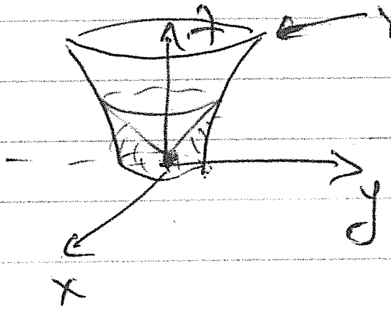
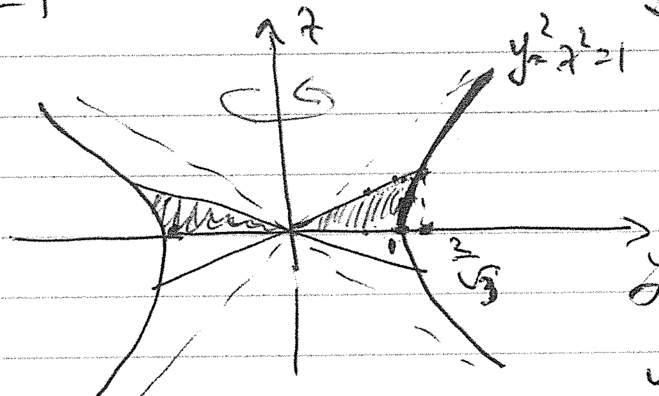
Dec. 3, 2019

#31, p. 959. Volume - ?

Region is bounded by $x^2+y^2-z^2=1$

$4z^2 = x^2+y^2$.





$y^2-z^2=1$

$4z^2=y^2 \Leftrightarrow z=\pm\frac{1}{2}y$

Find projection onto the xy-plane.

Projection onto xy-plane $= \{(x,y) \mid$ there is z corresp to (x,y) s.t. (x,y,z) is inside the solid $\}$.

$$\begin{cases} z^2 = x^2+y^2-1 \\ z^2 = \frac{1}{4}(x^2+y^2) \end{cases} \Rightarrow \frac{1}{4}(x^2+y^2) = x^2+y^2-1 \Leftrightarrow \frac{3}{4}(x^2+y^2) =$$

$$\Leftrightarrow x^2+y^2 = \frac{4}{3}.$$

$$\left(z^2 = \frac{1}{4}\cdot\frac{4}{3} = \frac{1}{3}\right).$$

Projection onto the xy-plane $= \{(x,y) \mid x^2+y^2 \le \frac{4}{3}\}$. " S_{xy}

$$V = \cancel{\iiint_{S_{xy}}} 2\iint_{x^2+y^2\le 1} \int_0^{\sqrt{\frac{x^2+y^2}{4}}} 1\, dz\, dx\, dy +$$

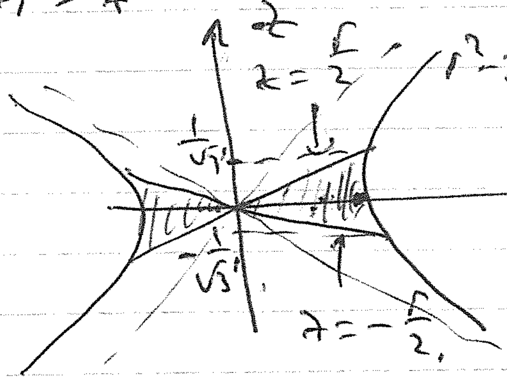
$$+ 2\iint_{1\le x^2+y^2 \le \frac{2}{\sqrt{3}}} \int_{\sqrt{x^2+y^2-1}}^{\sqrt{\frac{x^2+y^2}{4}}} 1\, dz\, dx\, dy.$$

In cylindrical coordinates,

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

Surface 1: $r^2 - z^2 = 1$

Surface 2: $4z^2 = r^2$



$$\text{Solid} = \{(r,\theta,z) \mid r^2 - z^2 \le 1,\ 4z^2 \le r^2\}$$

$$= \{(r,\theta,z) \mid r^2 - 1 \le z^2 \le \frac{r^2}{4}\}.$$

$$r^2 - 1 \le \frac{r^2}{4} \iff \frac{3}{4}r^2 \le 1 \iff r^2 \le \frac{4}{3}.$$

$$\text{Volume} = \iiint_{\text{Solid}} 1 \cdot r \, dz\, dr\, d\theta =$$

$$= 2\int_0^{2\pi}\int_1^{\frac{2}{\sqrt{3}}}\int_{\sqrt{r^2-1}}^{r/2} r \, dz\, dr\, d\theta \quad + $$

$\left\{ \text{The part of the solid above the } xy\text{-plane is } \{(r,\theta,z) \mid \sqrt{r^2-1} \le z \le \frac{r}{2}\}. \right\}$

$$+ \quad 2\int_0^{2\pi}\int_0^1\int_0^{r/2} r \, dz\, dr\, d\theta.$$

method 2: Solid $= \{(r,\theta,z) \mid r^2 - z^2 \le 1,\ 4z^2 \le r^2\} =$

$$= \{(r,\theta,z) \mid 4z^2 \le r^2 \le 1+z^2\} =$$

$$4z^2 \le 1+z^2 \Leftrightarrow 3z^2 \le 1 \Leftrightarrow z^2 \le \frac{1}{3} \Leftrightarrow |z| \le \frac{1}{\sqrt{3}}$$

$$= \{(r,\theta,z) \mid 2|z| \le r \le \sqrt{1+z^2},\ |z| \le \frac{1}{\sqrt{3}}\}$$

The part above the xy-plane:

$$\{(r,\theta,z) \mid 2z \le r \le \sqrt{1+z^2},\ 0 \le z \le \frac{1}{\sqrt{3}}\}.$$

$$\text{Volume} = 2\int_0^{2\pi}\int_0^{1/\sqrt{3}}\int_{2z}^{\sqrt{1+z^2}} r \, dr\, dz\, d\theta =$$

$$= 4\pi \int_0^{1/\sqrt{3}} \left[\frac{1}{2}r^2 \Big|_{r=2z}^{r=\sqrt{1+z^2}}\right] dz =$$

$$= 2\pi \int_0^{\frac{1}{\sqrt{3}}} (1+z^2-4z^2)\, dz = 2\pi \int_0^{\frac{1}{\sqrt{3}}} (1-3z^2)\, dz =$$

$$= 2\pi \left(z - z^3\right)\Big|_0^{\frac{1}{\sqrt{3}}} = 2\pi\left(\frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}}\right) =$$

$$= 2\pi \cdot \frac{2}{3} \cdot \frac{1}{\sqrt{3}} = \frac{4\pi}{3\sqrt{3}}.$$

#32. Region is bounded by $z = x^2+y^2$, $z=0$ and $(x^2+y^2)^2 = x^2-y^2$. Find the volume.

Cylindrical coordinates: $(r^2)^2 = r^2\cos^2\theta - r^2\sin^2\theta$

$\Leftrightarrow r^2 = \cos^2\theta - \sin^2\theta \Leftrightarrow r^2 = \cos 2\theta$

$\Leftrightarrow r = \sqrt{\cos 2\theta}$, $\cos 2\theta \geq 0$.

$\cos 2\theta = 0 \Leftrightarrow 2\theta = \frac{\pi}{2} + \pi n$, $n \in \mathbb{Z}$

$\Leftrightarrow \theta = \frac{\pi}{4} + \frac{\pi n}{2}$, $n \in \mathbb{Z}$.

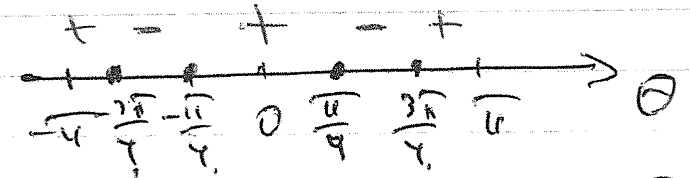
$\boxed{-\pi \leq \theta \leq \pi}$

$\therefore \theta_1 = \frac{\pi}{4}$,

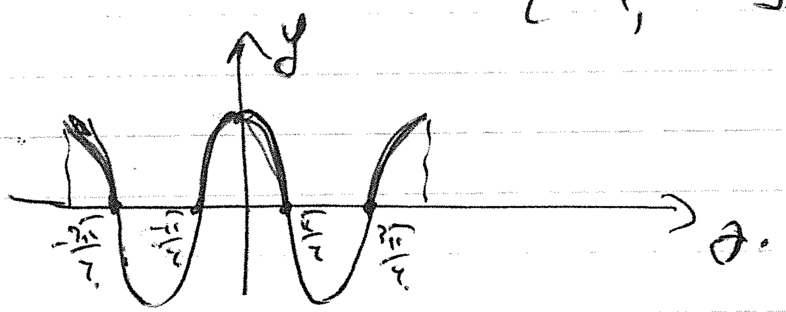
$\theta_2 = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$,

$\theta_3 = -\frac{\pi}{4}$

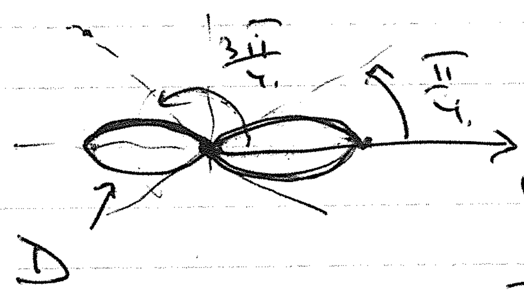
$\theta_4 = -\frac{3\pi}{4}$,



$\therefore \theta$ is in $\left[-\pi, -\frac{3\pi}{4}\right] \cup \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \pi\right]$.



$y = \cos 2\theta$.



Solid $= \{(r,\theta,z) \mid (r,\theta) \in D$ and $0 \leq z \leq r^2\} =$

$= \{(r,\theta,z) \mid 0 \leq z \leq r^2$, $\underline{-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \; 0 \leq r \leq \sqrt{\cos 2\theta}}$

and, if $\frac{3\pi}{4} \leq \theta \leq \pi$ or $-\pi \leq \theta \leq -\frac{3\pi}{4}$,

$0 \leq r \leq \sqrt{\cos 2\theta}\}$.

$$\text{Volume} = \int_A \int_0^{\sqrt{\cos 2\theta}} \int_0^{r^2} r \, dz \, dr \, d\theta, \text{ where}$$

$$A = \left[-\pi, -\frac{3\pi}{4}\right] \cup \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \pi\right].$$

$$\therefore \text{Volume} = \int_A \int_0^{\sqrt{\cos 2\theta}} r^3 \, dr \, d\theta =$$

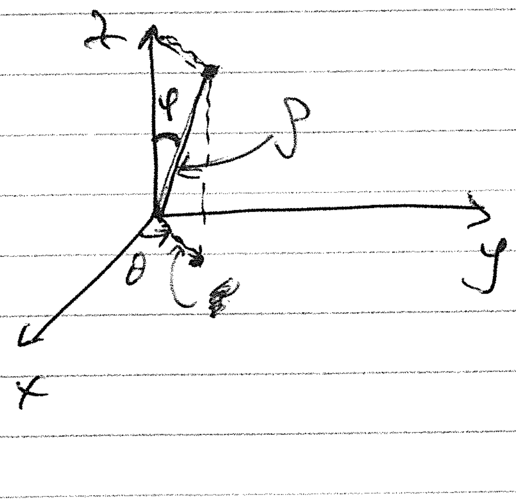
$$= \int_A \frac{1}{4} \cos^2 2\theta \, d\theta = \left\{ \begin{array}{l} \cos 2t = 2\cos^2 t - 1 \\ \therefore \cos^2 t = \frac{\cos 2t + 1}{2} \end{array} \right\} \text{etc.}$$

$= \ldots$ HW.

<u>Note:</u> by symmetry,

$$\text{Volume} = 2 \int_{-\frac{\pi}{4}}^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} \int_0^{r^2} r \, dz \, dr \, d\theta = \ldots \text{HW}$$

<u>Section 13.12</u> Spherical Coordinates.



$$\underline{x^2 + y^2 + z^2 = \rho^2}$$

$$z = \rho \cos\varphi, \quad \underline{0 \le \varphi \le \pi}$$

$$r = \rho \sin\varphi$$

$$\underline{x = \rho \sin\varphi \cos\theta}$$

$$\underline{y = \rho \sin\varphi \sin\theta}, \quad \underline{0 \le \theta \le 2\pi}.$$

If $P(x,y,z)$, where x,y,z are Cartesian coordinates, the its spherical coordinates are ρ, φ, θ where

$$x^2+y^2+z^2=\rho^2$$

$$\boxed{\begin{aligned} z &= \rho\cos\varphi \\ x &= \rho\sin\varphi\cos\theta \\ y &= \rho\sin\varphi\sin\theta \quad \text{with } 0\le\varphi\le\pi,\ 0\le\theta\le 2\pi. \end{aligned}}$$

Note: $x^2+y^2=\rho^2\sin^2\varphi$.

$$\boxed{dV = \rho^2\sin\varphi \; d\rho\, d\varphi\, d\theta} \quad (!!!)$$

$$\therefore \quad \boxed{\iiint\limits_{\substack{\text{solid}\\ \text{in terms of}\\ x,y,z}} f(x,y,z)\,dV = \iiint\limits_{\substack{\text{solid in}\\ \text{terms of}\\ \rho,\varphi,\theta}} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi) \cdot \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta}$$

Ex: Volume of a sphere of radius R.

$$\text{Solid} = \{(x,y,z) \mid x^2+y^2+z^2\le R^2\}$$
$$= \{(\rho,\varphi,\theta) \mid 0\le\rho\le R,\ 0\le\varphi\le\pi,\ 0\le\theta\le 2\pi\}.$$

$$\text{Volume} = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta =$$

$$= 2\pi\int_0^{\pi} \frac{1}{3}R^3\sin\varphi\, d\varphi = \frac{2\pi}{3}R^3\left(-\cos\varphi\right)\Big|_0^{\pi} =$$

$$= \frac{2\pi}{3}R^3(1+1) = \frac{4}{3}\pi R^3.$$

#2, p. 964 Spherical coordinates: $x^2+y^2=1$.

$\{ x^2+y^2=r^2,\ r=\rho\sin\varphi \quad \therefore \quad x^2+y^2=\rho^2\sin^2\varphi \}$

$\therefore\ \rho^2\sin^2\varphi=1 \iff \rho\sin\varphi=1$ or $\cancel{\rho\sin\varphi=-1}$.

$\left\{ \begin{array}{l} 0\le\varphi\le\pi \Rightarrow \underline{\underline{\sin\varphi\ge 0}} \\ \rho\ge 0 \end{array} \right\}$ $\quad \boxed{\rho\sin\varphi=1}$

#3, p. 964. $3z=\sqrt{x^2+y^2}$

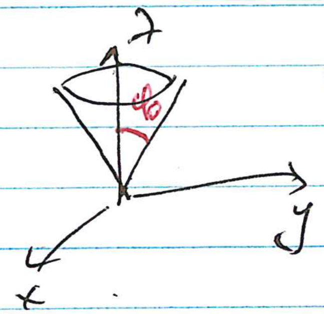
$z=\rho\cos\varphi, \quad x^2+y^2=\rho^2\sin^2\varphi$

$\Downarrow$

$\sqrt{x^2+y^2}=\rho|\sin\varphi|=\rho\sin\varphi$

$\therefore\ 3\rho\cos\varphi=\rho\sin\varphi \iff 3\cos\varphi=\sin\varphi$

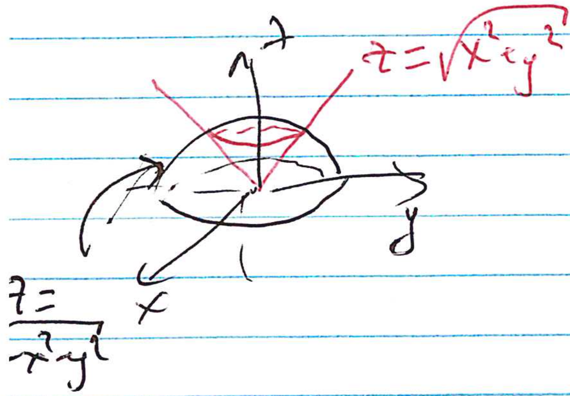
$\iff \tan\varphi=3 \iff \boxed{\varphi=\mathrm{Tan}^{-1}3.}$



$\varphi_0=\mathrm{Tan}^{-1}3$

#8, p. 964 Volume bounded by $z=\sqrt{x^2+y^2}$,

$z=\sqrt{1-x^2-y^2}$.



$S=\{(x,y,z)\,|\,\sqrt{x^2+y^2}\le z\le\sqrt{1-x^2-y^2}\}$.

In spherical coordinates:

$z = \rho\cos\varphi$, $x^2+y^2 = \rho^2\sin^2\varphi$

$\sqrt{x^2+y^2} = \rho\sin\varphi$

$\therefore\ \underbrace{\rho\sin\varphi \le \rho\cos\varphi} \le \sqrt{1-\rho^2\sin^2\varphi}$

$\sin\varphi \le \cos\varphi \iff \left\{ \sin\varphi \ge 0 \Rightarrow \cos\varphi \ge 0 \right\}$

$\iff \tan\varphi \le 1$ and $0 \le \varphi \le \pi$.

$\therefore\ \boxed{0 \le \varphi \le \frac{\pi}{4}}$

$\rho^2\cos^2\varphi \le 1-\rho^2\sin^2\varphi \iff \rho^2 \le 1 \iff \boxed{0 \le \rho \le 1}$

$\therefore\ S = \left\{ (\rho,\varphi,\theta) \,\middle|\, 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \frac{\pi}{4},\ 0 \le \rho \le 1 \right\}$.

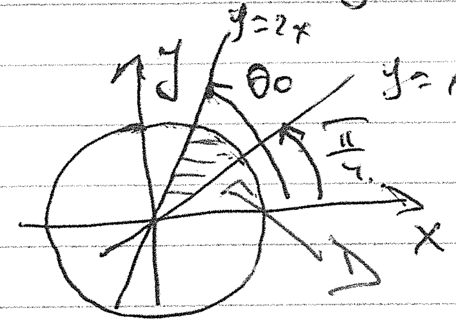
$$\text{Volume} = \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^2\sin\varphi \, d\rho\, d\varphi\, d\theta =$$

$$= 2\pi \int_0^{\pi/4} \frac{1}{3}\sin\varphi\, d\varphi = \frac{2\pi}{3}\left(-\cos\varphi\right)\Big|_0^{\frac{\pi}{4}} =$$

$$= \frac{2\pi}{3}\left(-\frac{1}{\sqrt{2}}+1\right)$$

HW: Do this proble using cylindrical coord.

#10, p. 964. Volume bounded by $x^2+y^2+z^2=1$, $y=x$, $y=2x$, $z=0$ (in the first octant).



$$S=\{(x,y,z)\mid (x,y)\in D,\ 0\le z\le\sqrt{1-x^2-y^2}\}.$$

Spherical coordinates: "above xy-plane": $0\le\varphi\le\frac{\pi}{2}$.

"inside $x^2+y^2+z^2=1$": $0\le\rho\le1$

"between lines $y=x$ and $y=2x$": $\frac{\pi}{4}\le\theta\le\theta_0$,

where θ_0 is s.t. $\tan\theta_0=2 \Leftrightarrow \theta_0=\mathrm{Tan}^{-1}2$.

$$\therefore \text{Solid}=\{(\rho,\varphi,\theta)\mid 0\le\rho\le1,\ 0\le\varphi\le\frac{\pi}{2},\ \frac{\pi}{4}\le\theta\le\mathrm{Tan}^{-1}2\}.$$

$$\text{Volume}=\int_{\frac{\pi}{4}}^{\mathrm{Tan}^{-1}2}\int_0^{\frac{\pi}{2}}\int_0^1\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta=$$

$$=\frac{1}{3}\cdot(-\cos\varphi)\Big|_0^{\frac{\pi}{2}}\cdot\left(\mathrm{Tan}^{-1}2-\frac{\pi}{4}\right)=$$

$$=\frac{1}{3}\left(\mathrm{Tan}^{-1}2-\frac{\pi}{4}\right).$$

#18, p. 967. Evaluate:

$$I = \int_0^9 \int_0^{\sqrt{81-y^2}} \int_0^{\sqrt{81-x^2-y^2}} \frac{1}{x^2+y^2+z^2}\, dz\, dx\, dy$$

$$S = \{(x,y,z) \mid 0 \le z \le \sqrt{81-x^2-y^2},\ 0 \le x \le \sqrt{81-y^2},\ 0 \le y \le 9\}$$

I. without visualizing:

$$\begin{cases} 0 \le \rho\cos\varphi \le \sqrt{81-\rho^2\sin^2\varphi} \\ 0 \le \rho\sin\varphi\cos\theta \le \sqrt{81-\rho^2\sin^2\varphi\sin^2\theta} \\ 0 \le \rho\sin\varphi\sin\theta \le 9. \end{cases}$$

$0 \le \varphi \le \frac{\pi}{2}$ $\quad \rho^2\cos^2\varphi \le 81-\rho^2\sin^2\varphi \Leftrightarrow \rho^2 \le 81$

$\Leftrightarrow 0 \le \rho \le 9$

$-\pi \le \theta \le \pi$

$\cos\theta \ge 0 \Rightarrow -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\rho^2\sin^2\varphi\cos^2\theta \le 81-\rho^2\sin^2\varphi\sin^2\theta$

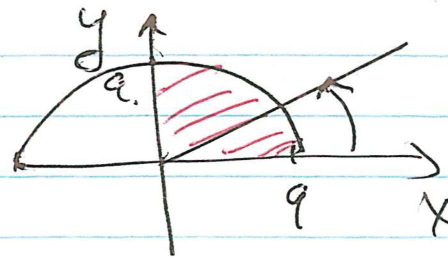
$\rho^2\sin^2\varphi \le 81 \Leftrightarrow \rho\sin\varphi \le 9$

$\sin\theta \ge 0 \Rightarrow 0 \le \theta \le \pi$ $\quad \therefore\ 0 \le \theta \le \frac{\pi}{2}$

$\rho\sin\varphi\sin\theta \le 9$

$\therefore S = \{(\rho,\varphi,\theta) \mid 0 \le \rho \le 9,\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le \varphi \le \frac{\pi}{2}\}.$

II. Visualize.



$\therefore$ Solid : part of the ~~sphere~~ ball $x^2+y^2+z^2 \le 81$ in the first octant,

$$\text{Solid} = \{(\rho, \varphi, \theta) \mid 0 \le \rho \le 9,\ 0 \le \varphi \le \tfrac{\pi}{2},\ 0 \le \theta \le \tfrac{\pi}{2}\}.$$

$$\therefore I = \underset{\substack{\text{Solid} \\ \text{in terms of} \\ \rho,\varphi,\theta}}{\iiint} \frac{1}{\rho^2} \cdot \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta =$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\pi/2} \int_0^{9} \sin\varphi \, d\rho \, d\varphi \, d\theta =$$

$$= \frac{\pi}{2} \cdot 9 \int_0^{\frac{\pi}{2}} \sin\varphi \, d\varphi = \frac{9}{2}\pi \left(-\cos\varphi\right)\Big|_0^{\frac{\pi}{2}} = \frac{9}{2}\pi.$$

Hw: #19, p. 964.

#21, p. 964 Find the volume bounded by

$$(x^2+y^2+z^2)^2 = x.$$

Spherical coordinates:

$$(\rho^2)^2 = \rho \sin\varphi \cos\theta$$

$$\rho^3 = \sin\varphi\cos\theta \iff \rho = (\sin\varphi\cos\theta)^{\frac{1}{3}}.$$

$\rho \geq 0 \Rightarrow \cos\theta \geq 0.$

If $-\pi \leq \theta \leq \pi$, then $\cos\theta \geq 0 \iff \underline{-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}}$

$$\left[\begin{array}{l} \text{Solid: } 0 \leq \rho \leq (\sin\varphi\cos\theta)^{\frac{1}{3}}, \\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \varphi \leq \pi. \end{array}\right.$$

$$\text{Volume} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\pi} \int_0^{(\sin\varphi\cos\theta)^{\frac{1}{3}}} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta =$$

$$= \frac{1}{3} \int_{-\frac{\pi}{2}}^{\pi/2} \int_0^{\pi} (\sin^2\varphi\cos\theta) \, d\varphi \, d\theta =$$

$$\cancel{= \frac{1}{3} (-\cos\varphi)\Big|_0^{\pi} \cdot (\sin\theta)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} =}$$

$$\cancel{= \frac{1}{3}(1+1)\cdot(1+1) = \frac{4}{3}.}$$

$$= \frac{1}{3}(\sin\theta)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdot \int_0^{\pi} \sin^2\varphi \, d\varphi = \left\{ \begin{array}{l} \cos 2\varphi = 1 - 2\sin^2\varphi \\ \sin^2\varphi = \frac{1-\cos 2\varphi}{2} \end{array} \right\}$$

$$= \frac{2}{3} \int_0^{\pi} \frac{1-\cos 2\varphi}{2} \, d\varphi = \frac{2}{3}\cdot\frac{1}{2}\left[\varphi - \frac{1}{2}\sin 2\varphi\right]\Big|_0^{\pi} =$$

$$= \frac{1}{3}\pi$$

#38, p. 959 Volume bounded by $z=0$, $x^2+y^2=1$, $z=e^{-x^2-y^2}$.

$$\text{Solid} = \{(x,y,z) \mid x^2+y^2 \le 1,\ 0 \le z \le e^{-x^2-y^2}\}$$

In cylindrical coordinates:

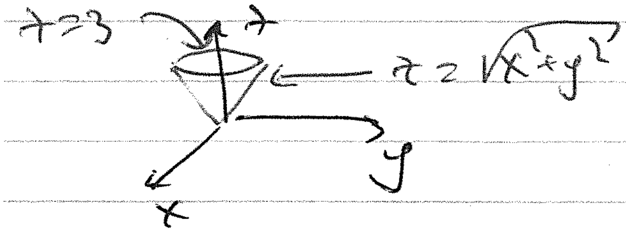
$$\text{Solid} = \{(r,\theta,z) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le z \le e^{-r^2}\}.$$

$$\text{Volume} = \int_0^{2\pi}\int_0^1\int_0^{e^{-r^2}} r\,dz\,dr\,d\theta =$$

$$= 2\pi \int_0^1 e^{-r^2}\cdot r\,dr = 2\pi\left(-\frac{1}{2}e^{-r^2}\right)\Big|_0^1 =$$

$$= -\pi\left(e^{-1}-e^{0}\right) = \pi\left(1-\frac{1}{e}\right).$$

#42, p. 959 S: bounded by $z=3$ and $z=\sqrt{x^2+y^2}$



$$\iiint_S \sqrt{x^2+y^2+z^2}\,dV = I.$$

$$S = \{(r,\theta,z) \mid \cancel{x^2+y^2 \le z}\ \ r \le z \le 3,\ 0 \le r \le 3,\ 0 \le \theta \le 2\pi\}.$$

$$\left\{ \sqrt{x^2+y^2} \le z \le 3 \iff r \le z \le 3 \right\}$$

$$I = \int_0^{2\pi}\int_0^3\int_r^3 \sqrt{r^2+z^2}\cdot r\,dz\,dr\,d\theta = \dots \text{ new}$$