that $\lim_{(x,y)\to(0,0)} f(x,y)$ exists?

Answer: NO (!!!).