Acknowledgment: this document is based on tutorials created by Prof. D. Trim a few years ago

Section 11.1: Rectangular Coordinates in Space

1. If $A(0,1,1)$ and $B(1,0,0)$, find all points $C(x,y,z)$ such that $|AB| = |AC|$. What surface does this equation describe?

2. If $A(0,1,1)$ and $B(1,0,0)$, find all points $C(x,y,z)$ such that $|AC| = |BC|$. What surface does this equation describe?

3. If $A(0,1,1)$ and $B(1,0,0)$, find all points $C(x,y,z)$ such that the triangle $\triangle ABC$ is equilateral. What curve does this equation describe?

4. Find the midpoint of the line segment joining the points $A(1,2,3)$ and $B(3,-2,-1)$.

5. If $A(1,2,3)$ and $B(3,-2,-1)$, find all points $C(x,y,z)$ on the line connecting A and B and such that $|AC| = 2|BC|$.

   Note: you can do all of the above problems by using only the formula for the distance between two points. However, the material covered in Section 11.5 (Planes and Lines) may simplify some calculations.

Answers:

1. $x^2 + (y-1)^2 + (z-1)^2 = 3$ (sphere)

2. $x - y - z + 1/2 = 0$ (plane)

3. one possible answer: $x^2 + (y-1)^2 + (z-1)^2 = 3$, $x - y - z + 1/2 = 0$ (circle)

4. $(2,0,1)$

5. $(7/3, -2/3, 1/3)$ and $(5,-6,-5)$

Section 11.2: Curves and Surfaces

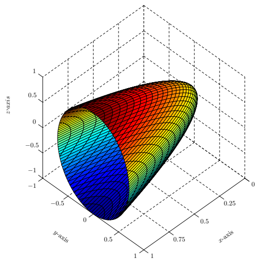
In questions 1–13, visualize (what's the intersection of the surface with coordinate planes? with planes parallel to coordinate planes? with planes containing coordinate axes?) and then draw the surface defined by the equation.

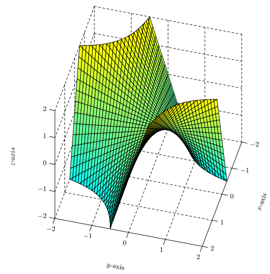
1. $x = 2y^2 + z^2$
2. $z = 2xy$
3. $z = |x + y|$
4. $x = z^3 + 1$
5. $|z| + |y| = 1$
6. $z^2 - x^2 = 3y^2$
7. $z^2 = y^2 - 2z + 3$
8. $x^2 + y^2 = 2x - 4y + 5$
9. $x^2 + y^2 = 2x - 4y - 5$
10. $4y^2 + x^2 = z^2 - 1$
11. $4y^2 + x^2 = z^2 + 1$
12. $2x^2 + 3y^2 + 4z^2 = 12$
13. $y^2 + 2x^2 = 4 - 2z$

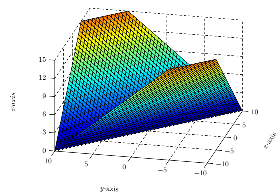
In questions 14–17 draw the curve and find equations for its projections in the $xy-$, $yz-$, and $xz-$coordinate planes.

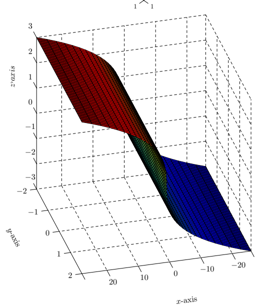
14. $z = 2x^2 + 4y^2, y + z = 1$
15. $x^2 + y^2 + 2z^2 = 2, x + y = 1$
16. $z = x^2 + y^2, z = 2x^2$
17. $z = x^2 + y^2, 2z = x^2$

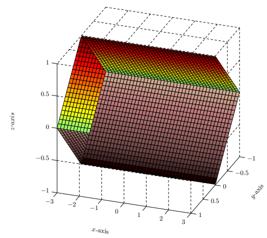
Answers:

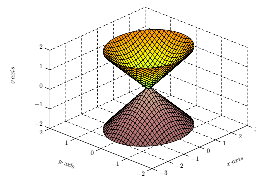
1. 

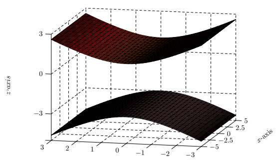
2. 

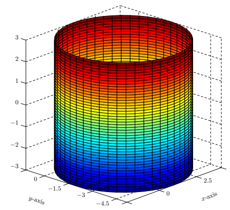
3. 

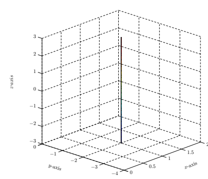
4. 

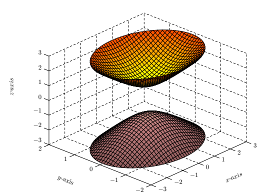
5. 

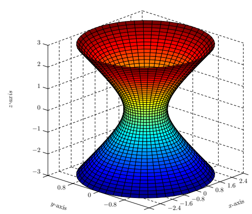
6. 

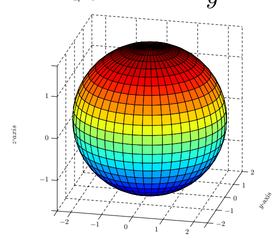
7. 

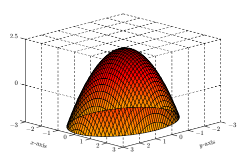
8. 

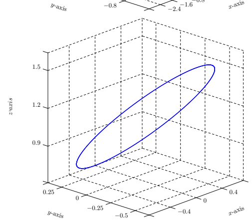
9.  (line: $x = 1$, $y = -2$, $z \in \mathbb{R}$)

10. 

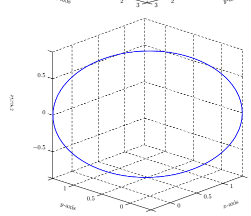
11. 

12. 

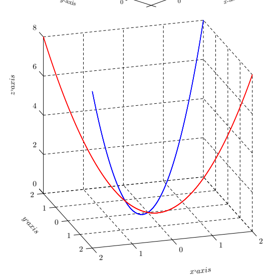
13. 

14. 

$2x^2 + 4y^2 + y = 1,\ z = 0;$
$y + z = 1,\ x = 0,\ -(1+\sqrt{17})/8 \leq y \leq (-1+\sqrt{17})/8;$
$4z^2 - 9z + 2x^2 + 4 = 0,\ y = 0$

15. 

$x + y = 1,\ z = 0,\ (1-\sqrt{3})/2 \leq x \leq (1+\sqrt{3})/2;$
$y^2 - y + z^2 = 1/2,\ x = 0$;
$x^2 - x + z^2 = 1/2,\ y = 0$

16. 

$y = \pm x,\ z = 0;$
$z = 2x^2,\ y = 0;$
$z = 2y^2,\ x = 0$

17. These surfaces only intersect at the point $(0, 0, 0)$

Section 11.5: Planes and Lines

1. Find the equation of the plane that contains $2x + 3y + 4z = 6$, $x - 2y + z = 3$ and
$$\frac{x - 1/2}{11} = \frac{y + 2}{2} = \frac{z - 1}{7}.$$
2. Find the equation of the plane that contains $2x + 3y + 4z = 6$, $x - 2y + z = 3$ and
$$\frac{2x - 1}{22} = \frac{y + 2}{2} = \frac{1 - z}{7}.$$
3. Find equations for the line perpendicular to the plane $x + 5y - 2z = 6$ and passing through the point of intersection of the lines
$$x = 2 + 3t, y = 1 - t, z = 4 + 2t \quad \text{and} \quad x = -1 + s, y = 2 + 3s, z = 2 + 2s.$$
4. Find equations for the line parallel to the line $x + y - z = 6$, $2x - y + z = 1$ and passing through the point of intersection of the lines
$$x = 2 + 3t, y = 1 - t, z = 4 + 2t \quad \text{and} \quad x = -1 + t, y = 2 + 3t, z = 2 + 2t.$$
5. Find equations for the line that is perpendicular to both the y-axis and the line $x - y = 2$, $3y + 4z = 6$, and intersects the z-axis at a point $\sqrt{11}$ units from the point $(1, -1, 2)$.
6. Find the equation of the plane that contains the points $(2, -1, 3)$ and $(1, 1, 4)$ and the line $2x - 3y + z = 3$, $x + 5y - z = 2$.

Answers:

1. $2x - 11y - 6 = 0$
2. $24x - 13y + 34z = 72$
3. $x = -1 + u, y = 2 + 5u, z = 2 - 2u$
4. $x = -1,\ y = 2 + t,\ z = 2 + t$
5. $(x, y, z) = (0, 0, -1) + t(3, 0, 4)$; or $(x, y, z) = (0, 0, 5) + t(3, 0, 4)$
6. $23x + 11y + z - 38 = 0$

Section 11.6: Geometric Applications of Scalar and Vector Products

1. Find the distance from the point $(3, -1, 5)$ to the line $x = 2 + 3t, y = 2t - 1, z = 4 + t$.
2. Find the distance between the planes $x = 2y - 3z + 1$ and $3x - 6y + 9z = 4$.
3. Find the distance between the lines $y = 2x + 3z - 4$, $3x + y - 2z = 6$, and $x = 2 + t, y = 3 - 2t, z = 1 + t$.
4. Find equations for the planes that are 2 units apart, equidistant from the point $(1, -1, 2)$ and parallel to the plane $x + 2y - 5z = 6$.
5. The three lines below form a triangle. Find the vertices of this triangle and its area.

$$\begin{cases} x = -11 + 5t, \\ y = t, \\ z = -2 + 2t; \end{cases} \qquad \begin{cases} x = 1 + 2s, \\ y = 1 - s, \\ z = -2 - 4s; \end{cases} \qquad \begin{cases} x = -2 + 3u, \\ y = -1 + 2u, \\ z = -8 + 6u. \end{cases}$$

6. If the vertices of the triangle in question 5 are three vertices of a parallelogram, what are the possibilities for the fourth vertex?
7. Find the centroid of the triangle in question 5 in two different ways: (i) using the fact that the centroid is the point of intersection of the three medians of the triangle, and (ii) using the fact that the centroid is the point on one of the medians, two-thirds of the way from the vertex to the midpoint of the opposite side.

Answers:

1. $\sqrt{6/7}$
2. $1/(3\sqrt{14})$
3. $1/\sqrt{14}$
4. $x + 2y - 5z + 11 \pm \sqrt{30} = 0$
5. Vertices: $(-1, 2, 2)$, $(4, 3, 4)$ and $(1, 1, -2)$. Area: $\sqrt{629}/2$
6. $(6, 2, 0)$, $(2, 4, 8)$, $(-4, 0, 4)$
7. $(4/3, 2, 4/3)$

Section 11.9: Differentiation and Integration of Vectors

1. Find $\mathbf{v}'(3)$ if $\mathbf{v}(t) = t^2\hat{\mathbf{i}} + \mathrm{Sin}^{-1}(t/4)\hat{\mathbf{j}} + \ln(2t+1)\hat{\mathbf{k}}$.
2. If $f(t) = t^2 + 1$ and $\mathbf{v}(t) = e^t\hat{\mathbf{i}} + [t/(t^2+1)^3]\hat{\mathbf{j}} - t\sqrt{t^2+1}\hat{\mathbf{k}}$, evaluate $\displaystyle\int f(t)\mathbf{v}(t)\,dt$.

Section 11.10: Parametric and Vector Representations of Curves

In questions 3-5, find parametric equations for the curve.

3. $z = 2\sqrt{x^2+y^2}, x^2+y^2 = 3-z$ from $(1,0,2)$ to $(-1,0,2)$ so that y is always nonpositive
4. first octant part of $x^2+z^2 = 4, x+y = 1$ directed so that z increases along the curve
5. $z = x^2+y^2$, $x^2+y^2-4y = 0$ directed clockwise as viewed from a point far up the z-axis

Answers:

1. $6\hat{\mathbf{i}} + \dfrac{1}{\sqrt{7}}\hat{\mathbf{j}} + \dfrac{2}{7}\hat{\mathbf{k}}$
2. $(t^2-2t+3)e^t\hat{\mathbf{i}} - \dfrac{1}{2(t^2+1)}\hat{\mathbf{j}} - \dfrac{1}{5}(t^2+1)^{5/2}\hat{\mathbf{k}} + \overline{C}$
3. $x = \cos t,\ y = -\sin t,\ z = 2,\ 0 \le t \le \pi$
4. $x = 2\cos t, y = 1 - 2\cos t, z = 2\sin t,\ \pi/3 \le t \le \pi/2$
5. $x = 2\cos t,\ y = 2 - 2\sin t,\ z = 8 - 8\sin t,\ 0 \le t \le 2\pi$

Section 11.11: Tangent Vectors and Lengths of Curves

1. Find all unit tangent vectors to the curve $x^2 + z^2 = 4$, $x + y = 1$ at the point $(\sqrt{2}, 1 - \sqrt{2}, \sqrt{2})$.
2. Find the angle between the tangent vectors to the curves
$$x^2 + y = z + 4,\ x + 2y = 5, \quad \text{and} \quad x + y^2 = 5,\ 2x + 3y + 4z = 4$$
at the point of intersection of the curves (assume that the curves are oriented in such a way that this angle is less than $\pi/2$).
3. Find the length of the curve
$$x = t + 1, \quad y = 2t^{3/2} - 3, \quad z = 4t - 2$$
between the points $(2, -1, 2)$ and $(1, -3, -2)$.
4. Show that it is impossible for the length of a curve joining the points $(1, -2, 3)$ and $(0, 4, 10)$ to be equal to 9.
5. Set up, but do not evaluate, a definite integral to find the length of the curve
$$x^2 + y^2 = z^2 - 4, \quad x + y = 4$$
joining the points $(4, 0, 2\sqrt{5})$ and $(2, 2, 2\sqrt{3})$. Simplify the integrand as much as possible.

Answers:

1. $\pm(-\hat{\mathbf{i}} + \hat{\mathbf{j}} + bk)/\sqrt{3}$
2. Point of intersection $(1, 2, -1)$; the angle is $\mathrm{Cos}^{-1}\left(\dfrac{21}{\sqrt{14}\sqrt{297}}\right)$
3. $\dfrac{2}{27}\left(26^{3/2} - 17^{3/2}\right)$
4. the length of the straight line segment connecting these points is $\sqrt{86}$ which is greater than 9
5. $2\displaystyle\int_2^4 \sqrt{\frac{t^2 - 4t + 7}{t^2 - 4t + 10}}\, dt$

Section 12.1: Multivariable Functions

1. Find the domains of the following functions:
   (i) $f(x,y) = \dfrac{x+y}{(x-y)\sqrt{xy}}$, (ii) $g(x,y) = \mathrm{Sin}^{-1}(x+y)$, (iii) $h(x,y,z) = \ln(1-x^2+y^2+z)$.

2. Find $f(x,y)$ if each level curve $f(x,y) = C$ is a circle centered at the origin and having radius
   (i) C, (ii) C^2, (iii) $\sqrt{C}$, (iv) $\ln C$.

Section 12.2: Limits and Continuity

Determine whether the limit exists. If it does not exist, explain why.

3. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-xy+y^2}{x^2+2y^2}$

4. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$

5. $\displaystyle\lim_{(x,y)\to(2,-3)} \frac{x^2-4x-y^2-6y-5}{x^2-4x+y^2+6y+13}$

6. $\displaystyle\lim_{(x,y)\to(3,2)} \frac{\sin(2x-3y)}{2x-3y}$

7. $\displaystyle\lim_{(x,y)\to(0,1)} \mathrm{Tan}^{-1}|y/x|$

8. $\displaystyle\lim_{(x,y)\to(0,1)} \mathrm{Tan}^{-1}(y/x)$

9. $\displaystyle\lim_{(x,y)\to(0,1)} \mathrm{Tan}^{-1}(x/y)$

Answers:

1. (i) $\{(x,y) \mid x>0, y>0, x\neq y\} \cup \{(x,y) \mid x<0, y<0, x\neq y\}$
   (ii) $\{(x,y) \mid -1-x \leq y \leq 1-x\}$
   (iii) $\{(x,y,z) \mid z > x^2-y^2-1\}$

2. (i) $\sqrt{x^2+y^2}$, (ii) $(x^2+y^2)^{1/4}$, (iii) x^2+y^2, (iv) $e^{\sqrt{x^2+y^2}}$

3. 4. 5. Does not exist 6. 1 7. $\pi/2$ 8. Does not exist 9. 0

Section 12.3: Partial Derivatives

1. Show that the function $f(x, y) = 3x^2 + y^2 \cos(2x/y)$ satisfies the equation

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2f.$$

2. Let

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0), \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

Show that f is not continuous at $(0, 0)$ and that $\dfrac{\partial f}{\partial x}(0, 0)$ and $\dfrac{\partial f}{\partial y}(0, 0)$ both exist. This example shows that the existence of partial derivatives does not imply that a function of several variables is continuous. This is in contrast to the single-variable case.

Section 12.4: Gradients

3. Find all functions $F(x, y, z)$, if there are any, such that

$$\nabla F(x, y, z) = (2xy^3 + yze^{xyz})\hat{\mathbf{i}} + (3x^2y^2 + xze^{xyz})\hat{\mathbf{j}} + (xye^{xyz} + z)\hat{\mathbf{k}}.$$

4. Find all functions $F(x, y, z)$, if there are any, such that

$$\nabla F(x, y, z) = (2xy^3 + yze^{xyz})\hat{\mathbf{i}} + (3x^2y^2 + xze^{xyz})\hat{\mathbf{j}} + (xye^{xyz} + y)\hat{\mathbf{k}}.$$

5. If $F(x, y) = x^3y^2 + 3xy - 4$, then the equation $F(x, y) = 0$ implicitly defines a curve in the xy-plane. Show that, at any point on the curve, the gradient ∇F is perpendicular to the curve.

Section 12.5: Higher-Order Partial Derivatives

6. For what value(s) of the constant b is the function $f(x, y) = e^{bx} \cos 5y$ harmonic in the entire xy-plane?

7. Show that $\mathrm{Tan}^{-1}(y/x)$ is harmonic everywhere in the xy-plane except for the points on the y-axis

Answers:

3. $x^2y^3 + e^{xyz} + z^2/2 + C$

4. None exist

6. ± 5

Section 12.6: Chain Rules for Partial Derivatives

1. If $z = f(u, v, t)$, $u = g(x, y, t)$, $v = h(x, y, t)$, and $y = k(t)$, what is the chain rule for $\left.\dfrac{\partial z}{\partial t}\right)_x$?
2. $If u = f(v)$, $v = g(x, y, z)$, $x = h(s, t)$, $y = k(s, t)$, and $z = m(t)$, what is the chain rule for $\left.\dfrac{\partial u}{\partial t}\right)_s$?
3. If $f(s)$ and $g(t)$ are differentiable functions, show that $\nabla f(x^2 - y^2) \cdot \nabla g(xy) = 0$.
4. If $z = x^2 + y^2$, $x = u \cos v$, and $y = u \sin v$, find and simplify $\left.\dfrac{\partial^2 z}{\partial v^2}\right)_u$.
5. If $f(v)$ is differentiable, show that $u(x, y) = x^3 f(x/y)$ satisfies the equation
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u.$$
6. If $z = f(u, v)$, $u = g(x, y)$, and $v = h(x, y)$, what is the chain rule for $\left.\dfrac{\partial^2 z}{\partial x^2}\right)_y$.

Answers:

1. $\dfrac{\partial z}{\partial u}\left(\dfrac{\partial u}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial u}{\partial t}\right) + \dfrac{\partial z}{\partial v}\left(\dfrac{\partial v}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial v}{\partial t}\right) + \left.\dfrac{\partial z}{\partial t}\right)_{u,v}$
2. $\dfrac{du}{dv}\left(\dfrac{\partial v}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial v}{\partial y}\dfrac{\partial y}{\partial t} + \dfrac{\partial v}{\partial z}\dfrac{dz}{dt}\right)$
4. 0
6. $\dfrac{\partial^2 z}{\partial u^2}\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial^2 z}{\partial v \partial u} + \dfrac{\partial^2 z}{\partial u \partial v}\right)\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial x} + \dfrac{\partial z}{\partial u}\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 z}{\partial v^2}\left(\dfrac{\partial v}{\partial x}\right)^2 + \dfrac{\partial z}{\partial v}\dfrac{\partial^2 v}{\partial x^2}$

Section 12.7: Implicit Differentiation

1. The equations
$$x^2 + y + 3s^2 + s = 2t - 1 \quad \text{and} \quad y^2 - x^4 + 2st + 7 = 6s^2t^2$$
define s and t as functions of x and y. Find $\dfrac{\partial s}{\partial x}$ when $s = 0$ and $t = 1$. Assume that $x > 0$. (Solve this problem using two approaches: (i) using Jacobian determinants and Cramer's rule, and (ii) without using Cramer's rule.)

2. The equations
$$x^3y^2 + uv = x + y + 2 \quad \text{and} \quad xy - y(u^2 + v^2) = 3u + 3,$$
define u and v as functions of x and y. Find $\dfrac{\partial u}{\partial y}$ when $x = 1$ and $y = 0$.

3. The equations
$$x = r\sin\phi\cos\theta, \quad y = r\sin\phi\sin\theta \quad \text{and} \quad z = r\cos\phi$$
define r, ϕ and θ as functions of x, y and z. Find $\dfrac{\partial \phi}{\partial y}$.

Section 12.8: Directional Derivatives

4. The function $f(x, y, z) = x^2y + z^3$ is defined at every point on the curve
$$x(y + z) = 3, \quad y - z = 4,$$
directed so that y increases along the curve. What is the rate of change of the function with respect to distance travelled along the curve at the point $(1, 7/2, -1/2)$?

5. In what direction(s) is the rate of change of the function $f(x, y) = x^2y - xy^2$ with respect to distance equal to (i) 1, (ii) 4 at the point $(1, 1)$?

6. At the point $(1, 2, -3)$, a vector $\mathbf{v}$ makes an angle of $\pi/3$ radians with the gradient of the function $f(x, y, z) = x^2yz - 3xy^3$. Find the rate of change of $f(x, y, z)$ in direction $\mathbf{v}$.

Answers:

1. 16
2. -3
3. $r^{-1}\cos\phi\sin\theta$
4. $-35/(4\sqrt{22})$
5. (i) $-\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$; (ii) None
6. $\sqrt{2821}/2$

Section 12.9: Tangent Lines and Tangent Planes

1. Find the rate of change of the function $f(x, y, z) = \sin(xy) - z^3$ at the point $(2, 0, 3)$ in the direction of the upward normal to the surface $xz^2 - x^2z = 6$.

2. Find equations for the tangent line to the curve

$$xyz + z^3 = 24, \quad x^3y^2z + y^3 = 4x - 2$$

at the point $(1, -1, 3)$.

3. Find the acute angle between the normal to the surface $x + z = 3$ and the tangent line to the curve

$$xy^3z + z^3 = 6, \quad xy + yz = -3$$

at their point of intersection.

Section 12.10: Relative Maxima and Minima

4. Find all critical points for the function $f(x, y) = x^3y^3 - x^2y^2 + 6$.

5. Find all critical points for the function $f(x, y) = x^3y^2 - xy + 3y$.

In questions 6-9, find and classify all critical points of the function as giving relative maxima, relative minima, saddle points, or none of these.

6. $f(x, y) = x^3 + xy + y^3$

7. $f(x, y) = x^3 - xy^2 + 3xy$

8. $f(x, y) = x^4 - 3x^2y^2 + y^4$

9. $f(x, y) = y^2 + |x - 1|$

Answers:

1. $-216/\sqrt{73}$

2. $x = 1 + 81t$, $y = -1 + 133t$, $z = 3 - 6t$

3. Point of intersection $(1, -1, 2)$; angle is $\mathrm{Cos}^{-1}\left(\dfrac{3\sqrt{5}}{10}\right)$

4. Every point on the x-axis, every point on the y-axis, and every point on the curve $y = 2/(3x)$.

5. $(3, 0)$ and $(9, 1/243)$

6. $(0, 0)$ gives a saddle point; $(-1/3, -1/3)$ gives a relative maximum

7. $(0, 0)$ gives a saddle point; $(0, 3)$ also gives a saddle point

8. $(0, 0)$ gives a saddle point (the test involving second partial derivatives does not work; consider, for example, curves $y = 0$ and $y = x$)

9. $(1, t)$, $t \in \mathbb{R}$ are critical points; $(1, 0)$ gives a relative minimum; if $t \neq 0$ then $(1, t)$ gives none of these

Section 12.11: Absolute Maxima and Minima

1. Find the maximum and minimum values of the function $f(x, y) = x^2 - y^2$ on the region $x^2 + y^2 \leq 1$.
2. Find the maximum value of the function $f(x, y) = xy(3 - x - 2y)$ on the triangle bounded by the positive $x-$ and $y-$axes and the line $x + y = 1$.
3. Find the maximum value of the function $f(x, y) = x^2 - y^2 + 2x + 9y/2$ considering only points inside and on the boundary of the region surrounded by the curves $x = 1 - y^2$ and $x = 0$.

Section 13.1: Double Integral and Double Iterated Integrals
Section 13.2: Evaluation of Double Integrals by Double Iterated Integrals

4. Evaluate the double iterated integral $\displaystyle\int_{-2}^{0} \int_{0}^{-x} \sqrt{y - x}\, dydx$.
5. Evaluate the double integral of $f(x, y) = x^3y^4 - 3xy^2 + y$ over the region bounded by the curves $y = -x^2$ and $y = x^2 - 1$.
6. Evaluate the double iterated integral $\displaystyle\int_{-2}^{0} \int_{-3x}^{6} e^{y^2}\, dydx$.
7. Evaluate the double integral $\displaystyle\iint_R \frac{1}{y - 1}\, dydx$, where R is the region bounded by the curves $y = 2x$, $y = x$, $x = 2$ and $x = 3$.

Answers:

1. Absolute maximum value 1 is attained at $(\pm 1, 0)$; absolute minimum value -1 is attained at $(0, \pm 1)$
2. Absolute maximum value $2\sqrt{3}/9$ is attained at $(1/\sqrt{3}, 1 - 1/\sqrt{3})$
3. Absolute maximum value $65/16$ is attained at $(3/4, 1/2)$
4. $16(4 - \sqrt{2})/15$
5. $-\sqrt{2}/3$
6. $(e^{36} - 1)/6$
7. $(5/2)\ln 5 - (3/2)\ln 3 - 2\ln 2$

Section 13.3: Area and Volumes of Solids of Revolution

1. Find the volumes of the solids of revolution when the area bounded by the curves

$$y = 2x - x^2, \quad y = x$$

is rotated around the lines: (i) $x = 3$, (ii) $y = 1$, (iii) $x + y = -1$.

Section 13.4: Fluid Pressure

2. A triangular plate has sides with lengths 3, 4 and 5 metres. It is submerged vertically in oil with density 950 kilograms per cubic metre. The side of length 4 metres is horizontal and is at the bottom, the side of length 3 metres is vertical, and the uppermost vertex is 1 metre below the surface of the oil. Find the force due to oil pressure on each side of the plate.

3. An elliptic plate has major axis of length $2a$ metres and minor axis of length $2b$ metres. Its major axis is horizontal and its minor axes is vertical. It is slowly being lowered into a tank of water. At the instant when only $b/2$ metres of the plate sticks out of the water, set up, but do NOT evaluate, a double iterated integral for the force due to the water on each side of the plate.

Section 13.5: Centres of Mass and Moments of Inertia

4. A thin plate with constant mass per unit area ρ has edges defined by the curves

$$x = \sqrt{a^2 - y^2}, \quad y = x \quad \text{and} \quad y = 0,$$

where $a > 0$ is a constant. Find the first moment of the plate about the x-axis.

5. A triangular plate has sides of lengths 2, 3 and 3, and constant mass per unit area ρ. Find its moment of inertia about the shorter side.

Answers:

1. (i) $5\pi/6$, (ii) $2\pi/15$, (iii) $7\sqrt{2}\pi/20$
2. $18\rho g \approx 1.68 \times 10^5 N$
3. $\displaystyle\int_{-b}^{b/2} \int_{-(a/b)\sqrt{b^2-y^2}}^{(a/b)\sqrt{b^2-y^2}} \rho g(b/2 - y)\, dx dy \approx 9810 \int_{-b}^{b/2} \int_{-(a/b)\sqrt{b^2-y^2}}^{(a/b)\sqrt{b^2-y^2}} (b/2 - y)\, dx dy\ N$
4. $\displaystyle\int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2-y^2}} \rho y\, dx dy = \frac{\sqrt{2} - 1}{3\sqrt{2}} \rho a^3$
5. $(8\sqrt{2}/3)\rho$

Section 13.6: Surface Area

1. Find the area of that part of the surface $z = xy$ inside the cylinder $x^2 + y^2 = a^2$.
2. Set up, but do NOT evaluate, a double iterated integral for the surface area of the ellipsoid
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$
3. Set up, but do NOT evaluate, a double iterated integral for the area of the surface $z = 2x^2 + y^2$ bounded by $y = 0$, $x = 0$ and $x + y = 1$.

Section 13.7: Double Iterated Integrals in Polar Coordinates

4. Find the area bounded by $(x^2 + y^2)^3 = 4a^2x^2y^2$.
5. Find the double integral of $f(x, y) = xy(x+y)$ over the region in the first quadrant bounded by $x^2+y^2 = 1$ and $x^2 + y^2 = 4$.

Section 13.8: Triple Integrals and Triple Iterated Integrals

6. Evaluate the triple integral of the function $f(x, y, z) = x$ over the solid bounded by the surfaces
$$2x + 3y + z = 6, \quad x = 0, \quad y = 0 \quad \text{and} \quad z = 0.$$

Answers:

1. $\displaystyle\iint_{x^2+y^2\le a^2} \sqrt{1 + x^2 + y^2}\, dA = \frac{2\pi}{3}\left((1 + a^2)^{3/2} - 1\right)$
2. $\displaystyle 2\iint_{(x/a)^2+(y/b)^2\le 1} \sqrt{1 + \left(\frac{-cx}{a^2\sqrt{1 - x^2/a^2 - y^2/b^2}}\right)^2 + \left(\frac{-cy}{b^2\sqrt{1 - x^2/a^2 - y^2/b^2}}\right)^2}\, dA$
$\displaystyle = 8\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \sqrt{1 + \left(\frac{-cx}{a^2\sqrt{1 - x^2/a^2 - y^2/b^2}}\right)^2 + \left(\frac{-cy}{b^2\sqrt{1 - x^2/a^2 - y^2/b^2}}\right)^2}\, dydx$
3. $\displaystyle\int_0^1 \int_0^{1-x} \sqrt{1 + 16x^2 + 4y^2}\, dydx$
4. $\displaystyle\iint_R 1\, dA = \int_0^{2\pi} \int_0^{a|\sin(2\theta)|} rdrd\theta = \pi a^2/2$
5. $\displaystyle\int_1^2 \int_0^{\pi/2} r^4 \sin\theta \cos\theta\, (\sin\theta + \cos\theta)\, d\theta dr = 62/15$
6. $\displaystyle\int_0^3 \int_0^{2-2x/3} \int_0^{6-2x-3y} x\, dzdydx = 9/2$

Section 13.9: Volumes

1. Set up, but do NOT evaluate, an integral for the volume in the first octant bounded by the surfaces

$$4x + 4y + z = 16, \quad z = 0, \quad y = x/2 \quad \text{and} \quad y = 2x.$$

2. Set up, but do NOT evaluate, a triple iterated integral for the volume in the first octant bounded by the surfaces

$$z = 2x + y, \quad 9x^2 + 4y^2 = 1, \quad x = 0, \quad y = 0 \quad \text{and} \quad z = 0.$$

3. Set up, but do NOT evaluate, a triple iterated integral for the volume bounded by the surfaces

$$z = 9 - x^2 - y^2 \quad \text{and} \quad z = x^2.$$

Section 13.11: Triple Iterated Integrals in Cylindrical Coordinates

4. Find the volume bounded by the surfaces $z = xy$, $x^2 + y^2 = 1$ and $z = 0$.
5. Find the volume bounded by the surfaces $z = 2\sqrt{x^2 + y^2}$ and $z = 9 - x^2 - y^2$. Get a numerical answer, but do not simplify it.
6. Set up, but do NOT evaluate, a triple iterated integral for the triple integral of the function $f(x, y, z) = x^2 + y^3$ over the solid bounded by the surfaces $(x^2 + y^2)^2 = 2xy$, $z = \sqrt{1 - x^2 - y^2}$ and $z = 0$.

Section 13.12: Triple Iterated Integrals in Spherical Coordinates

7. Find $\iiint_R x\,dV$ and $\iiint_R z\,dV$ over the solid R which is the part of the hemisphere $0 \le z \le \sqrt{a^2 - x^2 - y^2}$ that lies in the first octant.

Answers:

1. $\left(\int_0^{4/3}\int_{x/2}^{2x} + \int_{4/3}^{8/3}\int_{x/2}^{4-x}\right) 4(4 - x - y)\,dydx$
2. $\int_0^{1/3}\int_0^{(1/2)\sqrt{1-9x^2}}\int_0^{2x+y} 1\,dzdydx$
3. $\int_{-\sqrt{3}/2}^{\sqrt{3}/2}\int_{-\sqrt{9-2x^2}}^{\sqrt{9-2x^2}}\int_{x^2}^{9-x^2-y^2} 1\,dzdydx = 4\int_0^{\sqrt{3}/2}\int_0^{\sqrt{9-2x^2}}\int_{x^2}^{9-x^2-y^2} 1\,dzdydx$
4. $4\int_0^{\pi/2}\int_0^1\int_0^{r^2\cos\theta\sin\theta} r\,dzdrd\theta = 1/2$
5. $\int_0^{2\pi}\int_0^{\sqrt{10}-1}\int_{2r}^{9-r^2} r\,dzdrd\theta = 2\pi\left(\frac{9(\sqrt{10}-1)^2}{2} - \frac{2(\sqrt{10}-1)^3}{3} - \frac{(\sqrt{10}-1)^4}{4}\right)$
6. $\int_0^{\pi/2}\int_0^{\sqrt{\sin 2\theta}}\int_0^{\sqrt{1-r^2}} r\left(r^2\cos^2\theta + r^3\sin^3\theta\right)\,dzdrd\theta$
7. $\iiint_R x\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \rho^3\sin^2\phi\cos\theta\,d\rho d\phi d\theta = \frac{\pi a^4}{16}$
   $\iiint_R z\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \rho^3\sin\phi\cos\phi\,d\rho d\phi d\theta = \frac{\pi a^4}{16}$ (note: integrals have the same value by symmetry)