Oct 3, 2019

#31, p. 808 If $f(x,y) = \frac{x^3 y}{x-y}$, show that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3f(x,y).$$

$$\frac{\partial f}{\partial x} = y\frac{\partial}{\partial x}\left[\frac{x^3}{x-y}\right] = y\,\frac{3x^2(x-y) - x^3\cdot 1}{(x-y)^2} =$$

$$= \frac{y}{(x-y)^2}(2x^3 - 3x^2y)$$

$$\frac{\partial f}{\partial y} = x^3\frac{\partial}{\partial y}\left[\frac{y}{x-y}\right] = x^3\cdot\frac{1\cdot(x-y) - y\cdot(-1)}{(x-y)^2} =$$

$$= \frac{x^3}{(x-y)^2}(x - y + y) = \frac{x^4}{(x-y)^2}$$

$$LHS = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = \frac{xy}{(x-y)^2}(2x^3 - 3x^2y) + \frac{yx^4}{(x-y)^2} =$$

$$= \frac{x^3y}{(x-y)^2}(2x - 3y) + \frac{yx^4}{(x-y)^2} = \frac{x^3y}{(x-y)^2}\left[2x - 3y + x\right] =$$

$$= \frac{3x^3y}{(x-y)^2}(x-y) = \frac{3x^3y}{x-y} = 3f(x,y) = RHS.$$

Sect. 12.4 Gradients.

Def-n: If a fn $f = f(x,y,z)$ has partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ at each point in some region D, then $\text{grad} f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ is the gradient of f at $(x,y,z) \in D$.

#2, p. 812 If $f(x,y,z) = x^2yz$, find the gradient of f.

$$\frac{\partial f}{\partial x} = 2xyz, \quad \frac{\partial f}{\partial y} = x^2 z, \quad \frac{\partial f}{\partial z} = x^2 y$$

$$\therefore \nabla f = \left(2xyz,\ x^2 z,\ x^2 y\right).$$

Notation: $\overline{\text{grad } f}$, $\overline{\nabla f}$

If $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$, then you can write

$$\nabla f = \frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j} + \frac{\partial f}{\partial z}\bar{k}$$

#15, p. 812 $f(x,y,z) = xy \ln(x+y)$. Find $\nabla f(4,-2,0)$.

$$\frac{\partial f}{\partial x} = y\left[\ln(x+y) + x \cdot \frac{1}{x+y} \cdot 1\right]$$

$$\frac{\partial f}{\partial x}(4,-2,0) = (-2)\left[\ln 2 + \frac{4}{4-2}\right] = -2(\ln 2 + 2)$$
$$= -2\ln 2 - 4.$$

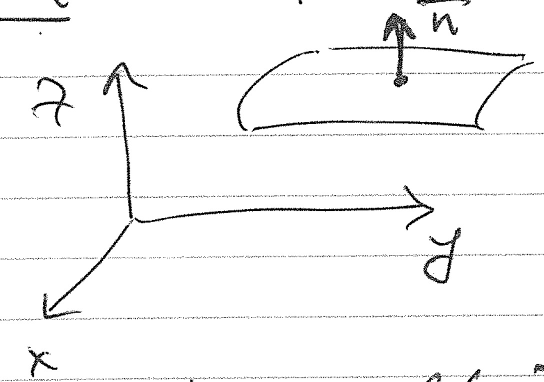
$$\frac{\partial f}{\partial y} = \left\{\frac{\partial f}{\partial y}(x,y,z)\right\} = x\left[\ln(x+y) + \frac{y}{x+y}\right]$$

$$\frac{\partial f}{\partial y}(4,-2,0) = 4\left[\ln 2 - \frac{2}{2}\right] = 4\ln 2 - 4.$$

$$\frac{\partial f}{\partial z} = 0$$

$$\therefore \nabla f(4,-2,0) = \left(-2\ln 2 - 4,\ 4\ln 2 - 4,\ 0\right).$$

Fact (will be proved later).



$f(x,y,z)=0$

$\bar{n}$ is normal to this surface

$\bar{n} \parallel \nabla f(x,y,z)$.

HW: Let $z=f(x,y)$ $\left(\Leftrightarrow \underbrace{z-f(x,y)}_{"F(x,y,z)"}=0 \right)$.

Show that $\nabla F(x,y,z) = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right)$

is $\perp$ to this surface at every point where it is defined.

#24, p. 813. If $\nabla f = (2xy-y,\ x^2-x)$, find $f(x,y)$.

$$\frac{\partial f}{\partial x} = 2xy-y \quad \text{and} \quad \frac{\partial f}{\partial y} = x^2-x$$

$\Downarrow$ integrate wrt x

$$f(x,y) = \int (2xy-y)\,dx = x^2y - xy + g(y)$$

$$\underset{x^2-x}{\overset{\parallel}{\frac{\partial f}{\partial y}}} = \frac{\partial}{\partial y}\left[x^2y-xy+g(y)\right] = x^2-x+g'(y)$$

$$\Rightarrow g'(y)=0 \Rightarrow g(y) = \text{const}.$$

$$\therefore f(x,y) = x^2y - xy + C, \text{ where } C \in \mathbb{R}.$$

#26, p. 813. If $f = f(x,y,z)$ and

$\nabla f = (yz,\ xz + 2yz,\ xy + y^2)$, find f.

$$\frac{\partial f}{\partial x} = yz \Rightarrow f(x,y,z) = \int yz\,dx = xyz + g(y,z)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[xyz + g(y,z)\right] = xz + \frac{\partial}{\partial y} g(y,z)$$

$$\overset{\shortparallel}{xz + 2yz}$$

$$\therefore \cancel{xz} + 2yz = \cancel{xz} + \frac{\partial}{\partial y} g(y,z)$$

$$\frac{\partial}{\partial y} g(y,z) = 2yz \Rightarrow g(y,z) = \int (2yz)\,dy =$$

$$= y^2 z + h(z)$$

$$\therefore f(x,y,z) = xyz + y^2 z + h(z)$$

$$xy + y^2 = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[xyz + y^2 z + h(z)\right]$$

$$= xy + y^2 + h'(z)$$

$$\therefore h'(z) = 0 \Rightarrow h(z) = C,\ C \in \mathbb{R}.$$

Answer: $f(x,y,z) = xyz + y^2 z + C,\ C \in \mathbb{R}.$

Q-n: Given a vector $(f(x,y,z), g(x,y,z), h(x,y,z))$, is there always a f-n $F = F(x,y,z)$ s.t.

$$\nabla F = \text{this vector} ?$$

Answer: NO.

Ex: If $\nabla f = (yz, xz+2yz, x)$, find f (if it exists).

As in the previous example,

$$f(x,y,z) = xyz + y^2z + h(z)$$

$$x = \frac{\partial f}{\partial z} = xy + y^2 + h'(z)$$

$$\therefore \; h'(z) = x - xy - y^2$$

makes no sense.

Is NOT a fn of only one variable z.

$\therefore$ Such f DNE.

$\boxed{\text{HW: \#30, p. 813}}$

#29, p. 813 If $\nabla f = \bar{0}$ for all points in $\mathbb{R}^3$, what can we say about f?

$$\nabla f = (0,0,0)$$

$$0 = \frac{\partial f}{\partial x} \Rightarrow f(x,y,z) = \int 0\,dx = g(y,z)$$

$$0 = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} g(y,z) \Rightarrow g(y,z) = \int 0\,dy = h(z)$$

$$\therefore \; f(x,y,z) = h(z)$$

$$0 = \frac{\partial f}{\partial z} = h'(z) \Rightarrow h(z) = C, \; C \in \mathbb{R}.$$

Answer: $f(x,y,z) = C, \quad C \in \mathbb{R}.$

Sect. 12.5 Higher-Order partial derivatives.

Ex: $f(x,y,z) = xyz$

$$\frac{\partial f}{\partial x} = yz \qquad \frac{\partial f}{\partial y} = xz$$

$$\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial x}[yz] = 0$$

$$\parallel \frac{\partial^2 f}{\partial x^2}$$

$$\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial y}[yz] = z \;:\; \frac{\partial^2 f}{\partial y \partial x} = z$$

$$\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial x}[xz] = z \;:\; \frac{\partial^2 f}{\partial x \partial y} = z.$$

Notation: $\frac{\partial^2 f}{\partial x \partial y} = \left\{ \frac{\partial^2}{\partial x \partial y} f = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] \right\}.$

$$\frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial^2}{\partial x^2}\left[\frac{\partial f}{\partial y}\right]$$

In general, for $f = (x,y,z)$, with $n_1 + n_2 + n_3 = n$,

$$\frac{\partial^n f}{\partial x^{n_1} \partial y^{n_2} \partial z^{n_3}} = \frac{\partial^{n_1}}{\partial x^{n_1}}\left[\frac{\partial^{n_2}}{\partial y^{n_2}}\left[\frac{\partial^{n_3} f}{\partial z^{n_3}}\right]\right].$$

Ex: $\frac{\partial^4 f}{\partial y \partial x \partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y} f\right]\right]\right].$

Notation: If $f = f(x,y,z)$

$$f_x = \frac{\partial f}{\partial x}, \quad f_y = \frac{\partial f}{\partial y}, \quad f_z = \frac{\partial f}{\partial z}$$

$$f_{xy^2zx} = \left(\left(\left((f_x)_y\right)_y\right)_z\right)_x$$

For example, $f_{x^2yz} = \dfrac{\partial^4 f}{\partial z \partial y \partial x^2}$

<u>Theorem:</u> If $f, \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial^2 f}{\partial x \partial y}, \dfrac{\partial^2 f}{\partial y \partial x}$ are all defined inside a circle centered at a point P and are continuous at P, then, at P,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

<u>Def-n:</u> 1) If $f = f(x,y,z)$, then eq-n

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

is called the <u>Laplace equation.</u>

2) If f satisfies the Laplace eq-n, then it is called a <u>harmonic f-n.</u>

<u>Notation:</u> $\Delta f(x,y,z) = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$

$$\left\{ \Delta f = \nabla^2 f \longleftarrow \text{notation} \right\}$$

<u>#1, p. 818</u> $f(x,y) = x^2y^2 - 2x^3y$. Find $\dfrac{\partial^2 f}{\partial x^2}$.

$$\frac{\partial f}{\partial x} = 2xy^2 - 6x^2y$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[2xy^2 - 6x^2y\right] = 2y^2 - 12xy.$$

#15, p. 818 $\quad \dfrac{\partial^{10} f}{\partial x^7 \partial y^3}$ if $f(x,y) = x^7 e^x y^2 + \dfrac{1}{y^6}$

$$\frac{\partial f}{\partial y} = x^7 e^x \cdot 2y - 6 \cdot y^{-7}$$

$$\frac{\partial^2 f}{\partial y^2} = 2x^7 e^x + 6 \cdot 7 \cdot y^{-8}$$

$$\frac{\partial^3 f}{\partial y^3} = -6 \cdot 7 \cdot 8\, y^{-9}$$

$$\frac{\partial^4 f}{\partial x \partial y^3} = \frac{\partial}{\partial x}\left[-6 \cdot 7 \cdot 8 \cdot y^{-9}\right] = 0$$

$$\therefore \quad \frac{\partial^{10} f}{\partial x^7 \partial y^3} = 0.$$

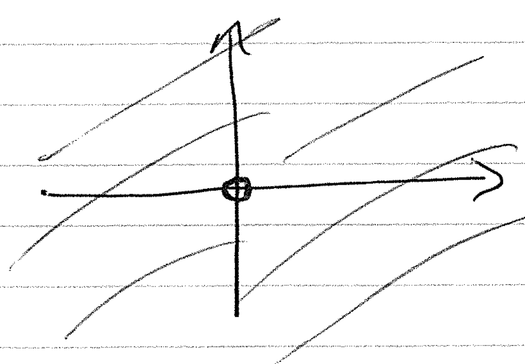
#24 Find the region where $f = f(x,y)$ is harmonic. $\quad f = \ln(x^2+y^2)$.

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = 2 \cdot \frac{1 \cdot (x^2+y^2) - x \cdot 2x}{(x^2+y^2)^2} = \frac{2}{(x^2+y^2)^2}(y^2-x^2)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{2}{(x^2+y^2)^2}(x^2-y^2)$$

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{2}{(x^2+y^2)^2}(y^2-x^2+x^2-y^2) = 0$$



$\longleftarrow$ Harmonic in

$$\{(x,y) \in \mathbb{R}^2 \mid (x,y) \neq (0,0)\}$$