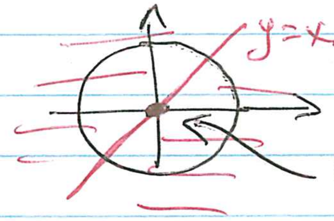
Ex: Find the limit if it exists or explain why it does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x-y}{x-y} \;=\; \left\{\begin{array}{l} f \text{ is not defined at } (0,0) \\ f \text{ is not defined along } y=x \end{array}\right.$$



$Dom(f) = \{(x,y) \mid x \neq y\}$.

$(x,y) \to (0,0)$ inside the domain of f

Recall the def-n ($\lim_{(x,y)\to(a,b)} f(x,y) = L$).
For any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that, for all (x,y) satisfying $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ and $(x,y) \in Dom(f)$, we have $|f(x,y) - L| < \varepsilon$

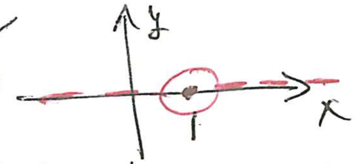
$$= \left\{\begin{array}{l} \text{Since } (x,y) \in Dom(f), \\ \quad x-y \neq 0 \text{ and so} \\ \frac{x-y}{x-y} = 1 \end{array}\right\} = \lim_{(x,y)\to(0,0)} 1 = 1.$$

Ex: Is $f(x,y) = \frac{x-y}{x-y}$ continuous at $(0,0)$?

NO since f is NOT defined at $(0,0)$.

(!): If f is NOT defined at (a,b), then f is NOT continuous at this point (i.e., no need to consider $\lim_{(x,y)\to(a,b)} f(x,y)$).

#5, p. 806. $\lim_{(x,y)\to(1,0)} \frac{x}{y} = \left\{ \right.$



$x \to 1$ as $x \to 1$

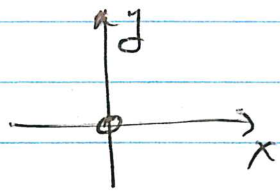
$y \to 0$ as $y \to 0$

$\frac{x}{y} \not\to L \in \mathbb{R}$ as $(x,y) \to (1,0)$ $\left. \right\}$

$= DNE$

Ex: $\lim_{(x,y)\to(0,0)} (x^2+y^2) \sin \frac{1}{x^2+y^2} = 0$

$= \left\{ \right.$



$0 \cdot \underbrace{\sin(\infty)}_{dne}$

Squeeze Theorem:

If $h(x,y) \le f(x,y) \le g(x,y)$ for all (x,y) close to (a,b) (but maybe not at (a,b)) and

$\lim_{(x,y)\to(a,b)} h(x,y) = \lim_{(x,y)\to(a,b)} g(x,y) = L$, then

$\lim_{(x,y)\to(a,b)} f(x,y) = L.$

~~[illegible]~~

$$-1 \le \sin \frac{1}{x^2+y^2} \le 1$$

$$\Downarrow$$

$$\underbrace{-(x^2+y^2)}_{h(x,y)} \le \underbrace{(x^2+y^2)\sin\frac{1}{x^2+y^2}}_{f(x,y)} \le \underbrace{(x^2+y^2)}_{g(x,y)}$$

$\lim_{(x,y)\to(0,0)} h(x,y) = \lim_{(x,y)\to(0,0)} -(x^2+y^2) = 0$

$\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{(x,y)\to(0,0)} (x^2+y^2) = 0$

$\therefore$ By the Squeeze Thm,

$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$

Ex: $\lim_{(x,y)\to(0,0)} xy \sin \frac{1}{xy}$.

WRONG SOL-N

$$-1 \le \sin \frac{1}{xy} \le 1$$

WRONG.

$$\Downarrow$$

$$-xy \le xy \sin \frac{1}{xy} \le xy$$

This only true if $xy \ge 0$.

Take x, y to be s.t. $xy = -1$

$LHS = -xy = -(-1) = 1$

$RHS = xy = -1$

$\therefore\ 1 \le -1$ wrong

Theorem:

$\lim_{(x,y)\to(a,b)} f(x,y) = 0$ if and only if $\lim_{(x,y)\to(a,b)} |f(x,y)| = 0$.

$$\lim_{(x,y)\to(0,0)} \left| xy \sin \frac{1}{xy} \right|$$

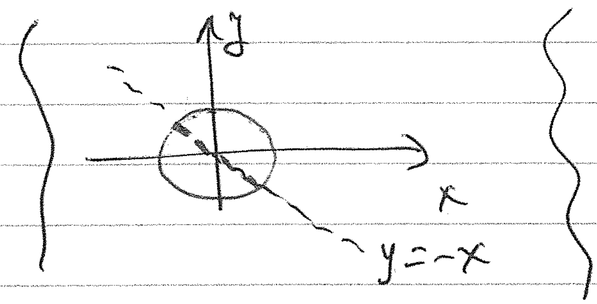
$$0 \le \left| xy \sin \frac{1}{xy} \right| = |xy| \cdot \left| \sin \frac{1}{xy} \right| \le |xy|$$

0 → 0 as $(x,y)\to(0,0)$; $|xy|$ → 0 as $(x,y)\to(0,0)$.

$\therefore$ By the S.T., $\lim_{(x,y)\to(0,0)} \left| xy \sin \frac{1}{xy} \right| = 0$

and so $\lim_{(x,y)\to(0,0)} xy \sin \frac{1}{xy} = 0$.

#14, p. 806 $\lim\limits_{(x,y)\to(0,0)} \dfrac{x-y}{x+y}$

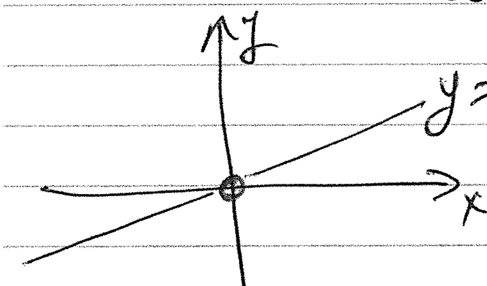


$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=0}} \frac{x-y}{x+y} = \lim_{\substack{x\to 0\\ y=0}} \frac{x-y}{x+y} = \lim_{x\to 0} \frac{x}{x} = 1$$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } x=0}} \frac{x-y}{x+y} = \lim_{\substack{y\to 0\\ x=0}} \frac{x-y}{x+y} = \lim_{y\to 0} \frac{-y}{y} = -1$$

Since $1 \neq -1$, $\lim\limits_{(x,y)\to(0,0)} \dfrac{x-y}{x+y}$ DNE.

#29, p. 806 $\lim\limits_{(x,y)\to(0,0)} \underbrace{\dfrac{x^6-2y^2}{3x^6+y^2}}_{= f(x,y)}$



$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=mx}} f(x,y) = \lim_{x\to 0} \frac{x^6-2(mx)^2}{3x^6+(mx)^2} =$$

$$= \lim_{x\to 0} \frac{x^6-2m^2x^2}{3x^6+m^2x^2} = \lim_{x\to 0} \frac{x^4-2m^2}{3x^4+m^2} = \frac{-2m^2}{m^2} = -2$$

only if $m \neq 0$

If $m=0$, then $\lim\limits_{\substack{(x,y)\to(0,0)\\ \text{along } y=0}} f(x,y) = \lim\limits_{x\to 0} \dfrac{x^4}{3x^4} = \dfrac{1}{3}$

limits along $y = mx$, $m \neq 0$ and $y = 0$ as $(x,y) \to (0,0)$ are different.

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ DNE.

<u>Ex:</u> $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=mx\\ m\in\mathbb{R}}} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x\cdot mx}{x^2+(mx)^2} = \lim_{x\to 0} \frac{mx^2}{x^2+m^2x^2} = \frac{m}{1+m^2},$$

i.e., limit depends on m.

$\therefore \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ DNE.

<u>Ex:</u> $\lim_{(x,y)\to(0,0)} \frac{x^2+y^4}{x^2-y^4}$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=mx,\ m\in\mathbb{R}}} f(x,y) = \lim_{x\to 0} \frac{x^2+(mx)^4}{x^2-(mx)^4} = \lim_{x\to 0} \frac{x^2+m^4x^4}{x^2-m^4x^4} = \lim_{x\to 0} \frac{1+m^4x^2}{1-m^4x^2} = 1$$

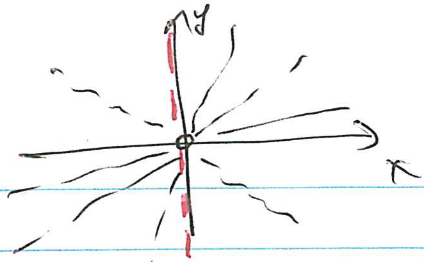
$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ along any line $y = mx$, $m \in \mathbb{R}$ is 1.

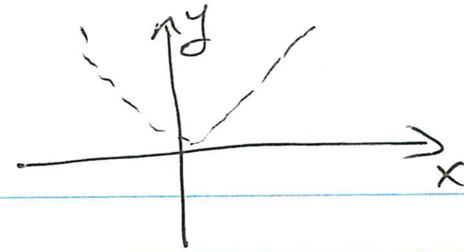
<u>Q-n:</u> Does this mean that $\lim_{(x,y)\to(0,0)} f(x,y) = 1$?

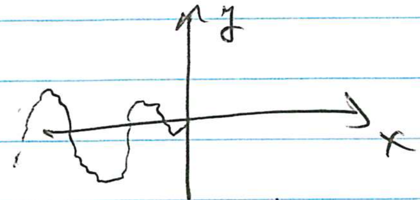
$$\left\{ \frac{x^2+m^4x^4}{x^2-m^4x^4} = \frac{\cancel{x^2}(1+m^4x^2)}{\cancel{x^2}(1-m^4x^2)} = \frac{1+m^4x^2}{1-m^4x^2} \right\}$$

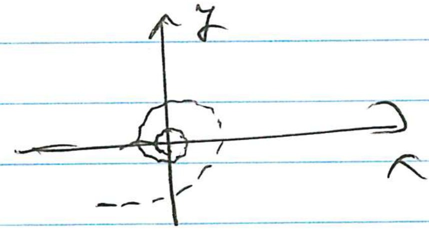
<u>Answer:</u> NO

(!):







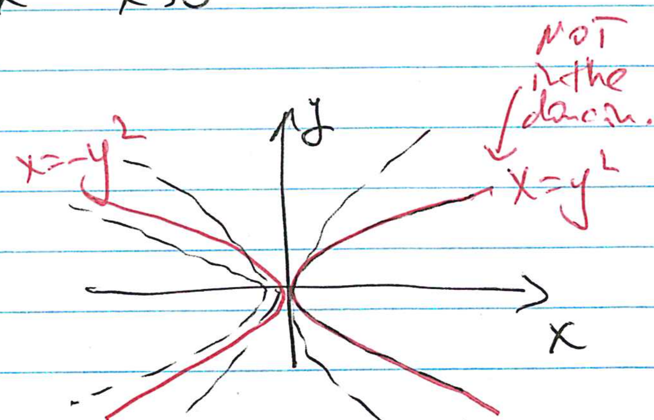


$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^4}{x^2-y^4} \quad \left\{ \begin{array}{l} y=\sqrt{x} \\ y^4=(\sqrt{x})^4 = x^2 \\ x^2-y^4=0 \longleftarrow \text{IS NOT in the domain of } f \end{array} \right\}$$

$$\left\{ \begin{array}{l} x=y^2 \\ x^2=y^4 \Rightarrow x^2-y^4=0 \\ \text{NOT in the domain} \end{array} \right\} \qquad \left\{ \begin{array}{l} y=x^2 \\ y^4=x^8 \text{, i.e. we \underline{are}} \\ \text{inside the domain} \end{array} \right\}$$

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=x^2}} \frac{x^2+y^4}{x^2-y^4} = \lim_{x\to 0} \frac{x^2+x^8}{x^2-x^8} = \lim_{x\to 0} \frac{1+x^6}{1-x^6} = 1$$

Consider $x=ay^2$, $a\in\mathbb{R}$, $a\neq\pm1$



$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } x=ay^2}} f(x,y) = \lim_{y\to 0} \frac{(ay^2)^2+y^4}{(ay^2)^2-y^4} = \lim_{y\to 0} \frac{a^2y^4+y^4}{a^2y^4-y^4} = \lim_{y\to 0} \frac{a^2+1}{a^2-1} =$$

$$= \frac{a^2+1}{a^2-1} \quad (\text{recall that } a\neq\pm1).$$

Limit depends on a

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ DNE.

Method 2: $\lim\limits_{(x,y)\to(0,0)} \frac{x^2+y^4}{x^2-y^4}$

$\lim\limits_{\substack{(x,y)\to(0,0)\\ \text{along } x=0}} f(x,y) = \lim\limits_{y\to 0} \frac{-y^4}{y^4} = -1$

$\lim\limits_{\substack{(x,y)\to(0,0)\\ \text{along } y=0}} f(x,y) = \lim\limits_{x\to 0} \frac{x^2}{x^2} = 1$

$1 \neq -1$ $\therefore$ lim DNE.

#32, p. 806 $\lim\limits_{(x,y)\to(1,1)} \frac{x^2-2x-y^2+2y}{x^2-2x+y^2-2y+2} =$

$= \left\{ \begin{array}{l|l} w = x-1 & x = w+1 \\ v = y-1 & y = v+1 \end{array} \right\}$

$= \lim\limits_{(w,v)\to(0,0)} \frac{(w+1)^2-2(w+1)-(v+1)^2+2(v+1)}{(w+1)^2-2(w+1)+(v+1)^2-2(v+1)+2} = \ldots$ HW.

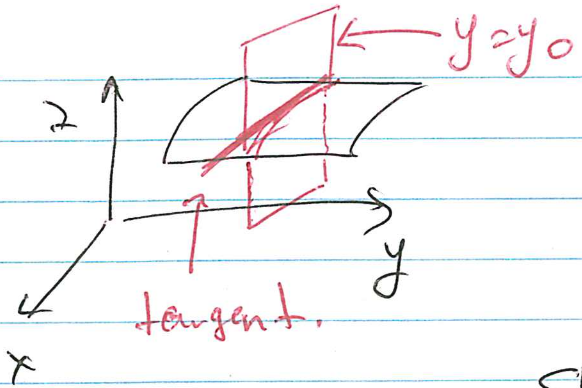
{ Note: In this problem, you could also complete squares }

HW: #34, p. 806

#22-26 (+ find all limits at pts where f-s are discontinuous)

#38, 39.

Section 12.3 Partial Derivatives.



$z=f(x,y)$ surface

Curve: $z=f(x,y)$, $y=y_0$

$\Downarrow$

$z=f(x,y_0)$, $y=y_0$.

Slope = derivative with respect to x

$$= \frac{d}{dx} f(x,y_0)$$

Def-n: The partial derivative of $f=f(x,y)$ with respect to x is

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$ and

the partial derivative with respect to y is

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}.$$

Notation: $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial x}(x,y)$, f_x, $\left.\frac{\partial f}{\partial x}\right)_y$,

$\frac{\partial z}{\partial x}$ if $z=f(x,y)$.

#1, p. 808 $f(x,y) = x^3y^2 + 2xy$

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial}{\partial x}(x^3y^2+2xy) = 3x^2y^2 + 2y$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{\partial}{\partial y}(x^3y^2+2xy) = 2x^3y + 2x$$

#24, p. 808 $\frac{\partial f}{\partial x}$ at $(1,-1,1,-1)$ if

$$f(x,y,z,t) = \frac{zt}{x^2+y^2-t^2}.$$

$$\frac{\partial f}{\partial x} = zt \cdot \frac{-1}{(x^2+y^2-t^2)^2} \cdot 2x$$

$$\frac{\partial f}{\partial x}(1,-1,1,-1) = -\frac{2\cdot 1\cdot 1\cdot(-1)}{(1+1-1)^2} = \frac{2}{1} = 2.$$