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ONE ESTIMATE FOR DIVIDED DIFFERENCES AND ITS APPLICATIONS

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We give an estimate for the general divided differences $[x_0,\ldots,x_m;f]$, where some points x_i are allowed to coalesce (in this case, f is assumed to be sufficiently smooth). This estimate is then applied to significantly strengthen the celebrated Whitney and Marchaud inequalities and their generalization to the Hermite interpolation. As one of numerous corollaries of this estimate, we can mention the fact that, given a function $f\in C^{(r)}(I)$ and a set $Z=\{z_j\}_{j=0}^{\mu}$ such that $z_{j+1}-z_j\ge\lambda|I|$ for all $0\le j\le\mu-1$, where $I:=[z_0,z_\mu]$, $|I|$ is the length of I, and λ is a positive number, the Hermite polynomial $\mathcal{L}(\cdot;f;Z)$ of degree $\le r\mu+\mu+r$ satisfying the equality $\mathcal{L}^{(j)}(z_\nu;f;Z)=f^{(j)}(z_\nu)$ for all $0\le\nu\le\mu$ and $0\le j\le r$ approximates f so that, for all $x\in I$,

$$|f(x)-\mathcal{L}(x;f;Z)|\le C\,(\operatorname{dist}(x,Z))^{r+1}\int\limits_{\operatorname{dist}(x,Z)}^{2|I|}\frac{\omega_{m-r}(f^{(r)},t,I)}{t^2}\,dt,$$

where $m:=(r+1)(\mu+1),\ C=C(m,\lambda)$ and $\operatorname{dist}(x,Z):=\min_{0\le j\le\mu}|x-z_j|$.

1. Introduction

V. K. Dzyadyk had a significant impact on the theory of extension of functions, and we start our presentation by recalling three of his most significant (in our opinion) results in this direction.

First, in 1956 (see [4]), he solved a problem posed by S. M. Nikolskii of extension of a function

$$f\in\operatorname{Lip}_M(\alpha,p),\quad 0<\alpha\le 1,\quad p\ge 1,$$

on a finite interval $[a,b]$, to a function $F\in\operatorname{Lip}_{M_1}(\alpha,p)$ on the entire real line, i.e., $F|_{[a,b]}=f$.

Then, in 1958 (see [5] or [6, p. 171, 172]), he showed that if $f\in C[0,1]$, then this function can be extended to a function $F\in C[-1,1]$ with controlled second modulus of smoothness on $[-1,1]$, i.e., $F|_{[0,1]}=f$, and the second moduli of smoothness of f and F satisfy the inequality $\omega_2(F,\delta;[-1,1])\le 5\omega_2(f,\delta;[0,1]),\ 0<\delta\le 1$. [This result was independently proved by Frey [9] (also in 1958).]

In the present paper, we mainly deal with the results related to the third main Dzyadyk's result. It can be described as follows:

Given a function $f\in C[a,b]$ and $a\le x_0<x_1<x_2\le b$, the second divided difference $[x_0,x_1,x_2;f]$ can be estimated as follows (see, e.g., [16, p. 176] and [8, p. 237]):

$$\big|[x_0,x_1,x_2;f]\big|\le\frac{c}{x_2-x_0}\int\limits_{h}^{x_2-x_0}\frac{\omega_2(f,t)}{t^2}\,dt,\tag{1.1}$$

where $c=\text{const}<18$ and $h:=\min\{x_1-x_0,x_2-x_1\}$.

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Now let ω_2 be an arbitrary function of the second-modulus-of-smoothness type, i.e., $\omega_2 \in C[0,\infty]$ is nondecreasing and such that $\omega_2(0) = 0$ and

$$t_1^{-2}\omega_2(t_1) \le 4t_2^{-2}\omega_2(t_2), \qquad 0 < t_2 < t_1.$$

In 1983, Dzyadyk and Shevchuk [7] proved that if f is defined on an arbitrary set $E \subset \mathbb{R}$ and satisfies (1.1) with $\omega_2(t)$ instead of $\omega_2(f,t)$ for every triple of points $x_0, x_1, x_2 \in E$ satisfying $x_0 < x_1 < x_2$, then f can be extended from E to a function $F \in C(\mathbb{R})$ such that $\omega_2(F,t;\mathbb{R}) \le c\omega_2(t)$. In other words, (1.1) with $\omega_2(t)$ instead of $\omega_2(f,t)$ is necessary and sufficient for a function f to be the trace, on the set $E \subset \mathbb{R}$, of a function $F \in C(\mathbb{R})$ satisfying $\omega_2(F,t;\mathbb{R}) \le c\omega_2(t)$. This result was independently proved by Brudnyi and Shvartsman [2] in 1982 (see also Jonsson [14] for $\omega_2(t) = t$).

Dzyadyk posed the problem of description of these traces for functions of the type of kth modulus of smoothness with $k > 2$. He conjectured that an analog of (1.1) must be a corollary of the Whitney and Marchaud inequalities. In 1984, this conjecture was confirmed by Shevchuk [19] and the corresponding (exact) analog of (1.1) with $k > 2$ was found [see (2.7) in what follows with $r = 0$]. Earlier, the case $\omega(t) = t^{k-1}$ was proved by Jonsson whose paper [14] was submitted in 1981, revised in 1983, and published in 1985.

What happens when we have differentiable functions? In 1934, Whitney [23] described the traces of r times continuously differentiable functions $F\colon \mathbb{R} \mapsto \mathbb{R}$ on arbitrary closed sets $E \subset \mathbb{R}$. A trace of this kind consists of all functions $f\colon E \mapsto \mathbb{R}$ whose rth differences converge on E (see [24] for the definition).

In 1975, de Boor [1] described the traces of functions $F\colon \mathbb{R} \mapsto \mathbb{R}$ with bounded rth derivative on arbitrary sets $E \subset \mathbb{R}$ of isolated points. A trace of this kind consists of all functions whose rth divided differences are uniformly bounded on E (in 1965, Subbotin [22] obtained exact constants in the case where the sets E consist of equidistant points).

Finally, for an arbitrary given set $E \subset \mathbb{R}$, the necessary and sufficient condition for a function f to be a trace (on E) of a function $F \in C^{(r)}(\mathbb{R})$ with prescribed kth modulus of continuity of the rth derivative was obtained by Shevchuk [19] in 1984 (see also Theorems 11.1 and 12.3 in [20], Theorems 3.2 and 4.3 in Chapter 4 in [8], and [21], where a linear extension operator was given).

In fact, this necessary and sufficient condition is an analog of (1.1) for the kth modulus of continuity of the rth derivative of f, i.e., inequality (2.7) in Theorem 2.1 in what follows. However, the original proof of Theorem 2.1 was distributed among several publications (see [10, 18, 19], as well as [20] and [8]), and there was an unfortunate misprint in the formulation of Theorem 6.4 in Section 3 of [8]: In relation (3.6.36), "k" should be replaced by "m." Hence, the main aim of the present paper is to properly formulate this theorem (Theorem 2.1), give its complete and self-contained proof, and discuss several important corollaries/applications that have been inadvertently overlooked in the past.

2. Definitions, Notation, and the Main Result

For $f \in C[a,b]$ and any $k \in \mathbb{N}$, we set

$$\Delta_u^k(f,x;[a,b]) := \begin{cases} \sum_{i=0}^{k}(-1)^i\binom{k}{i} f(x+(k/2-i)u), & x \pm (k/2)u \in [a,b], \\ 0, & \text{otherwise}, \end{cases}$$

and denote by

$$\omega_k(f,t;[a,b]) := \sup_{0<u\le t} \|\Delta_u^k(f,\cdot;[a,b])\|_{C[a,b]} \tag{2.1}$$

the kth modulus of smoothness of f on $[a,b]$.

We now recall the definition of Lagrange–Hermite divided differences (see, e.g., [3, p. 118]). Let

$$X = \{x_j\}_{j=0}^m$$

be a collection of $m+1$ points with possible repetitions. For each j, the multiplicity m_j of x_j is the number of x_i such that $x_i = x_j$. Let l_j be the number of $x_i = x_j$ with $i \le j$. We say that a point x_j is a simple knot if its multiplicity is 1. Suppose that a real-valued function f is defined at all points in X and, moreover, for each $x_j \in X$, $f^{(l_j-1)}(x_j)$ is also defined (i.e., f has $m_j - 1$ derivatives at every point with multiplicity m_j).

By

$$[x_0; f] := f(x_0),$$

we denote the divided difference of f of order 0 at the point x_0.

Definition 2.1. *Let $m \in \mathbb{N}$. If $x_0 = \ldots = x_m$, then we denote*

$$[x_0, \ldots, x_m; f] = \Big[\underbrace{x_0, \ldots, x_0}_{m+1}; f\Big] := \frac{f^{(m)}(x_0)}{m!}.$$

Otherwise, $x_0 \ne x_{j^*}$ for some number j^* and

$$[x_0, \ldots, x_m; f] := \frac{1}{x_{j^*} - x_0}\big([x_1, \ldots, x_m; f] - [x_0, \ldots, x_{j^*-1}, x_{j^*+1}, \ldots, x_m; f]\big),$$

denotes the divided (Lagrange–Hermite) difference of f of order m at the knots $X = \{x_j\}_{j=0}^m$.

Note that $[x_0, \ldots, x_m; f]$ is symmetric in $x_0, \ldots, x_m$ (i.e., does not depend on the enumeration of points from X). We recall that

$$L_m(x; f) := L_m(x; f; x_0, \ldots, x_m) := f(x_0) + \sum_{j=1}^{m} [x_0, \ldots, x_j; f](x - x_0)\ldots(x - x_{j-1}) \tag{2.2}$$

is an (Hermite) polynomial of degree $\le m$ such that

$$L_m^{(l_j-1)}(x_j; f) = f^{(l_j-1)}(x_j), \quad \text{for all} \quad 0 \le j \le m. \tag{2.3}$$

Hence, in particular, if x_{j_*} is a simple knot, then we can write

$$[x_0, \ldots, x_m; f] := \frac{f(x_{j_*}) - L_{m-1}(x_{j_*}; f; x_0, \ldots, x_{j_*-1}, x_{j_*+1}, \ldots, x_m)}{\prod_{j=0, j\ne j_*}^{m} (x_{j_*} - x_j)}. \tag{2.4}$$

From now on, for the sake of convenience, we assume that all interpolation points are numbered from left to right, i.e., the set of interpolation points $X = \{x_j\}_{j=0}^m$ is such that $x_0 \le x_1 \le \ldots \le x_m$. We also assume that the maximum multiplicity of each point is $r+1$ with $r \in \mathbb{N}_0$ and, hence,

$$x_j < x_{j+r+1}, \quad \text{for all} \quad 0 \le j \le m - r - 1. \tag{2.5}$$

Also let

$$\mathcal{Q}_{m,r} := \{(p,q) \mid 0 \le p, q \le m \text{ and } q - p \ge r+1\}$$
$$= \{(p,q) \mid 0 \le p \le m-r-1 \text{ and } p+r+1 \le q \le m\}. \tag{2.6}$$

Note that $\mathcal{Q}_{m,r} = \varnothing$ if $m \le r$.

Further, for all $(p,q) \in \mathcal{Q}_{m,r}$, we set

$$d(p,q) := d(p,q;X) := \min\{x_{q+1} - x_p, x_q - x_{p-1}\},$$

where $x_{-1} := x_0 - (x_m - x_0)$ and $x_{m+1} := x_m + (x_m - x_0)$. In particular, we also note that

$$d := d(X) := d(0,m;X) = 2(x_m - x_0).$$

Everywhere in what follows, Φ is the set of nondecreasing functions $\varphi \in C[0,\infty]$ satisfying $\varphi(0) = 0$. We also denote

$$\Lambda_{p,q,r}(x_0,\dots,x_m;\varphi) := \frac{\displaystyle\int_{x_q-x_p}^{d(p,q)} u^{p+r-q-1}\varphi(u)du}{\prod_{i=0}^{p-1}(x_q - x_i)\prod_{i=q+1}^{m}(x_i - x_p)}, \quad (p,q) \in \mathcal{Q}_{m,r},$$

and

$$\Lambda_r(x_0,\dots,x_m;\varphi) := \max_{(p,q)\in\mathcal{Q}_{m,r}} \Lambda_{p,q,r}(x_0,\dots,x_m;\varphi).$$

Here, we have used the ordinary convention that

$$\prod_{i=0}^{-1} := 1 \qquad \text{and} \qquad \prod_{i=m+1}^{m} := 1.$$

The following theorem is the main result of the present paper.

Theorem 2.1. *Let $r \in \mathbb{N}_0$ and $m \in \mathbb{N}$ be such that $m \ge r+1$. Suppose that a set $X = \{x_j\}_{j=0}^m$ is such that $x_0 \le x_1 \le \ldots \le x_m$ and (2.5) is satisfied. If $f \in C^{(r)}[x_0, x_m]$, then*

$$\big|[x_0,\dots,x_m;f]\big| \le c\Lambda_r(x_0,\dots,x_m;\omega_k), \tag{2.7}$$

where $k := m - r$, $\omega_k(t) := \omega_k\big(f^{(r)}, t; [x_0, x_m]\big)$, and the constant c depends only on m.

3. Auxiliary Lemmas

Throughout this section, we assume that $r \in \mathbb{N}_0$, $m \in \mathbb{N}$, $m \ge r+1$, the set $X = \{x_j\}_{j=0}^m$ is such that $x_0 \le x_1 \le \ldots \le x_m$ and (2.5) is satisfied, and that $(p,q) \in \mathcal{Q}_{m,r}$. For the sake of convenience, we also denote $k := m - r$.

We first show that Theorem 2.1 is true in the case $m = r+1$ (i.e., $k = 1$).

Lemma 3.1. *Theorem 2.1 holds for $m = r + 1$.*

Proof. If $m = r + 1$, then $\mathcal{Q}_{m,r} = \{(0, r+1)\}$ and, therefore,

$$\Lambda_r(x_0, \ldots, x_m; \varphi) = \Lambda_{0,r+1,r}(x_0, \ldots, x_m; \varphi) = \int\limits_{d/2}^{d} u^{-2}\varphi(u)du.$$

Hence, since $x_0 \neq x_m$ by assumption (2.5), inequality (2.7) follows from the identity

$$[x_0, \ldots, x_m; f] = \frac{[x_1, \ldots, x_{r+1}; f] - [x_0, \ldots, x_r; f]}{x_m - x_0} = \frac{f^{(r)}(\theta_1) - f^{(r)}(\theta_2)}{r!d/2},$$

where $\theta_1 \in (x_1, x_{r+1}),\ \theta_2 \in (x_0, x_r),$ and the following estimate is true:

$$\frac{\left|f^{(r)}(\theta_1) - f^{(r)}(\theta_2)\right|}{d} \leq \frac{\omega_1(d/2)}{d} \leq \int\limits_{d/2}^{d} \frac{\omega_1(u)}{u^2}\, dt = \Lambda_r(x_0, \ldots, x_m; \omega_1).$$

Lemma 3.1 is proved.

For $k > 2,$ we need the following lemma:

Lemma 3.2. *Let $(p, q) \in \mathcal{Q}_{m,r}$ be such that $q - p + 2 \leq m$. If $\varphi \in \Phi$ and $\omega \in \Phi$ are such that*

$$\varphi(t) \leq t^{k-1} \int\limits_{t}^{d} u^{-k}\omega(u)du, \quad t \in (0, d/2], \tag{3.1}$$

then

$$\Lambda_{p,q,r}(x_0, \ldots, x_m; \varphi) \leq 2^{k^2}\Lambda_r(x_0, \ldots, x_m; \omega). \tag{3.2}$$

Proof. Let $(p, q) \in \mathcal{Q}_{m,r}$ with $q - p + 2 \leq m$ be fixed. Consider a collection $\big\{(p_\nu, q_\nu)\big\}_{\nu=0}^{m-q+p}$, which is defined as follows: Let $(p_0, q_0) := (p, q)$ and, for $\nu \geq 1$,

$$(p_\nu, q_\nu) := \begin{cases} (p_{\nu-1} - 1, q_{\nu-1}) & \text{if} \quad x_{q_{\nu-1}} - x_{p_{\nu-1}-1} \leq x_{q_{\nu-1}+1} - x_{p_{\nu-1}}, \\ (p_{\nu-1}, q_{\nu-1} + 1), & \text{otherwise.} \end{cases}$$

It is clear that $q_\nu - p_\nu = q_{\nu-1} - p_{\nu-1} + 1,$ and, hence,

$$q_\nu - p_\nu = q - p + \nu. \tag{3.3}$$

One can easily show (e.g., by induction) that, for all $1 \leq \nu \leq m - q + p,$ we have

$$0 \leq p_\nu \leq p_{\nu-1} < q_{\nu-1} \leq q_\nu \leq m.$$

Hence, in particular,

$$(p_{m-q+p}, q_{m-q+p}) = (0, m).$$

In the remaining part of the proof, we use the notation

$$d_\nu := d(p_\nu, q_\nu), \quad 0 \le \nu \le m - q + p.$$

Note that

$$d_\nu \ge d_{\nu-1} = x_{q_\nu} - x_{p_\nu}, \quad 1 \le \nu \le m + q - p,$$

and

$$d_{m-q+p-1} = x_m - x_0 = d/2.$$

We now show that, for all $1 \le \nu \le m - q + p$,

$$\frac{d_{\nu-1}}{\prod_{i=0}^{p_{\nu-1}-1}(x_{q_{\nu-1}} - x_i)\prod_{i=q_{\nu-1}+1}^{m}(x_i - x_{p_{\nu-1}})} \le \frac{2^k}{\prod_{i=0}^{p_\nu-1}(x_{q_\nu} - x_i)\prod_{i=q_\nu+1}^{m}(x_i - x_{p_\nu})}. \tag{3.4}$$

Indeed, if $x_{q_{\nu-1}} - x_{p_{\nu-1}-1} \le x_{q_{\nu-1}+1} - x_{p_{\nu-1}}$, then

$$(p_\nu, q_\nu) = (p_{\nu-1} - 1, q_{\nu-1}), \qquad d_{\nu-1} = x_{q_{\nu-1}} - x_{p_{\nu-1}-1},$$

and, for $q_{\nu-1} + 1 \le j \le m$,

$$\begin{aligned} x_j - x_{p_\nu} &= (x_j - x_{q_{\nu-1}}) + (x_{q_{\nu-1}} - x_{p_{\nu-1}-1}) \\ &\le (x_j - x_{p_{\nu-1}}) + (x_{q_{\nu-1}+1} - x_{p_{\nu-1}}) \le 2(x_j - x_{p_{\nu-1}}), \end{aligned}$$

whence it follows that

$$\prod_{i=q_{\nu-1}+1}^{m}(x_i - x_{p_{\nu-1}}) \ge 2^{q_{\nu-1}-m} \prod_{i=q_\nu+1}^{m}(x_i - x_{p_\nu}).$$

This yields (3.4) because $m - q_{\nu-1} \le m - q \le k$.

Similarly, if $x_{q_{\nu-1}} - x_{p_{\nu-1}-1} > x_{q_{\nu-1}+1} - x_{p_{\nu-1}}$, then

$$(p_\nu, q_\nu) = (p_{\nu-1}, q_{\nu-1} + 1), \qquad d_{\nu-1} = x_{q_{\nu-1}+1} - x_{p_{\nu-1}},$$

and, for $0 \le j \le p_{\nu-1} - 1$, we get

$$\begin{aligned} x_{q_\nu} - x_j &= (x_{q_{\nu-1}+1} - x_{p_{\nu-1}}) + (x_{p_{\nu-1}} - x_j) \\ &< (x_{q_{\nu-1}} - x_{p_{\nu-1}-1}) + (x_{q_{\nu-1}} - x_j) \le 2(x_{q_{\nu-1}} - x_j) \end{aligned}$$

and, therefore,

$$\prod_{i=0}^{p_{\nu-1}-1}(x_{q_{\nu-1}}-x_i) \geq 2^{-p_{\nu-1}} \prod_{i=0}^{p_\nu-1}(x_{q_\nu}-x_i).$$

This also yields (3.4) because $p_{\nu-1} \leq p < k$.

Inequality (3.4) implies that, for all $1 \leq \nu \leq m-q+p$, we get

$$\frac{\prod_{i=0}^{\nu-1} d_i}{\prod_{i=0}^{p-1}(x_q-x_i)\prod_{i=q+1}^{m}(x_i-x_p)} \leq \frac{2^{k\nu}}{\prod_{i=0}^{p_\nu-1}(x_{q_\nu}-x_i)\prod_{i=q_\nu+1}^{m}(x_i-x_{p_\nu})}. \tag{3.5}$$

It is clear that $d(p,q) \leq x_m - x_0 = d/2$ and, hence, condition (3.1) implies that

$$\int\limits_{x_q-x_p}^{d(p,q)} u^{p+r-q-1}\varphi(u)du \leq \int\limits_{x_q-x_p}^{d(p,q)} u^{p+m-q-2}\left(\int\limits_u^d v^{-k}\omega(v)dv\right)du.$$

As a result of integration by parts, we obtain

$$(m-q+p-1)\int\limits_{x_q-x_p}^{d(p,q)} u^{p+r-q-1}\varphi(u)du - \int\limits_{x_q-x_p}^{d(p,q)} u^{p+r-q-1}\omega(u)du$$

$$\leq d^{m-q+p-1}(p,q)\int\limits_{d(p,q)}^{d}\frac{\omega(u)}{u^k}\,du = d^{m-q+p-1}(p,q)\sum_{\nu=1}^{m-q+p}\int\limits_{d_{\nu-1}}^{d_\nu}\frac{\omega(u)}{u^k}\,du$$

$$\leq 2\sum_{\nu=1}^{m-q+p}\prod_{i=0}^{\nu-1} d_i \int\limits_{d_{\nu-1}}^{d_\nu} u^{p+r-q-1-\nu}\omega(u)du.$$

The last estimate is obvious for $1 \leq \nu \leq m-q+p-1$. For $\mu = m-q+p$, it follows from the inequality

$$d_0^{m-q+p-1}d_{m-q-p} \leq 2\prod_{i=0}^{m-q+p-1} d_i,$$

which is true because

$$d_0^{m-q+p-1} \leq \prod_{i=0}^{m-q+p-2} d_i$$

and

$$d_{m-q-p} = d(0,m) = d = 2d_{m-q+p-1}.$$

Finally, taking into account (3.3) and (3.5) and recalling that $d_{\nu-1} = x_{q_\nu} - x_{p_\nu},\ 1 \le \nu \le m-q+p$, we obtain

$$
\begin{aligned}
&(m-q+p-1)\Lambda_{p,q,r}(x_0,\dots,x_m;\varphi) \\
&\qquad \le \Lambda_{p,q,r}(x_0,\dots,x_m;\omega) + 2\sum_{\nu=1}^{m-q+p} 2^{k\nu}\Lambda_{p_\nu,q_\nu,r}(x_0,\dots,x_m;\omega),
\end{aligned}
$$

which implies (3.2).

Lemma 3.2 is proved.

Lemma 3.3. *If $k = m - r \ge 2,\ \varphi \in \Phi$ and $\omega \in \Phi$ are such that*

$$\varphi(t) \le t^{k-1}\int_t^d u^{-k}\omega(u)du, \quad t \in (0, d/2], \tag{3.6}$$

and $\varphi(t) \le \omega(t),\ t \in [d/2, d]$, then

$$\Lambda_r(x_0,\dots,x_{m-1};\varphi) \le c(x_m - x_0)\Lambda_r(x_0,\dots,x_m;\omega) \tag{3.7}$$

and

$$\Lambda_r(x_1,\dots,x_m;\varphi) \le c(x_m - x_0)\Lambda_r(x_0,\dots,x_m;\omega), \tag{3.8}$$

where the constants c depend only on k.

Proof. We first note that (3.8) follows from (3.7). Indeed, given $X = \{x_i\}_{i=0}^m$, we define the set $Y = \{y_i\}_{i=0}^m$ as follows:

$$y_i := -x_{m-i}, \quad 0 \le i \le m.$$

Then $y_0 \le y_1 \le \ldots \le y_m,\ y_m - y_0 = x_m - x_0$ (thus, in particular, $d(Y) = d(X) = d$),

$$
\begin{aligned}
d(p,q;Y) &= \min\{y_{q+1} - y_p, y_q - y_{p-1}\} \\
&= \min\{x_{m-p} - x_{m-q-1}, x_{m-p+1} - x_{m-q}\} \\
&= d(m-q, m-p; X) = d(m-q, m-p),
\end{aligned}
$$

and it is not difficult to check that, for any $\psi \in \Phi$,

$$\Lambda_{p,q,r}(y_0,\dots,y_m;\psi) = \Lambda_{m-q,m-p,r}(x_0,\dots,x_m;\psi)$$

and

$$\Lambda_{p,q,r}(y_0,\dots,y_{m-1};\psi) = \Lambda_{m-q-1,m-p-1,r}(x_1,\dots,x_m;\psi).$$

Hence, by using the fact that $(p,q) \in \mathcal{Q}_{\mu,r}$ iff $(\mu-q,\mu-p) \in \mathcal{Q}_{\mu,r},\ \mu = m-1, m,$ we find

$$\begin{aligned}\Lambda_r(x_0,\dots,x_m;\omega) &= \max_{(p,q)\in\mathcal{Q}_{m,r}} \Lambda_{p,q,r}(x_0,\dots,x_m;\omega)\\ &= \max_{(m-q,m-p)\in\mathcal{Q}_{m,r}} \Lambda_{m-q,m-p,r}(y_0,\dots,y_m;\omega)\\ &= \Lambda_r(y_0,\dots,y_m;\omega)\end{aligned}$$

and

$$\begin{aligned}\Lambda_r(x_1,\dots,x_m;\varphi) &= \max_{(p,q)\in\mathcal{Q}_{m-1,r}} \Lambda_{p,q,r}(x_1,\dots,x_m;\varphi)\\ &= \max_{(m-q-1,m-p-1)\in\mathcal{Q}_{m-1,r}} \Lambda_{m-q-1,m-p-1,r}(y_0,\dots,y_{m-1};\varphi)\\ &= \Lambda_r(y_0,\dots,y_{m-1};\varphi).\end{aligned}$$

This means that (3.8) follows from (3.7) applied to the set Y.

We are now ready to prove (3.7). Let $(p^*,q^*) \in \mathcal{Q}_{m-1,r}$ be such that

$$\Lambda^* := \Lambda_{p^*,q^*,r}(x_0,\dots,x_{m-1};\varphi) = \Lambda_r(x_0,\dots,x_{m-1};\varphi),$$

and, for the sake of convenience, we denote

$$X_m := \{x_0,\dots,x_m\} \qquad \text{and} \qquad X_{m-1} := \{x_0,\dots,x_{m-1}\}.$$

We consider four cases.

Case I: $(p^*,q^*) = (0,m-1)$.

We set $h := x_{m-1} - x_0$ and note that $\Lambda^* = \int_h^{2h} u^{-k}\varphi(u)du.$

If $h \le d/4$, then

$$\begin{aligned}2^{1-k}\Lambda^* \le (2h)^{1-k}\varphi(2h) &\le \int_{2h}^{d} u^{-k}\omega(u)du\\ &\le \int_{h}^{d/2} u^{-k}\omega(u)du + \int_{d/2}^{d} u^{-k}\omega(u)du\\ &\le \int_{h}^{d/2} u^{-k}\omega(u)du + d\int_{d/2}^{d} u^{-k-1}\omega(u)du\end{aligned}$$

$$= (x_m - x_0)\left(\Lambda_{0,m-1,r}(x_0,\dots,x_m;\omega) + 2\Lambda_{0,m,r}(x_0,\dots,x_m;\omega)\right)$$

$$\le 3(x_m - x_0)\Lambda_r(x_0,\dots,x_m;\omega).$$

Further, if $h > d/4$, then

$$\Lambda^* = \int_h^{d/2} u^{-k}\varphi(u)du + \int_{d/2}^{2h} u^{-k}\varphi(u)du$$

$$\le (4/d)^{k-1}\varphi(d/2) + \int_{d/2}^{2h} u^{-k}\varphi(u)du$$

$$< 4^k \int_{d/2}^{d} u^{-k}\varphi(u)du \le 4^k \int_{d/2}^{d} u^{-k}\omega(u)du$$

$$\le 4^k d \int_{d/2}^{d} u^{-k-1}\omega(u)du = 2\cdot 4^k (x_m - x_0)\Lambda_{0,m,r}(x_0,\dots,x_m;\omega)$$

$$\le 2\cdot 4^k (x_m - x_0)\Lambda_r(x_0,\dots,x_m;\omega).$$

Case II: Either (i) $q^* \ne m-1$ or (ii) $q^* = m-1,\ p^* > 0$, and $x_m - x_{p^*} > x_{m-1} - x_{p^*-1}$.

In this case, $d(p^*, q^*; X_{m-1}) = d(p^*, q^*; X_m) = x_{m-1} - x_{p^*-1}$ and, therefore

$$\Lambda^* = (x_m - x_{p^*})\Lambda_{p^*,q^*,r}(x_0,\dots,x_m;\varphi) \le (x_m - x_0)\Lambda_{p^*,q^*,r}(x_0,\dots,x_m;\varphi).$$

Since $q^* - p^* + 2 \le m$, we can apply Lemma 3.2 and obtain (3.7).

Case III: $q^* = m-1,\ p^* \ge 2$ and $x_m - x_{p^*} \le x_{m-1} - x_{p^*-1}$.

In this case, $d(p^*, q^*; X_{m-1}) = x_{m-1} - x_{p^*-1}$ and $d(p^*, q^*; X_m) = x_m - x_{p^*}$. Hence, in view of the fact that, for $0 \le i \le p^*-1$, we can write

$$x_m - x_i = x_m - x_{p^*} + x_{p^*} - x_i \le x_{m-1} - x_{p^*-1} + x_{p^*} - x_i \le 2(x_{m-1} - x_i),$$

we obtain

$$\Lambda_{p^*,m-1,r}(x_0,\dots,x_{m-1};\varphi) - (x_m - x_{p^*})\Lambda_{p^*,m-1,r}(x_0,\dots,x_m;\varphi)$$

$$= \prod_{i=0}^{p^*-1}(x_{m-1} - x_i)^{-1} \int_{x_m - x_{p^*}}^{x_{m-1} - x_{p^*-1}} u^{p^*+r-m}\varphi(u)du$$

$$\leq 2^{p^*} \prod_{i=0}^{p^*-1} (x_m - x_i)^{-1} (x_m - x_{p^*-1}) \int_{x_m - x_{p^*}}^{x_m - x_{p^*-1}} u^{p^*+r-m-1} \varphi(u) du$$

$$= 2^{p^*} (x_m - x_{p^*-1}) \Lambda_{p^*,m,r}(x_0, \dots, x_m; \varphi).$$

Since $m - p^* + 2 \leq m$, we can apply Lemma 3.2 to get (3.7).

Case IV: $(p^*, q^*) = (1, m-1)$ and $x_m - x_1 \leq x_{m-1} - x_0$.

In this case, we have

$$\Lambda^* = \frac{1}{x_{m-1} - x_0} \int_{x_{m-1}-x_1}^{x_{m-1}-x_0} u^{1-k} \varphi(u) du$$

$$\leq \frac{1}{x_{m-1} - x_0} \int_{x_{m-1}-x_1}^{x_{m-1}-x_0} \left(\int_u^d v^{-k} \omega(v) dv \right) du$$

$$\leq \int_{x_{m-1}-x_0}^{d} u^{-k} \omega(u)\, du + \frac{1}{x_{m-1} - x_0} \int_{x_{m-1}-x_1}^{x_{m-1}-x_0} u^{1-k} \omega(u) du =: \mathcal{A}_1 + \mathcal{A}_2.$$

Thus,

$$\mathcal{A}_1 = \int_{x_{m-1}-x_0}^{d/2} u^{-k} \omega(u) du + \int_{d/2}^{d} u^{-k} \omega(u) du$$

$$\leq \int_{x_{m-1}-x_0}^{d/2} u^{-k} \omega(u) du + d \int_{d/2}^{d} u^{-k-1} \omega(u) du$$

$$= (x_m - x_0) \left(\Lambda_{0,m-1,r}(x_0, \dots, x_m; \omega) + 2\Lambda_{0,m,r}(x_0, \dots, x_m; \omega) \right)$$

$$\leq 3(x_m - x_0) \Lambda_r(x_0, \dots, x_m; \omega)$$

and

$$\mathcal{A}_2 = \frac{1}{x_{m-1} - x_0} \int_{x_{m-1}-x_1}^{x_m - x_1} u^{1-k} \omega(u) du + \frac{1}{x_{m-1} - x_0} \int_{x_m - x_1}^{x_{m-1}-x_0} u^{1-k} \omega(u) du$$

$$\leq (x_m - x_1) \Lambda_{1,m-1,r}(x_0, \dots, x_m; \omega) + \int_{x_m - x_1}^{x_{m-1}-x_0} u^{-k} \omega(u) du$$

$$\leq (x_m - x_0)\Lambda_{1,m-1,r}(x_0,\dots,x_m;\omega) + \int\limits_{x_m-x_1}^{x_m-x_0} u^{-k}\omega(u)du$$

$$= (x_m - x_0)\left(\Lambda_{1,m-1,r}(x_0,\dots,x_m;\omega) + \Lambda_{1,m,r}(x_0,\dots,x_m;\omega)\right)$$

$$\leq 2(x_m - x_0)\Lambda_r(x_0,\dots,x_m;\omega).$$

Lemma 3.3 is proved.

4. Proof of Theorem 2.1

We use induction on $k = m - r$. The base case $k = 1$ is addressed in Lemma 3.1. Suppose that $k \geq 2$ is given. We assume that Theorem 2.1 holds for $k-1$ and prove it for k.

Denote by P_{k-1} the polynomial of the best uniform approximation of $f^{(r)}$ on $[x_0, x_m]$ of degree at most $k-1$. Let g be such that

$$g^{(r)} := f^{(r)} - P_{k-1}.$$

Then

$$\omega_k\big(g^{(r)}, t; [x_0, x_m]\big) = \omega_k\big(f^{(r)}, t; [x_0, x_m]\big) =: \omega_k^f(t)$$

and, in view of the Whitney inequality, we find

$$\big\|g^{(r)}\big\|_{[x_0,x_m]} \leq c\omega_k\left(f^{(r)}, x_m - x_0; [x_0, x_m]\right) = c\omega_k^f(x_m - x_0). \tag{4.1}$$

We now use the well-known Marchaud inequality formulated as follows:

If $F \in C[a,b]$ and $1 \leq \ell < k$, then, for all $0 < t \leq b - a$,

$$\omega_\ell(F, t; [a,b]) \leq c(k)t^\ell \left(\int\limits_t^{b-a} \frac{\omega_k(F, u; [a,b])}{u^{\ell+1}}\, du + \frac{\|F\|_{[a,b]}}{(b-a)^\ell} \right).$$

This inequality implies that, for $0 < t \leq x_m - x_0$,

$$\omega_{k-1}^g(t) := \omega_{k-1}\left(g^{(r)}, t; [x_0, x_m]\right)$$

$$\leq ct^{k-1} \left(\int\limits_t^{x_m-x_0} \frac{\omega_k^f(u)}{u^k}\, du + \frac{\omega_k^f(x_m - x_0)}{(x_m - x_0)^{k-1}} \right)$$

$$\leq ct^{k-1} \int\limits_t^{2(x_m-x_0)} \frac{\omega_k^f(u)}{u^k}\, du. \tag{4.2}$$

We also note that (4.1) implies, in particular, that, for all $t \in [x_m - x_0, 2(x_m - x_0)]$,

$$\omega^g_{k-1}(t) \le c\|g^{(r)}\|_{[x_0,x_m]} \le c\omega^f_k(x_m - x_0) \le c\omega^f_k(t). \tag{4.3}$$

We now represent the divided difference in the form

$$\begin{aligned}(x_m - x_0)[x_0, \dots, x_m; f] &= (x_m - x_0)[x_0, \dots, x_m; g] \\ &= [x_1, \dots, x_m; g] - [x_0, \dots, x_{m-1}; g] \\ &= [y_0, \dots, y_{m-1}; g] - [x_0, \dots, x_{m-1}; g],\end{aligned}$$

where $y_j := x_{j+1},\ 0 \le j \le m-1$. By the induction hypothesis,

$$\big|[x_0, \dots, x_{m-1}; g]\big| \le c\Lambda_r(x_0, \dots, x_{m-1}; \omega^g_{k-1})$$

and

$$\big|[y_0, \dots, y_{m-1}; g]\big| \le c\Lambda_r\big(y_0, \dots, y_{m-1}; \omega^g_{k-1}\big).$$

Further, in view of (4.2), (4.3), and the homogeneity of $\Lambda_r(z_0, \dots, z_m; \psi)$ with respect to ψ, Lemma 3.3 with $\varphi := \omega^g_{k-1}$ and $\omega := K\omega^f_k$, where K is the maximum of constants c in relations (4.2) and (4.3), implies that

$$\Lambda_r\big(x_0, \dots, x_{m-1}; \omega^g_{k-1}\big) \le c(x_m - x_0)\Lambda_r\Big(x_0, \dots, x_m; \omega^f_k\Big)$$

and

$$\Lambda_r\big(y_0, \dots, y_{m-1}; \omega^g_{k-1}\big) = \Lambda_r\big(x_1, \dots, x_m; \omega^g_{k-1}\big) \le c(x_m - x_0)\Lambda_r\Big(x_0, \dots, x_m; \omega^f_k\Big),$$

which yields (2.7).

Theorem 2.1 is proved.

5. Applications

Throughout this section, the set $X = \{x_j\}_{j=0}^{m-1}$ is assumed to be such that $x_0 \le x_1 \le \ldots \le x_{m-1}$ (unless otherwise specified). We denote

$$I := [x_0, x_{m-1}] \qquad \text{and} \qquad |I| = x_{m-1} - x_0.$$

Moreover, all constants written in the form $C(\mu_1, \mu_2, \ldots)$ may depend only on the parameters $\mu_1, \mu_2, \ldots$ and not on anything else.

We first recall that the classical Whitney interpolation inequality can be written in the following form:

Theorem 5.1 (Whitney inequality, [25]). *Let $r \in \mathbb{N}_0$ and $m \in \mathbb{N}$ be such that $m \ge \max\{r+1, 2\}$. Suppose that a set $X = \{x_j\}_{j=0}^{m-1}$ is such that*

$$x_{j+1} - x_j \ge \lambda|I| \quad \textit{for all} \quad 0 \le j \le m-2, \tag{5.1}$$

where $0 < \lambda \le 1$. *If* $f \in C^{(r)}(I)$, *then*

$$|f(x) - L_{m-1}(x; f; x_0, \dots, x_{m-1})| \le C(m, \lambda)|I|^r \omega_{m-r}(f^{(r)}, |I|, I), \quad x \in I,$$

where $L_{m-1}(\cdot; f; x_0, \dots, x_{m-1})$ *is the (Lagrange) polynomial of degree* $\le m - 1$ *interpolating* f *at the points of* X.

We emphasize that condition (5.1) implies that the points of the set X in this theorem are assumed to be sufficiently well separated from each other. It is natural to ask what happens if condition (5.1) is not satisfied and, moreover, if some of the points in X are allowed to coalesce. In this, case, $L_{m-1}(\cdot\,; f; x_0, \dots, x_{m-1})$ is the Hermite polynomial whose derivatives interpolate the corresponding derivatives of f at points with multiplicities greater than 1, and Theorem 5.1 does not give any information on the error of approximation of f.

It turns out that it is possible to use Theorem 2.1 to answer this question and significantly strengthen Theorem 5.1. As far as we know, the formulation of the following theorem (which is, in fact, a corollary of a more general Theorem 5.3 presented below) is new and did not appear anywhere in the literature.

Theorem 5.2. *Let* $r \in \mathbb{N}_0$ *and* $m \in \mathbb{N}$ *be such that* $m \ge r + 2$. *Suppose that a set* $X = \{x_j\}_{j=0}^{m-1}$ *is such that*

$$x_{j+r+1} - x_j \ge \lambda|I|, \quad \textit{for all } \ 0 \le j \le m - r - 2, \tag{5.2}$$

where $0 < \lambda \le 1$. *If* $f \in C^{(r)}(I)$, *then*

$$\bigl|f(x) - L_{m-1}(x; f; x_0, \dots, x_{m-1})\bigr| \le C(m, \lambda)|I|^r \omega_{m-r}(f^{(r)}, |I|, I), \quad x \in I,$$

where $L_{m-1}(\cdot; f; x_0, \dots, x_{m-1})$ *is the Hermite polynomial defined in (2.2) and (2.3).*

Theorem 5.2 is an immediate corollary of the next more general theorem. Prior to formulating this theorem, it is necessary to introduce the following notation: Given $X = \{x_j\}_{j=0}^{m-1}$ with $x_0 \le x_1 \le \dots \le x_{m-1}$ and $x \in [x_0, x_{m-1}]$, we renumber the points x_j so that the distance from these points to x becomes nondecreasing. In other words, we use a permutation $\sigma = (\sigma_0, \dots, \sigma_{m-1})$ of the points $(0, \dots, m - 1)$ such that

$$|x - x_{\sigma_{\nu-1}}| \le |x - x_{\sigma_\nu}|, \quad \text{for all } \ 1 \le \nu \le m - 1. \tag{5.3}$$

Note that this permutation σ depends on x and is not unique if there are at least two points from X located at the same distance from x. We also denote

$$\mathcal{D}_r(x, X) := \prod_{\nu=0}^{r} |x - x_{\sigma_\nu}|, \quad 0 \le r \le m - 1. \tag{5.4}$$

Theorem 5.3. *Let* $r \in \mathbb{N}_0$ *and* $m \in \mathbb{N}$ *be such that* $m \ge r + 2$. *Suppose that a set* $X = \{x_j\}_{j=0}^{m-1}$ *is such that*

$$x_{j+r+1} - x_j \ge \lambda|I|, \quad \textit{for all } \ 0 \le j \le m - r - 2, \tag{5.5}$$

where $0 < \lambda \le 1$. *If* $f \in C^{(r)}(I)$, *then, for each* $x \in I$,

$$\begin{aligned}&\left|f(x) - L_{m-1}(x; f; x_0, \dots, x_{m-1})\right| \\ &\qquad \le C(m, \lambda)\mathcal{D}_r(x, X) \int\limits_{|x-x_{\sigma_r}|}^{2|I|} \frac{\omega_{m-r}(f^{(r)}, t, I)}{t^2}\, dt,\end{aligned} \tag{5.6}$$

where $\mathcal{D}_r(x, X)$ is defined in (5.4) and $L_{m-1}(\cdot; f; x_0, \dots, x_{m-1})$ is the Hermite polynomial defined in (2.2) and (2.3).

Prior to proving Theorem 5.3 we formulate its corollary. First, if $k \in \mathbb{N}$ and $\mathrm{w}(t) := \omega_k(f^{(r)}, t; I)$, then

$$t_2^{-k}\mathrm{w}(t_2) \le 2^k t_1^{-k}\mathrm{w}(t_1) \quad \text{for} \quad 0 < t_1 < t_2.$$

Hence, if we denote

$$\lambda_x := |I| \sqrt[k]{|x - x_{\sigma_r}|/|I|}$$

and note that $|x - x_{\sigma_r}| \le \lambda_x \le |I|$, then, for $k \ge 2$, we get

$$\begin{aligned}\int\limits_{|x-x_{\sigma_r}|}^{2|I|} \frac{\mathrm{w}(t)}{t^2}\, dt &= \left(\int\limits_{|x-x_{\sigma_r}|}^{\lambda_x} + \int\limits_{\lambda_x}^{2|I|} \right) \frac{\mathrm{w}(t)}{t^2}\, dt \\ &\le \mathrm{w}(\lambda_x) \int\limits_{|x-x_{\sigma_r}|}^{\infty} t^{-2}\, dt + 2^k \lambda_x^{-k} \mathrm{w}(\lambda_x) \int\limits_0^{2|I|} t^{k-2}\, dt = \frac{\mathrm{w}(\lambda_x)}{|x - x_{\sigma_r}|} \left(1 + \frac{2^{2k-1}}{k-1}\right).\end{aligned}$$

Therefore, we immediately get the following consequence of Theorem 5.3:

Corollary 5.1. *Let $r \in \mathbb{N}_0$ and let $m \in \mathbb{N}$ be such that $m \ge r + 2$. Suppose that a set $X = \{x_j\}_{j=0}^{m-1}$ is such that condition (5.5) is satisfied.*

If $f \in C^{(r)}(I)$, then, for each $x \in I$,

$$\begin{aligned}\left|f(x) - L_{m-1}(x; f; x_0, \dots, x_{m-1})\right| &\le C(m, \lambda)\mathcal{D}_{r-1}(x, X)\omega_{m-r}(f^{(r)}, \lambda_x, I) \\ &\le C(m, \lambda)\mathcal{D}_{r-1}(x, X)\omega_{m-r}(f^{(r)}, |I|, I),\end{aligned} \tag{5.7}$$

where

$$\lambda_x := |I|\big(|x - x_{\sigma_r}|/|I|\big)^{1/(m-r)}.$$

We are now ready to prove Theorem 5.3.

Proof of Theorem 5.3. Note that all constants C encountered in what follows may depend only on m and λ and are different even if they appear in the same line. Clearly, we can assume that x is different from all x_j.

Hence, we suppose that $1 \le i \le m-1$ and $x \in (x_{i-1}, x_i)$ are fixed and denote

$$y_j := \begin{cases} x_j & \text{for} \quad 0 \le j \le i-1, \\ x & \text{for} \quad j = i, \\ x_{j-1} & \text{for} \quad i+1 \le j \le m, \end{cases}$$

$$Y := \{y_j\}_{j=0}^m, \qquad d(Y) := 2(y_m - y_0) = 2(x_{m-1} - x_0) = 2|I|,$$

$$k := m - r, \qquad \text{and} \qquad \omega_k(t) := \omega_k\left(f^{(r)}, t, [y_0, y_m]\right) = \omega_k\left(f^{(r)}, t, I\right).$$

Condition (5.5) implies that

$$y_j < y_{j+r+1} \quad \text{for all} \quad 0 \le j \le m-r-1$$

and, hence, we can use Theorem 2.1 to estimate $\left|[y_0, \dots, y_m; f]\right|$. Thus, identity (2.4) with $j_* := i$, which yields $y_{j_*} = x$, implies that

$$\begin{aligned} \left|f(x) - L_{m-1}(x; f; x_0, \dots, x_{m-1})\right| &= \left|f(x) - L_{m-1}(x; f; y_0, \dots, y_{i-1}, y_{i+1}, \dots, y_m)\right| \\ &= \left|[y_0, \dots, y_m; f]\right| \prod_{j=0, j\ne i}^{m} |x - y_j| \\ &\le c\Lambda_r(y_0, \dots, y_m; \omega_k) \prod_{j=0}^{m-1} |x - x_j| \\ &\le c\mathcal{D}_r(x, X)|I|^{k-1}\Lambda_r(y_0, \dots, y_m; \omega_k). \end{aligned} \tag{5.8}$$

We also note that it is possible to show that

$$\prod_{j=0}^{m-1} |x - x_j| \ge (\lambda/2)^{k-1}\mathcal{D}_r(x, X)|I|^{k-1},$$

and, hence, the estimate presented above cannot be improved.

In order to estimate Λ_r, we suppose that $(p, q) \in \mathcal{Q}_{m,r}$ and estimate $\Lambda_{p,q,r}$. Since $q - p \ge r + 1$, we obtain

$$y_q - y_i \ge y_q - y_{p-1} \ge y_{p+r+1} - y_{p-1} \ge \lambda|I| \quad \text{for} \quad 0 \le i \le p-1,$$

and

$$y_i - y_p \ge y_{q+1} - y_p \ge y_{p+r+2} - y_p \ge \lambda|I| \quad \text{for} \quad q+1 \le i \le m.$$

Hence,

$$\Lambda_{p,q,r}(y_0, \dots, y_m; \omega_k) \le C|I|^{q-m-p} \int_{y_q - y_p}^{2|I|} u^{p+r-q-1}\omega_k(u)du. \tag{5.9}$$

We consider the following two cases:

Case I: $q \ge p+r+2$, or $q = p+r+1$ and $x \notin [y_p, y_q]$.

It is clear that $y_q - y_p \ge \lambda|I|$. Thus, it follows from (5.9) that

$$\Lambda_{p,q,r}(y_0,\dots,y_m;\omega_k) \le C|I|^{-k}\omega_k(|I|) \le C|I|^{1-k}\int\limits_{|I|}^{2|I|}\frac{\omega_k(u)}{u^2}\,du.$$

Case II: $q = p+r+1$ and $x \in [y_p, y_q]$.

If $x = y_p$, then $p = i,\ q = i+r+1$, and $y_q - y_p = x_{i+r} - x \ge |x - x_{\sigma_r}|$.
If $x = y_q$, then $q = i,\ p = i-r-1$, and $y_q - y_p = x - x_{i-r-1} \ge |x - x_{\sigma_r}|$.
If $x \in (y_p, y_q)$, then $y_q - y_p = x_{p+r} - x_p$. Note that it is impossible that

$$|x - x_{\sigma_r}| > \max\{x - x_p, x_{p+r} - x\},$$

because this would imply that $\{p,\dots,p+r\} \subset \{\sigma_0,\dots,\sigma_{r-1}\}$, which cannot happen because these sets have the cardinalities $r+1$ and r, respectively. Hence, we conclude that

$$|x - x_{\sigma_r}| \le \max\{x - x_p, x_{p+r} - x\} \le x_{p+r} - x_p.$$

Therefore, in this case, inequality (5.9) implies that

$$\Lambda_{p,q,r}(y_0,\dots,y_m;\omega_k) \le C|I|^{1-k}\int\limits_{|x-x_{\sigma_r}|}^{2|I|}\frac{\omega_k(u)}{u^2}du.$$

Hence,

$$\Lambda_r(y_0,\dots,y_m;\omega_k) \le C|I|^{1-k}\int\limits_{|x-x_{\sigma_r}|}^{2|I|}\frac{\omega_k(u)}{u^2}du.$$

Together with (5.8), this implies (5.6).
Theorem 5.3 is proved.

We now formulate one more corollary to illustrate the power of Theorem 5.3.
Suppose that $Z = \{z_j\}_{j=0}^{\mu}$ with $z_0 < z_1 < \ldots < z_\mu$, and that $X = \{x_j\}_{j=0}^{m-1}$ with $m := (r+1)(\mu+1)$ is such that $x_{\nu(r+1)+j} := z_\nu$, for all $0 \le \nu \le \mu$ and $0 \le j \le r$. In other words,

$$X = \left\{\underbrace{z_0,\dots,z_0}_{r+1},\underbrace{z_1,\dots,z_1}_{r+1},\dots,\underbrace{z_\mu,\dots,z_\mu}_{r+1}\right\}.$$

Further, given $f \in C^{(r)}[z_0, z_\mu]$, let

$$\mathcal{L}(x; f; Z) := L_{m-1}(x, f; x_0, \ldots, x_{m-1})$$

be the Hermite polynomial of degree $\le m-1 = r\mu + \mu + r$ such that

$$\mathcal{L}^{(j)}(z_\nu; f; Z) = f^{(j)}(z_\nu), \quad \text{for all} \quad 0 \le \nu \le \mu \quad \text{and} \quad 0 \le j \le r. \tag{5.10}$$

Moreover,

$$\operatorname{dist}(x, Z) := \min_{0 \le j \le \mu} |x - z_j|, \quad x \in \mathbb{R}.$$

Corollary 5.2. *Let $r \in \mathbb{N}_0$ and $\mu \in \mathbb{N}$. Suppose that a set $Z = \{z_j\}_{j=0}^{\mu}$ is such that*

$$z_{j+1} - z_j \ge \lambda |I|, \quad \textit{for all } \ 0 \le j \le \mu - 1,$$

where $0 < \lambda \le 1,\ I := [z_0, z_\mu]$, and $|I| := z_\mu - z_0$. If $f \in C^{(r)}(I)$, then, for each $x \in I$,

$$\begin{aligned} \left| f(x) - \mathcal{L}(x; f; Z) \right| &\le C \left(\operatorname{dist}(x, Z)\right)^{r+1} \int\limits_{\operatorname{dist}(x,Z)}^{2|I|} \frac{\omega_{m-r}(f^{(r)}, t, I)}{t^2} dt \\ &\le C \left(\operatorname{dist}(x, Z)\right)^{r} \omega_{m-r}\left(f^{(r)}, |I| \left(\operatorname{dist}(x, Z)/|I|\right)^{1/(m-r)}, I\right) \\ &\le C \left(\operatorname{dist}(x, Z)\right)^{r} \omega_{m-r}(f^{(r)}, |I|, I), \end{aligned}$$

where $m := (r+1)(\mu+1),\ C = C(m, \lambda)$ and the polynomial $\mathcal{L}(\cdot; f; Z)$ of degree $\le m-1$ satisfies (5.10).

As a final remark, we note that some results published in the literature follow from the results discussed in the presented paper. Thus,

(i) the main theorem in [12] immediately follows from Corollary 5.2 with $\mu = 1,\ z_0 = -1$, and $z_1 = 1$,

(ii) Corollary 5.1 is much stronger than the main theorem in [13],

(iii) a special case in Lemmas 8 and 9 from [15] for $k = 0$ follows from Corollary 5.1,

(iv) several propositions in the unconstrained case considered in [11] follow from Corollary 5.1,

(v) Lemma 3.3 and Corollaries 3.4–3.6 in [17] follow from Corollary 5.1

and

(vi) the proof of Lemma 3.1 in [16] can be simplified by using Corollary 5.1.

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