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ScienceDirect
Journal of Approximation Theory 162 (2010) 2168–2183
Journal of Approximation Theory
www.elsevier.com/locate/jat

Three-monotone spline approximation

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Received 20 January 2010; accepted 8 July 2010
Available online 14 July 2010

Communicated by Dany Leviatan

Abstract

For $r \geq 3$, $n \in \mathbb{N}$ and each 3-monotone continuous function f on $[a, b]$ (*i.e.*, f is such that its third divided differences $[x_0, x_1, x_2, x_3]f$ are nonnegative for all choices of distinct points $x_0, \ldots, x_3$ in $[a, b]$), we construct a spline s of degree r and of minimal defect (*i.e.*, $s \in \mathbb{C}^{r-1}[a, b]$) with $n - 1$ equidistant knots in (a, b), which is also 3-monotone and satisfies

$$\|f - s\|_{\mathbb{L}_\infty[a,b]} \leq c\omega_4(f, n^{-1}, [a, b])_\infty,$$

where $\omega_4(f, t, [a, b])_\infty$ is the (usual) fourth modulus of smoothness of f in the uniform norm. This answers in the affirmative the question raised in [8, Remark 3], which was the only remaining unproved Jackson-type estimate for uniform 3-monotone approximation by piecewise polynomial functions (ppfs) with uniformly spaced fixed knots.

Moreover, we also prove a similar estimate in terms of the Ditzian–Totik fourth modulus of smoothness for splines with Chebyshev knots, and show that these estimates are no longer valid in the case of 3-monotone spline approximation in the $\mathbb{L}_p$ norm with $p < \infty$. At the same time, positive results in the $\mathbb{L}_p$ case with $p < \infty$ are still valid if one allows the knots of the approximating ppf to depend on f while still being controlled.

These results confirm that 3-monotone approximation is the transition case between monotone and convex approximation (where most of the results are "positive") and k-monotone approximation with $k \geq 4$ (where just about everything is "negative").

Keywords: 3-monotone approximation by piecewise polynomials and splines; Degree of approximation; Jackson-type estimates

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doi:10.1016/j.jat.2010.07.004

1. Introduction and main results

Let $\mathbb{S}_r(\mathbf{z}_n)$ be the (linear) space of all piecewise polynomial functions (ppfs) of degree r (order $r+1$) with the knots $\mathbf{z}_n := (z_i)_{i=0}^n$, $z_0 < z_1 < \cdots < z_{n-1} < z_n$, *i.e.*, for each $0 \le i \le n-1$, $s|_{(z_i, z_{i+1})} \in \Pi_r$, where Π_r denotes the space of algebraic polynomials of degree $\le r$. Also, let $\widetilde{\mathbb{S}}_r(\mathbf{z}_n) := \mathbb{S}_r(\mathbf{z}_n) \cap \mathbb{C}^{r-1}$ be the corresponding space of splines of minimal defect (highest smoothness). Additionally, $\mathbb{S}_{N,r}[a,b]$ is the (nonlinear) space of free knot ppfs of degree r with at most N pieces in $[a,b]$ ($N-1$ knots in (a,b)). (Clearly, for any $\mathbf{z}_n := (z_i)_{i=0}^n$, $\mathbb{S}_r(\mathbf{z}_n) \subseteq \mathbb{S}_{n,r}[z_0, z_n]$.)

Throughout this paper, "$\mathbf{z}_n$ is a partition of $[a,b]$" always means that $\mathbf{z}_n$ is an ordered set $(z_i)_{i=0}^n$, $a =: z_0 < z_1 < \cdots < z_{n-1} < z_n := b$, and additionally we set $z_{-i} := z_0$ and $z_{n+i} := z_n$ for $i \in \mathbb{N}$ (a similar convention is used for partitions $\mathbf{x}_n$, $\mathbf{y}_n$, etc.). In particular, we denote by $\mathbf{u}_n$ and $\mathbf{t}_n$ the uniform and Chebyshev partitions of $[-1,1]$, respectively, *i.e.*, $\mathbf{u}_n := (-1+2i/n)_{i=0}^n$ and $\mathbf{t}_n := (\cos((n-i)\pi/n))_{i=0}^n$.

Now, with $J_j := [z_j, z_{j+1}]$ let

$$\eta(\mathbf{z}_n) := \max_{0 \le j \le n-1} |J_{j\pm1}|/|J_j| \tag{1}$$

be the scale of the partition $\mathbf{z}_n$. Also, let

$$\mu(\mathbf{z}_n) = \max_{0 \le i < j \le n} \frac{(j-i)(z_{i+1}-z_i)}{z_j - z_i} \tag{2}$$

and

$$\vartheta(\mathbf{z}_n) = \max_{\substack{0 \le i < j \le n;\\ \max\{3i-2j,0\} \le k \le \min\{3j-2i,n\}-1}} \frac{(j-i)(z_{k+1}-z_k)}{z_j - z_i}. \tag{3}$$

Clearly, $\mu(\mathbf{z}_n) \le \vartheta(\mathbf{z}_n)$ (consider $k=i$ in (3)), and $1 \le \eta(\mathbf{z}_n) \le \vartheta(\mathbf{z}_n)$ (consider $j=i+1$ in (3)). It is obvious that $\eta(\mathbf{u}_n) = \mu(\mathbf{u}_n) = \vartheta(\mathbf{u}_n) = 1$, and it is not difficult to show that $\eta(\mathbf{t}_n) \le 3$, $\mu(\mathbf{t}_n) \le 2$, and $\vartheta(\mathbf{t}_n) \le 6$.

As usual, $\omega_m(f,t,J)_p$ is the mth modulus of smoothness of $f \in \mathbb{L}_p(J)$ on an interval J, and $\omega_m(f,J)_p := \omega_m(f,|J|,J)_p$.

Given $k \in \mathbb{N}$ and an open interval $I = (a,b)$, let $\Delta^k(I)$ (or $\Delta^k(a,b)$ with a slight abuse of the notation) be the class of all k-monotone functions on $I = (a,b)$, *i.e.*, all functions $f : I \mapsto \mathbb{R}$ such that their kth divided differences $[x_0,\ldots,x_k]f$ are nonnegative for all choices of $(k+1)$ distinct points $x_0,\ldots,x_k$ in I. Recall that, if $f \in \mathbb{C}^k(I)$, then $f \in \Delta^k(I)$ if and only if $f^{(k)} \ge 0$ on I. Functions from $\Delta^k(a,b)$ are not assumed to be defined at the endpoints of the interval (a,b), and, hence, have to be neither bounded nor integrable on (a,b). For example, if $f(x) = (-1)^k x^{-1-1/p}$, then $f \in \Delta^k(0,1)$ for $k \in \mathbb{N}$, but $f \notin \mathbb{L}_p[0,1]$, $0 < p \le \infty$.

It is well known (see [9,11]) that, if $k \ge 2$, then $f \in \Delta^k(a,b)$ iff $f^{(k-2)}$ exists and is convex on (a,b). Therefore, $f^{(k-2)}$ satisfies a Lipschitz condition on any closed subinterval of (a,b), is absolutely continuous there, and has left and right nondecreasing derivatives $f_-^{(k-1)}$ and $f_+^{(k-1)}$ everywhere on (a,b). Moreover, the set E where $f^{(k-1)}$ fails to exist is countable, and $f^{(k-1)}$ is continuous on $(a,b) \setminus E$.

Throughout this paper, $c(\gamma_1,\gamma_2,\ldots)$ denote positive constants which depend only on the parameters $\gamma_1,\gamma_2,\ldots$ (note that $c(p,\ldots)$ depends on p only as $p \to 0$) and which may be different for different occurrences (even if they appear in the same line). At the same time, $\mathfrak{c}_i(\gamma_1,\gamma_2,\ldots)$, $i \in \mathbb{N}$, denote positive constants which are fixed throughout the paper. If the interval $[a,b]$ is

$[-1, 1]$, it will be dropped from the notation. For example, $\mathbb{C}^m := \mathbb{C}^m[-1, 1]$, $\mathbb{L}_p := \mathbb{L}_p[-1, 1]$, $\mathbb{S}_{N,r} := \mathbb{S}_{N,r}[-1, 1]$, etc. Also, whenever we write $\mathbb{L}_\infty$ we mean $\mathbb{C}$. Furthermore, we denote $\Delta^k := \Delta^k(-1, 1)$ and, for readers' convenience, emphasize one more time that $\Delta^k(-1, 1)$ is different from $\Delta^k[-1, 1]$.

For a function $f \in \Delta^k$, it is natural to require that the objects used to approximate it also belong to Δ^k, *i.e.*, the shape of the function is preserved. Problems of monotone ($k = 1$) and convex ($k = 2$) approximation by ppfs with fixed knots and polynomials have been extensively investigated with a "good" pattern of the results, *i.e.*, in many situations it is possible to obtain the same order of approximation as in the unconstrained case. Surprisingly, for k-monotone approximation with $k \geq 4$, the order of approximation is much worse, as was first shown by Konovalov and Leviatan [3] in the context of shape-preserving widths.

Studies of 3-monotone approximation by ppfs with fixed knots in the uniform norm in [8] indicated that this case also somewhat fits a pattern of a "good" one, but the proofs turned out to be more complicated. The question of validity of Jackson-type estimates for approximation by ppfs with uniformly spaced fixed knots has been answered in all but one case as discussed in [8, Remark 3]. Namely, it was unknown whether, for any $f \in \Delta^3 \cap \mathbb{C}$, it is possible to construct a cubic ppf $s \in \Delta^3$ with $n - 1$ equidistant knots such that

$$\|f - s\|_{\mathbb{L}_\infty} \leq c\omega_4(f, n^{-1}, [-1, 1])_\infty. \tag{4}$$

In this paper, we answer this question in the affirmative. This turned out to be the most difficult case of Jackson-type estimates for k-monotone approximation by ppfs with fixed knots and required an application of some very recent results on shape-preserving spline smoothing [6]. Note that a weaker estimate with the third modulus of smoothness of the derivative has been established in [8], where one can also find a detailed discussion on Jackson-type estimates involving the derivatives of the function. We believe that the difficulty with (4) is that it is a "boundary" case between the "good" cases and the "bad" ones.

The first step in establishing (4) is the following result. For 3-monotone approximation of $f \in \Delta^3$ in the $\mathbb{L}_p$ (quasi-) norm by cubic splines, we can achieve the best possible order of approximation (see (6)), but the location of the knots may depend on f. At the same time, we can still guarantee that the knots are not too close to each other (see (5)), which makes this result different from a constrained free-knot spline approximation (see [7] or [10]).

Theorem 1.1. *For every $\boldsymbol{\eta} \geq 1$, there exists a constant $\mathfrak{c}_1(\boldsymbol{\eta}) > 0$ so that the following statement is valid. Let $f \in \Delta^3 \cap \mathbb{L}_p$, $0 < p \leq \infty$, and let $\mathbf{x}_n$ be a partition of $[-1, 1]$ such that $\eta(\mathbf{x}_n) \leq \boldsymbol{\eta}$.*

Then there exist a partition $\mathbf{y}_m$ of $[-1, 1]$, $m \leq 20n$, and a cubic ppf $s \in \mathbb{S}_3(\mathbf{y}_m) \cap \Delta^3$ such that, for each $0 \leq k \leq m - 1$, there exists $1 \leq j \leq n - 1$ such that $[y_k, y_{k+1}] \subseteq [x_{j-1}, x_{j+1}]$ and

$$y_{k+1} - y_k \geq \mathfrak{c}_1(\boldsymbol{\eta})(x_{j+1} - x_{j-1}). \tag{5}$$

Also, for each $0 \leq j \leq n - 1$,

$$\|f - s\|_{\mathbb{L}_p[x_j, x_{j+1}]} \leq c(\boldsymbol{\eta}, p)\omega_4(f, [x_{j-1}, x_{j+2}])_p. \tag{6}$$

For 3-monotone approximation in $\mathbb{L}_p$, $p < \infty$, by a ppf with *fixed knots* we cannot even get an analog of estimate (6) with ω_3 instead of ω_4 due to [3] (see also [2, Remark 5]). At the same time, for $p = \infty$, we can move the knots to the right place and make them independent of the function using recent results on shape-preserving smoothing [6]. However, in order to be able to apply [6] we need to guarantee one additional degree of smoothness, *i.e.*, we need to ensure that our ppf is in $\mathbb{C}^2$, and this can be achieved for $p = \infty$ by [8, Theorem 5]. We would like to remark that it is impossible to gain this extra degree of smoothness if approximation takes place in $\mathbb{L}_p$ with $p < \infty$ as, otherwise, one could follow the proof of Theorem 1.2 and obtain a Jackson-type estimate in $\mathbb{L}_p$ with ω_4, which is invalid.

In the case $p = \infty$, we have the following theorem.

Theorem 1.2. *Let $\boldsymbol{\vartheta} \ge 1$ and $r \ge 3$. For any $f \in \Delta^3 \cap \mathbb{C}$ and every partition $\mathbf{x}_n$ of $[-1, 1]$ such that $\vartheta(\mathbf{x}_n) \le \boldsymbol{\vartheta}$, there exists a spline $s \in \widetilde{\mathbb{S}}_r(\mathbf{x}_n) \cap \Delta^3$ of minimal defect such that*

$$\|f - s\|_{\mathbb{L}_\infty} \le c(r, \boldsymbol{\vartheta}) \max_{1 \le j \le n-1} \omega_4(f, [x_{j-1}, x_{j+1}])_\infty.$$

The next two results are immediate corollaries of Theorem 1.2.

Theorem 1.3. *Let $r \ge 3$ and $n \in \mathbb{N}$. For any $f \in \Delta^3 \cap \mathbb{C}$, there exists a spline $s \in \widetilde{\mathbb{S}}_r(\mathbf{u}_n) \cap \Delta^3$ of minimal defect such that*

$$\|f - s\|_{\mathbb{L}_\infty} \le c(r)\omega_4(f, n^{-1}, [-1, 1])_\infty.$$

Theorem 1.4. *Let $r \ge 3$ and $n \in \mathbb{N}$. For any $f \in \Delta^3 \cap \mathbb{C}$, there exists a spline $s \in \widetilde{\mathbb{S}}_r(\mathbf{t}_n) \cap \Delta^3$ of minimal defect such that*

$$\|f - s\|_{\mathbb{L}_\infty} \le c(r)\omega_4^{\varphi}(f, n^{-1})_\infty,$$

where $\omega_4^{\varphi}(f, n^{-1})_\infty$ is the Ditzian–Totik modulus of smoothness of order 4.

Note that one cannot replace ω_4 with ω_5 in the above estimates (see Theorem 7.1), and recall that these estimates are not valid for 3-monotone approximation in the $\mathbb{L}_p$ norm with $p < \infty$.

At the same time, for k-monotone approximation, $k \ge 4$, the situation is much worse. For instance, one cannot have estimates with $\omega_4(f, n^{-1}, [-1, 1])_p$ (see [3, Remark (iii), p. 241]). Moreover, as a simple corollary of the results from [2], we show in Theorem 7.4 that even ω_3 does not work.

Remark 1.5. It is possible to verify the validity of Theorem 1.3 completely bypassing Theorem 1.2 and only using Theorem 1.1 (and the fact that $\mathbf{y}_m$ is quasi-uniform if $\mathbf{x}_n = \mathbf{u}_n$), [4, Corollary 1.5, Lemma 5.1] and [8, Theorem 6].

2. Special free-knot spline approximation

Recall that $f_+^{(i)}(x)$ and $f_-^{(i)}(x)$ denote the right and left ith derivatives of f at x, respectively. By $\Delta_*^k(a, b)$ we denote the subclass of those functions $f \in \Delta^k(a, b)$ for which the values $\{f_+^{(i)}(a)\}_{i=0}^{k-1}$ and $\{f_-^{(i)}(b)\}_{i=0}^{k-1}$ are finite.

For $f \in \Delta_*^k(a, b)$, we define by $\Delta^k[f](a, b)$ the set of all functions $h \in \Delta_*^k(a, b)$ such that

$$h_+^{(i)}(a) = f_+^{(i)}(a), \qquad h_-^{(i)}(b) = f_-^{(i)}(b), \quad 0 \le i \le k-2,$$
$$h_+^{(k-1)}(a) \ge f_+^{(k-1)}(a), \quad \text{and} \quad h_-^{(k-1)}(b) \le f_-^{(k-1)}(b).$$

Remark 2.1. Suppose that $f \in \Delta_*^k(a, b)$ and $\mathbf{z}_m$ is a partition of $[a, b]$. The fact that $h \in \Delta^k(z_i, z_{i+1})$ (or $h \in \Delta_*^k(z_i, z_{i+1})$) for all $0 \le i \le m-1$, does NOT imply that h is k-monotone on (a, b). At the same time, if h is such that $h \in \Delta^k[f](z_i, z_{i+1})$ for all $0 \le i \le m-1$, then $h \in \Delta^k(a, b)$.

The following lemma shows that, instead of an arbitrary $f \in \Delta^k(a,b) \cap \mathbb{L}_p[a,b]$, we may consider $f \in \Delta^k_*(a,b)$, *i.e.*, we may assume that the function f and its derivatives are bounded at the endpoints of (a,b) and, hence, at all interior points as well.

Lemma 2.2 (*[7, Lemma 4.4]*). *Let $k \in \mathbb{N}$, $0 < p \le \infty$, and $f \in \Delta^k(a,b) \cap \mathbb{L}_p[a,b]$. Then, for any $\varepsilon > 0$, there exists $f_\varepsilon \in \Delta^k_*(a,b)$ such that*

$$\|f - f_\varepsilon\|_{\mathbb{L}_p[a,b]} < \varepsilon.$$

Moreover, f_ε coincides with f everywhere except perhaps near the endpoints of (a,b).

Proposition 2.3 (*[7, Proposition 4.3]*). *Let $k, r \in \mathbb{N}$, $k \ge 2$, $r \ge k-1$, $0 < p \le \infty$, $f \in \Delta^k_*(a,b) \cap \mathbb{L}_p[a,b]$, and let q be such that either $q \in \Pi_r \cap \Delta^k(a,b)$ or $(-q) \in (\Pi_r \setminus \Pi_k) \cap \Delta^k(a,b)$. Then there exists s such that*

$$s \in \mathcal{S}_{c(k),r}[a,b] \cap \Delta^k[f](a,b)$$

and

$$\|f - s\|_{\mathbb{L}_p[a,b]} \le c(p,r,k)\,\|f - q\|_{\mathbb{L}_p[a,b]}.$$

Theorem 2.4. *Let $k, r \in \mathbb{N}$, $k \ge 2$, $r \ge k-1$, $0 < p \le \infty$, $f \in \Delta^k \cap \mathbb{L}_p$, $\mathbf{x}_n$ be a partition of $[-1,1]$, and let σ be any ppf from $\mathcal{S}_r(\mathbf{x}_n)$. Then there exist a constant $\mathfrak{c}_2 = \mathfrak{c}_2(k,r) \in \mathbb{N}$ and a ppf $s \in \mathcal{S}_{\mathfrak{c}_2 n,r} \cap \Delta^k$, such that*

(i) *s has $\le \mathfrak{c}_2$ pieces in each interval $[x_j, x_{j+1}]$, $0 \le j \le n-1$, and*
(ii) *$\|f - s\|_{\mathbb{L}_p[x_j,x_{j+1}]} \le c(k,r,p)\|f - \sigma\|_{\mathbb{L}_p[x_j,x_{j+1}]}$, $0 \le j \le n-1$.*

Proof. In view of Lemma 2.2, we can assume that $f \in \Delta^k_*$ and, hence, $f \in \Delta^k_*(a,b)$, for any $(a,b) \subseteq (-1,1)$. The restriction p_j of σ to each interval $[x_j, x_{j+1}]$, $0 \le j \le n-1$, is a polynomial of degree $\le r$ whose kth derivative is a polynomial of degree $\le r-k$ and, hence, has at most $r-k$ real zeros inside $[x_j, x_{j+1}]$ (or is identically zero there). These zeros partition $[x_j, x_{j+1}]$ into at most $r-k+1$ subintervals $\mathcal{J}^j_1, \dots, \mathcal{J}^j_m$, $1 \le m \le \max\{1, r-k+1\}$, and $p_j^{(k)}$ is either nonnegative or negative in the interior of each $\mathcal{J}^j_i$, $1 \le i \le m$. This implies that either $p_j \in \Pi_r \cap \Delta^k(\mathcal{J}^j_i)$ or $(-p_j) \in (\Pi_r \setminus \Pi_k) \cap \Delta^k(\mathcal{J}^j_i)$, for each $1 \le i \le m$. Proposition 2.3 implies that, for each $1 \le i \le m$, there exists a spline s^j_i such that $s^j_i \in \mathcal{S}_{c(k),r}(\mathcal{J}^j_i) \cap \Delta^k[f](\mathcal{J}^j_i)$ and $\|f - s^j_i\|_{\mathbb{L}_p(\mathcal{J}^j_i)} \le c(p,r,k)\,\|f - p_j\|_{\mathbb{L}_p(\mathcal{J}^j_i)}$.

We now “glue” all pieces s^j_i together, obtaining the spline s defined on $[-1,1]$ and whose restriction to $\mathcal{J}^j_i$ is s^j_i. Using Remark 2.1, it is easy to see that $s \in \mathcal{S}_{\mathfrak{c}_2 n,r} \cap \Delta^k$, $\mathfrak{c}_2 = \mathfrak{c}_2(k,r)$, and (i) and (ii) are satisfied. □

3. Local approximation by splines with controlled knots and the proof of Theorem 1.1

Lemma 3.1. *For any interval I, $k \in \mathbb{N}_0$, $0 < p \le \infty$, and $q \in \Pi_r$,*

$$\left\|q^{(k)}\right\|_{\mathbb{L}_\infty(I)} \le c(r,p)|I|^{-k-1/p}\,\|q\|_{\mathbb{L}_p(I)} \le c|I|^{-k}\,\|q\|_{\mathbb{L}_\infty(I)}.$$

Lemma 3.2. *For any intervals $I \subseteq J$, $0 < p \le \infty$ and $q \in \Pi_r$,*

$$\|q\|_{\mathbb{L}_p(J)} \le c(r,p)(|J|/|I|)^{r+1/p} \|q\|_{\mathbb{L}_p(I)}.$$

Lemma 3.3 ([1, Lemma 3.2]). *Let $r \in \mathbb{N}$, $d := 2r^2$. For any $q_1, q_2 \in \Pi_r$ and knot sequence $\mathbf{x}_d = (x_i)_{i=0}^d$, $x_0 < x_1 < \cdots < x_d$, there exists a spline $s \in \mathbb{S}_r(\mathbf{x}_d)$ such that $s(x)$ is a number between $q_1(x)$ and $q_2(x)$ for all $x \in [x_0, x_d]$, $s \equiv q_1$ on $(-\infty, x_0]$ and $s \equiv q_2$ on $[x_d, \infty)$.*

Lemma 3.4. *Let $y_0 < y_1 < y_2 < y_3$, $h := y_3 - y_0$ and for some $\theta > 0$*

$$y_1 - y_0 \ge \theta h \quad \text{and} \quad y_3 - y_2 \ge \theta h.$$

Suppose that $f \in \Delta^3[y_0, y_3]$ is such that

$$f|_{[y_0,y_1]} =: q_1 \in \Pi_3 \quad \text{and} \quad f|_{[y_2,y_3]} =: q_2 \in \Pi_3$$

(i.e., q_1 and q_2 are cubic polynomials), and so $f \in \mathbb{C}^1[y_0, y_3]$. Then, for every $0 < p \le \infty$, there exists a cubic spline $\mathfrak{z} \in \mathbb{S}_{N,3}[y_0, y_3]$ satisfying

(i) $\mathfrak{z} \in \Delta^3[y_0, y_3]$;
(ii) *$\mathfrak{z}$ has ≤ 19 knots in (y_0, y_3), i.e., $N \le 20$;*
(iii) *the distance between any two knots of $\mathfrak{z}$ is $\ge \mathfrak{c}_3(\theta)h$;*
(iv) *$\mathfrak{z} \equiv f$ in some neighborhoods of y_0 and y_3 (i.e., the left- and right-most pieces of $\mathfrak{z}$ are q_1 and q_2, respectively);*
(v) $\|f - \mathfrak{z}\|_{\mathbb{L}_p[y_0,y_3]} \le c(\theta, p)\omega_4(f, [y_0, y_3])_p$.

Proof. Everywhere in this proof, for simplicity, we set $\omega_4 := \omega_4(f, [y_0, y_3])_p$. If q_* is a cubic polynomial satisfying Whitney's inequality $\|f - q_*\|_{\mathbb{L}_p[y_0,y_3]} \le c\omega_4$, then, using Lemma 3.2 and recalling that $\|f + g\|_p \le 2^{\max\{0,(1-p)/p\}}(\|f\|_p + \|g\|_p)$, we have

$$\begin{aligned}
\|f - q_1\|_{\mathbb{L}_p[y_0,y_3]} &\le c\|f - q_*\|_{\mathbb{L}_p[y_0,y_3]} + c\|q_* - q_1\|_{\mathbb{L}_p[y_0,y_3]} \\
&\le c\omega_4 + c\|q_* - q_1\|_{\mathbb{L}_p[y_0,y_1]} \\
&= c\omega_4 + c\|q_* - f\|_{\mathbb{L}_p[y_0,y_1]} \le c\omega_4.
\end{aligned}$$

The same estimate is clearly also valid for q_2 in place of q_1, and so

$$\left\|f - q_j\right\|_{\mathbb{L}_p[y_0,y_3]} \le c\omega_4, \quad j = 1, 2, \tag{7}$$

and

$$\|q_1 - q_2\|_{\mathbb{L}_p[y_0,y_3]} \le c\omega_4. \tag{8}$$

Denote $a_j := q_j'''$, $j = 1, 2$, and note that $a_j \ge 0$ are constants. We consider two cases depending on how large a_1 is.

Case I. Suppose that

$$a_1 \ge A_1 h^{-3-1/p}\omega_4, \tag{9}$$

where $A_1 = A_1(\theta, p)$ will be chosen shortly. Take $x_0 := (y_0 + y_1)/2$, $x_{18} := (y_2 + y_3)/2$ and $x_i := x_0 + i(x_{18} - x_0)/18$, $1 \le i \le 17$, and apply Lemma 3.3 to obtain a spline $\mathfrak{z}$. Conditions (ii), (iii) and (iv) of the lemma are clearly satisfied for this $\mathfrak{z}$. Taking into account (8) and Lemma 3.3, we obtain

$$\|q_1 - \mathfrak{z}\|_{\mathbb{L}_p[y_0,y_3]} \le \|q_1 - q_2\|_{\mathbb{L}_p[y_0,y_3]} \le c\omega_4. \tag{10}$$

Together with (7), this implies that

$$\|f-\mathfrak{z}\|_{\mathbb{L}_p[y_0,y_3]} \le c\,\|f-q_1\|_{\mathbb{L}_p[y_0,y_3]} + c\,\|q_1-\mathfrak{z}\|_{\mathbb{L}_p[y_0,y_3]} \le c\omega_4,$$

which is (v). It remains to verify (i). Since $\mathfrak{z} \in \mathbb{C}^2$ and $\mathfrak{z}'''$ exists everywhere on $[y_0, y_3]$ except possibly at x_i, $0 \le i \le 18$, it is sufficient to prove that $\mathfrak{z}'''(x) \ge 0$, $x \ne x_i$. For $x \in [y_0, x_0) \cup (x_{18}, y_3]$ this is obvious because of (iv), and for $x \in (x_i, x_{i+1})$, $0 \le i \le 17$, taking into account that $x_{i+1} - x_i \ge \theta h/18$ and using Lemma 3.1 and (10), we have

$$\begin{aligned}|a_1-\mathfrak{z}'''(x)| &\le \left\|(q_1-\mathfrak{z})'''\right\|_{\mathbb{L}_\infty[x_i,x_{i+1}]} \le ch^{-3-1/p}\,\|q_1-\mathfrak{z}\|_{\mathbb{L}_p[x_i,x_{i+1}]}\\ &\le \mathfrak{c}_4 h^{-3-1/p}\omega_4.\end{aligned}$$

If we select $A_1 := \mathfrak{c}_4$, then (9) guarantees that $\mathfrak{z}'''(x) \ge 0$.

Case II. Suppose now that (9) does not hold, *i.e.*, $a_1 < \mathfrak{c}_4 h^{-3-1/p}\omega_4$. Lemma 3.1 and (8) yield

$$|a_1-a_2| = \left\|(q_1-q_2)'''\right\|_{\mathbb{L}_\infty[y_0,y_3]} \le ch^{-3-1/p}\,\|q_1-q_2\|_{\mathbb{L}_p[y_0,y_3]} \le ch^{-3-1/p}\omega_4,$$

and so we have

$$a_j \le ch^{-3-1/p}\omega_4, \quad j=1,2. \tag{11}$$

Take $z_0 := (y_0+y_1)/2$, $z_1 := y_1$, $z_2 := y_2$, $z_3 := (y_2+y_3)/2$, denote

$$l_j(x) := q_1''(z_j)(x-z_j) + q_1'(z_j), \quad j=0,1,$$

and

$$l_j(x) := q_2''(z_j)(x-z_j) + q_2'(z_j), \quad j=2,3$$

and define

$$s_1(x) := \begin{cases} f'(x), & x \notin [z_0,z_3],\\ \max\limits_{j=0,1,2,3} l_j(x), & x \in [z_0,z_3].\end{cases}$$

By convexity of f', s_1 is a convex quadratic ppf satisfying

$$s_1(x) \le f'(x), \quad x \in [y_0,y_3]. \tag{12}$$

Since the tangent lines to any quadratic polynomial at points $x=a$ and $x=b$ intersect at $x=(a+b)/2$, we conclude that the knots of s_1 are z_0, $(z_0+z_1)/2$, $\tilde{z}$, $(z_2+z_3)/2$, z_3, where $\tilde{z} \in [z_1,z_2] = [y_1,y_2]$, and, consequently, they are not closer than $\theta h/4$ to one another.

Now, let

$$s_2(x) := \begin{cases} f'(x), & x \notin [z_0,z_3],\\ l(x), & x \in [z_0,z_3],\end{cases}$$

where l is the linear function interpolating f' at z_0 and z_3. The convexity of f' implies that s_2 is a convex quadratic spline with knots z_0 and z_3 such that

$$f'(x) \le s_2(x), \quad x \in [y_0,y_3]. \tag{13}$$

Inequalities (12) and (13) now guarantee that we can choose $\alpha \in [0,1]$ so that

$$\mathfrak{z}(x) := f(z_0) + \int_{z_0}^{x} (\alpha s_1(t) + (1-\alpha)s_2(t))\mathrm{d}t$$

satisfies $\mathfrak{z}(z_3) = f(z_3)$ and, hence, $\mathfrak{z}$ is a cubic spline satisfying (iv). Clearly, (i)–(iii) are fulfilled as well, and so we only need to verify (v).

Let $\tilde{y} := (y_0 + y_3)/2$; then Lemma 3.1 and (8) imply that

$$|q_1''(\tilde{y}) - q_2''(\tilde{y})| \le \|q_1'' - q_2''\|_{\mathbb{L}_\infty[y_0,y_3]} \le ch^{-2-1/p}\omega_4. \tag{14}$$

Inequality (11) yields

$$q_1''(\tilde{y}) - q_1''(y_0) \le ch^{-2-1/p}\omega_4$$

and

$$q_2''(y_3) - q_2''(\tilde{y}) \le ch^{-2-1/p}\omega_4,$$

which combined with (14) provide (recall that $q_1''(y_0) = f''(y_0)$ and $q_2''(y_3) = f''(y_3)$)

$$f''(y_3) - f''(y_0) \le ch^{-2-1/p}\omega_4. \tag{15}$$

Recall that, if $g \in \Delta^3(I)$, then $g''(x)$ exists for all $x \in I$ with a set of exceptions which is at most countable, and g'' is nondecreasing on its domain of definition. Hence, since $f \in \Delta^3[y_0, y_3]$, we have

$$f''(y_0) \le f''(x) \le f''(y_3) \quad \text{a.e. on } [y_0, y_3].$$

Similarly, since $\mathfrak{z} \in \Delta^3[y_0, y_3]$, (iv) implies that

$$f''(y_0) \le \mathfrak{z}''(x) \le f''(y_3) \quad \text{a.e. on } [y_0, y_3],$$

and so we obtain, by (15),

$$|f''(x) - \mathfrak{z}''(x)| \le ch^{-2-1/p}\omega_4 \quad \text{a.e. on } [y_0, y_3].$$

Integrating twice, we arrive at

$$\|f - \mathfrak{z}\|_{\mathbb{L}_\infty[y_0,y_3]} \le ch^{-1/p}\omega_4,$$

which implies (v). □

Remark 3.5. It may appear that in Case II we have a lot of freedom to construct $\mathfrak{z}$. However, for some f, the required $\mathfrak{z}$ is unique; for example, when $f(x) = (x - y)_+^2$ for a fixed $y \in [y_1, y_2]$, our only choice is $\mathfrak{z} \equiv f$.

Proof of Theorem 1.1. First, let σ be such that $\sigma|_{[x_j,x_{j+1}]}$ is a cubic polynomial of best $\mathbb{L}_p$ unconstrained approximation to f on $[x_j, x_{j+1}]$. Then, by Whitney's inequality, we have

$$\|f - \sigma\|_{\mathbb{L}_p[x_j,x_{j+1}]} \le c\omega_4(f, [x_j, x_{j+1}])_p, \quad 0 \le j \le n-1.$$

Theorem 2.4 implies that there exists $\tilde{s} \in \mathbb{S}_{\mathfrak{c}_2 n,3} \cap \Delta^3$ such that

$$\|f - \tilde{s}\|_{\mathbb{L}_p[x_j,x_{j+1}]} \le c\omega_4(f, [x_j, x_{j+1}])_p, \quad 0 \le j \le n-1,$$

and $\tilde{s}$ has $\le \mathfrak{c}_2 = \mathfrak{c}_2(3,3)$ pieces in each $[x_j, x_{j+1}]$. Hence, for each $0 \le j \le n-1$, we can find an interval $I_j := [a_j, b_j] \subseteq [x_j, x_{j+1}]$ of length $\ge (x_{j+1} - x_j)/\mathfrak{c}_2$ such that $\tilde{s}$ has no knots in I_j. With $t_j := (a_j + b_j)/2$, for every $0 \le j \le n-2$, we apply Lemma 3.4 to $\tilde{s}$ with $y_0 := t_j$, $y_1 := b_j$, $y_2 := a_{j+1}$, $y_3 := t_{j+1}$ and $\theta := 1/(2\boldsymbol{\eta}\mathfrak{c}_2)$ to obtain $\mathfrak{z}_j$ on $[t_j, t_{j+1}]$. We now define s so that $s|_{[t_j,t_{j+1}]} := \mathfrak{z}_j$, $0 \le j \le n-2$, and near the endpoints of $[-1, 1]$, *i.e.*, on the intervals

$[-1, t_0]$ and $[t_{n-1}, 1]$, the required spline s is defined extending the polynomials $\mathfrak{z}|_{[t_0,t_0+\varepsilon]}$ and $\mathfrak{z}|_{[t_{n-1}-\varepsilon,t_{n-1}]}$, where $\varepsilon > 0$ is small enough. (Note that (iv) of Lemma 3.4 implies that these $\mathfrak{z}|_{[t_0,t_0+\varepsilon]}$ and $\mathfrak{z}|_{[t_{n-1}-\varepsilon,t_{n-1}]}$ are the same polynomials as $\tilde{s}|_{[a_0,b_0]}$ and $\tilde{s}|_{[a_{n-1},b_{n-1}]}$, respectively.)

It follows from Lemma 3.4 that $s \in \Delta^3$ (by (i) and (iv)), s is in $\mathbb{S}_{20(n-1)+1,3}$, has at most 40 pieces in each interval $[x_j, x_{j+1}]$ (by (ii)), and the distance between any two knots of s in $[x_j, x_{j+2}]$ is not less than $c(\boldsymbol{\eta})(x_{j+2} - x_j)$ (by (iii)). Now, (v) of Lemma 3.4 implies that

$$\|\tilde{s} - s\|_{\mathbb{L}_p[t_j,t_{j+1}]} \le c\omega_4(\tilde{s}, [t_j, t_{j+1}])_p,$$

and so, for $1 \le j \le n-2$,

$$\begin{aligned}\|f - s\|_{\mathbb{L}_p[x_j,x_{j+1}]} &\le c\,\|f - \tilde{s}\|_{\mathbb{L}_p[x_j,x_{j+1}]} + c\,\|\tilde{s} - s\|_{\mathbb{L}_p[t_{j-1},t_j]} + c\,\|\tilde{s} - s\|_{\mathbb{L}_p[t_j,t_{j+1}]}\\ &\le c\omega_4(f, [x_j, x_{j+1}])_p + c\omega_4(\tilde{s}, [t_{j-1}, t_j])_p + c\omega_4(\tilde{s}, [t_j, t_{j+1}])_p\\ &\le c\omega_4(f, [x_{j-1}, x_{j+2}])_p.\end{aligned}$$

Finally, it remains to prove the above estimate for $j = 1$ and $j = n-1$. We only consider the case $j = 1$ (*i.e.*, approximation on $[-1, x_1]$) since the proof in the case for $j = n-1$ is similar. Let q_* be the cubic polynomial satisfying Whitney's inequality $\|f - q_*\|_{\mathbb{L}_p[-1,x_1]} \le c\omega_4(f, [-1, x_1])_p$, and recall that $\tilde{s}$ has no knots inside $[a_0, b_0]$ of length $\ge c(x_1 + 1)$, and $\tilde{s}|_{[a_0,b_0]} = s|_{[-1,t_0]} =: p_0 \in \Pi_3$. We have

$$\begin{aligned}\|f - s\|_{\mathbb{L}_p[-1,x_1]} &\le c\,\|f - s\|_{\mathbb{L}_p[-1,t_0]} + c\,\|f - s\|_{\mathbb{L}_p[t_0,t_1]}\\ &\le c\,\|f - q_*\|_{\mathbb{L}_p[-1,t_0]} + c\,\|q_* - p_0\|_{\mathbb{L}_p[-1,t_0]}\\ &\quad + c\,\|f - \tilde{s}\|_{\mathbb{L}_p[t_0,t_1]} + c\,\|\tilde{s} - s\|_{\mathbb{L}_p[t_0,t_1]}\\ &\le c\omega_4(f, [-1, x_2])_p + c\,\|q_* - p_0\|_{\mathbb{L}_p[-1,t_0]}.\end{aligned}$$

Now, Lemma 3.2 implies that

$$\begin{aligned}\|q_* - p_0\|_{\mathbb{L}_p[-1,t_0]} &\le c\,\|q_* - p_0\|_{\mathbb{L}_p[a_0,t_0]} = c\,\|q_* - \tilde{s}\|_{\mathbb{L}_p[a_0,t_0]}\\ &\le c\,\|q_* - f\|_{\mathbb{L}_p[a_0,t_0]} + c\,\|f - \tilde{s}\|_{\mathbb{L}_p[a_0,t_0]}\\ &\le c\omega_4(f, [-1, x_1])_p,\end{aligned}$$

which implies that

$$\|f - s\|_{\mathbb{L}_p[-1,x_1]} \le c\omega_4(f, [-1, x_2])_p,$$

and the proof of Theorem 1.1 is now complete. □

4. Smoothing and moving knots to the right place

Given a partition $\mathbf{z}_n$ of $[-1, 1]$, we recall (see [6]) that a partition $\tilde{\mathbf{z}}_m$ of $[-1, 1]$ is called a δ-*remesh* of $\mathbf{z}_n$ if, for each $0 \le j \le n-1$,

$$\max\{\tilde{z}_{i+1} - \tilde{z}_i \mid [\tilde{z}_i, \tilde{z}_{i+1}] \cap (z_j, z_{j+1}) \ne \emptyset\} \le \delta \min_{\nu=j-1,j,j+1} |z_{\nu+1} - z_\nu|$$

with z_{-1} and z_{n+1} defined to be (in this definition only) $-\infty$ and $+\infty$, respectively.

Lemma 4.1 (*[6, Theorem 1.1 (q = 3)]*). *Let $r \ge 3$, and let $\mathbf{y}_l$ be a partition of $[-1, 1]$. There exists a constant $\delta = \delta(r)$ such that, for each $s_* \in \mathbb{S}_r(\mathbf{y}_l) \cap \Delta^3$ such that*

$$s_* \in \mathbb{C}^2, \tag{16}$$

and any partition $\mathbf{x}_n$ *which is a* δ*-remesh of* $\mathbf{y}_l$*, there exists a spline* $\tilde{s} \in \widetilde{\mathbb{S}}_r(\mathbf{x}_n) \cap \Delta^3$ *of minimal defect satisfying*

$$\|s_* - \tilde{s}\|_{\mathbb{L}_p(\mathcal{I}_j)} \le c(r, p)\omega_{r+1}(s_*, \mathcal{I}_j)_p, \quad 0 \le j \le l,$$

for all $0 < p \le \infty$*, where* $\mathcal{I}_j := [(y_{j-1} + y_j)/2, (y_j + y_{j+1})/2]$.

It can be noted that the case for $r = 3$ in Lemma 4.1 was not included in the statement of [6, Theorem 1.1], but it is not difficult to check that the same proof works and is, in fact, simpler.

As will be shown below, Lemma 4.1 allows us to smooth 3-monotone splines to achieve minimal defect and to change the location of the knots. However, an arbitrary 3-monotone ppf, while being in $\mathbb{C}^1$, is not necessarily in $\mathbb{C}^2$, and so the extra smoothness condition (16) has to be taken care of before Lemma 4.1 can be applied. The following lemma provides this smoothing to $\mathbb{C}^2$ in the case $p = \infty$.

Lemma 4.2 (*[8, Theorem 5]*). *Suppose that* $\mathbf{y}_l$ *is a partition of* $[-1, 1]$ *and* $s \in \mathbb{S}_3(\mathbf{y}_l) \cap \Delta^3$. *Then, there is a ppf* $s_* \in \mathbb{S}_3(\mathbf{y}_l) \cap \Delta^3$ *such that*

$$s_* \in \mathbb{C}^2$$

and

$$\|s - s_*\|_{\mathbb{L}_\infty} \le c(\eta(\mathbf{y}_l), \mu(\mathbf{y}_l)) \max_{1 \le j \le l-1} \omega_4(s, [y_{j-1}, y_{j+1}])_\infty.$$

5. Auxiliary results

Given $\delta > 0$ and a partition $\mathbf{x}_n$ of $[-1, 1]$ with bounded $\vartheta(\mathbf{x}_n)$, we will show that there exists a partition $\mathbf{z}_m$ such that $\mathbf{x}_n$ is its δ-remesh and, at the same time, $\vartheta(\mathbf{z}_n)$ is still bounded.

Lemma 5.1. *For any* $\delta > 0$, $\boldsymbol{\vartheta} \ge 1$ *and any partition* $\mathbf{x}_n$ *of* $[a, b]$ *with* $\vartheta(\mathbf{x}_n) \le \boldsymbol{\vartheta}$*, there is a partition* $\mathbf{z}_m$ *satisfying*

(i) $\mathbf{x}_n$ *is a* δ*-remesh of* $\mathbf{z}_m$,
(ii) $\eta(\mathbf{z}_m) \le 2\boldsymbol{\vartheta}$,
(iii) $\vartheta(\mathbf{z}_m) \le c(\delta, \boldsymbol{\vartheta})$,
(iv) *for any* $0 \le j \le m - 1$ *and* $0 \le k \le n - 1$ *such that* $(x_k, x_{k+1}) \cap [z_j, z_{j+1}] \ne \emptyset$

$$z_{j+1} - z_j \le c(\delta, \boldsymbol{\vartheta})(x_{k+1} - x_k).$$

Proof. Let $\mathcal{K} := \max\{\lceil \boldsymbol{\vartheta}/\delta \rceil, 1\}$, $m := \lfloor n/\mathcal{K} \rfloor$ and define $\mathbf{z}_m = (z_j)_{j=0}^m$ as follows:

$$z_j := x_{\mathcal{K} j}, \quad 0 \le j \le m - 1, \quad \text{and} \quad z_m := x_n.$$

(Note that $z_{m-1} = x_{\mathcal{K}(m-1)} < x_{\mathcal{K} m} \le x_n = z_m$.)

We will now show that $\mathbf{x}_n$ is a δ-remesh of $\mathbf{z}_m$. Let $J_j := [z_j, z_{j+1}]$, and suppose that k is such that $[x_k, x_{k+1}] \cap (z_j, z_{j+1}) \ne \emptyset$ (*i.e.*, $[x_k, x_{k+1}] \subseteq J_j$). We need to show that

$$x_{k+1} - x_k \le \delta \begin{cases} \min\{|J_0|, |J_1|\}, & \text{if } j = 0, \\ \min\limits_{\nu = j-1, j, j+1} |J_\nu|, & \text{if } 1 \le j \le m-2, \\ \min\{|J_{m-2}|, |J_{m-1}|\}, & \text{if } j = m - 1. \end{cases} \tag{17}$$

First, if $0 \le j \le m - 2$, the fact that $\vartheta(\mathbf{x}_n) \le \boldsymbol{\vartheta}$ implies that

$$\frac{x_{k+1} - x_k}{|J_j|} \le \boldsymbol{\vartheta} \frac{x_{\mathcal{K}(j+1)} - x_{\mathcal{K} j}}{\mathcal{K}|J_j|} = \frac{\boldsymbol{\vartheta}}{\mathcal{K}} \le \delta$$

and, if $j = m-1$,

$$\frac{x_{k+1}-x_k}{|J_{m-1}|} \le \frac{\boldsymbol{\vartheta}}{n - \mathcal{K}(m-1)} \le \frac{\boldsymbol{\vartheta}}{\mathcal{K}} \le \delta.$$

If $1 \le j \le m-2$, then we also have

$$\frac{x_{k+1}-x_k}{|J_{j\pm1}|} \le \boldsymbol{\vartheta}\frac{x_{\mathcal{K}(j\pm1+1)} - x_{\mathcal{K}(j\pm1)}}{\mathcal{K}|J_{j\pm1}|} \le \frac{\boldsymbol{\vartheta}}{\mathcal{K}} \le \delta.$$

Finally, if $j = m-1$, then

$$\frac{x_{k+1}-x_k}{|J_{m-2}|} \le \boldsymbol{\vartheta}\frac{x_{\mathcal{K}(m-1)} - x_{\mathcal{K}(m-2)}}{\mathcal{K}|J_{m-2}|} = \frac{\boldsymbol{\vartheta}}{\mathcal{K}} \le \delta.$$

Now, for any $0 \le j \le m-1$, let k_j be such that

$$x_{k_j+1} - x_{k_j} = \max_{[x_i,x_{i+1}]\subseteq[z_j,z_{j+1}]}(x_{i+1}-x_i).$$

If $0 \le j \le m-2$, we have

$$\frac{|J_{j+1}|}{|J_j|} \le \frac{2\mathcal{K}(x_{k_{j+1}+1} - x_{k_{j+1}})}{x_{\mathcal{K}(j+1)} - x_{\mathcal{K}j}} \le 2\boldsymbol{\vartheta},$$

and

$$\frac{|J_j|}{|J_{j+1}|} \le \frac{\mathcal{K}(x_{k_j+1} - x_{k_j})}{x_{\mathcal{K}(j+2)} - x_{\mathcal{K}(j+1)}} \le \boldsymbol{\vartheta},$$

and so (ii) is verified.

Now, suppose that $k \ge 0$ is such that $[x_k, x_{k+1}] \subseteq [z_j, z_{j+1}]$, for some $0 \le j \le m-1$. Then,

$$\begin{aligned} z_{j+1} - z_j &\le x_{\min\{\mathcal{K}j+2\mathcal{K},n\}} - x_{\mathcal{K}j} = \sum_{\nu=\mathcal{K}j}^{\min\{\mathcal{K}j+2\mathcal{K},n\}-1}(x_{\nu+1}-x_\nu) \\ &\le \sum_{\nu=\mathcal{K}j}^{\min\{\mathcal{K}j+2\mathcal{K},n\}-1}\boldsymbol{\vartheta}^{|k-\nu|}(x_{k+1}-x_k) \le (x_{k+1}-x_k)\sum_{\nu=\mathcal{K}j}^{\mathcal{K}j+2\mathcal{K}-1}\boldsymbol{\vartheta}^{|k-\nu|} \\ &\le 2\mathcal{K}\boldsymbol{\vartheta}^{2\mathcal{K}-1}(x_{k+1}-x_k), \end{aligned}$$

and so (iv) follows.

We will now show that $\vartheta(\mathbf{z}_m)$ is bounded. Suppose that i, j and k are such that $0 \le i < j \le m$, $\max\{3i-2j, 0\} \le k \le \min\{3j-2i, m\}-1$, and

$$\vartheta(\mathbf{z}_m) = \frac{(j-i)(z_{k+1}-z_k)}{z_j - z_i},$$

and let $i' := \mathcal{K}i$, $j' := \mathcal{K}j$, $k' := \mathcal{K}k$. Clearly, $0 \le i' < j' \le \mathcal{K}m \le n$ and $\max\{3i'-2j', 0\} \le k' \le \min\{3j'-2i', \mathcal{K}m\} - \mathcal{K} \le \min\{3j'-2i', n\}-1$, and so

$$\begin{aligned} \vartheta(\mathbf{z}_m) &= \frac{(j'-i')(z_{k+1}-z_k)}{\mathcal{K}(z_j - z_i)} \le \frac{(j'-i')(z_{k+1}-z_k)}{\mathcal{K}(x_{j'} - x_{i'})} \\ &\le \frac{(j'-i')}{\mathcal{K}(x_{j'}-x_{i'})}(x_{\min\{k'+2\mathcal{K},n\}} - x_{k'}) \le \frac{\vartheta(\mathbf{x}_n)}{\mathcal{K}}\frac{(x_{\min\{k'+2\mathcal{K},n\}} - x_{k'})}{x_{k'+1} - x_{k'}} \\ &\le 2\boldsymbol{\vartheta}^{2\mathcal{K}}. \end{aligned}$$

Hence, the proof is now complete. □

Lemma 5.2. *For any* $y_0 < y_1 < \cdots < y_N$ *such that*

$$y_N - y_0 \le \lambda \min_{0\le j\le N-1}(y_{i+1} - y_i)$$

and $f \in \mathbb{C}[y_0, y_N]$, *we have*

$$\omega_k(f, [y_0, y_N])_\infty \le c(k,\lambda) \max_{1\le j\le N-1} \omega_k(f, [y_{j-1}, y_{j+1}])_\infty.$$

Proof. Let $\beta := k^{-1}\min_{0\le j\le N-1}(y_{i+1} - y_i)$. Then

$$\omega_k(f, [y_0, y_N])_\infty \le c(k,\lambda)\omega_k(f, \beta, [y_0, y_N])_\infty = c(k,\lambda)\omega_k(f, \beta, [y_{\nu-1}, y_{\nu+1}])_\infty,$$

for some $1 \le \nu \le N-1$, and so the lemma is proved. $\square$

6. Proof of Theorem 1.2

Let $f \in \Delta^3 \cap \mathbb{C}$, $r \ge 3$, and suppose that $\mathbf{x}_n$ is a partition of $[-1, 1]$ such that $\vartheta(\mathbf{x}_n) \le \boldsymbol{\vartheta}$.

Step 1. We set $\mathfrak{c}_5 := \mathfrak{c}_1(2\boldsymbol{\vartheta})$ and $\delta_1 := (\delta\mathfrak{c}_5^2)/(8\boldsymbol{\vartheta}^2)$ (where $\delta = \delta(r)$ is given in Lemma 4.1) and use Lemma 5.1 to construct a partition $\mathbf{z}_m$ of $[-1, 1]$ such that $\mathbf{x}_n$ is a δ_1-remesh of $\mathbf{z}_m$, $\eta(\mathbf{z}_m) \le 2\boldsymbol{\vartheta}$, $\vartheta(\mathbf{z}_m) \le c(r, \boldsymbol{\vartheta})$, and, for any $0 \le j \le m-1$ and any $0 \le k \le n-1$ such that $(x_k, x_{k+1}) \cap [z_j, z_{j+1}] \ne \emptyset$,

$$z_{j+1} - z_j \le c(r, \boldsymbol{\vartheta})(x_{k+1} - x_k). \tag{18}$$

Step 2. It follows from Theorem 1.1 (with $\mathbf{x}_n := \mathbf{z}_m$) that there exists a partition $\mathbf{y}_l$ of $[-1, 1]$ with $l \le 20m$, and a cubic ppf $s \in \mathbb{S}_3(\mathbf{y}_l) \cap \Delta^3$ such that, for each $0 \le k \le l-1$, there exists $1 \le j \le m-1$ such that $[y_k, y_{k+1}] \subseteq [z_{j-1}, z_{j+1}]$ and

$$y_{k+1} - y_k \ge \mathfrak{c}_5(z_{j+1} - z_{j-1}).$$

Also,

$$\|f - s\|_{\mathbb{L}_\infty[z_j, z_{j+1}]} \le c(\boldsymbol{\vartheta})\omega_4(f, [z_{j-1}, z_{j+2}])_\infty, \quad 0 \le j \le m-1.$$

It is easy to see that

$$\eta(\mathbf{y}_l) \le 2\eta^2(\mathbf{z}_m)/\mathfrak{c}_5 \le 8\boldsymbol{\vartheta}^2/\mathfrak{c}_5.$$

It is also rather straightforward to show that $\mu(\mathbf{y}_l)$ is bounded by $c(r, \boldsymbol{\vartheta})$.

We also note that $\mathbf{x}_n$ is a δ-remesh of $\mathbf{y}_l$. Indeed, suppose that $[x_i, x_{i+1}] \cap (y_k, y_{k+1}) \ne \emptyset$, where $0 \le k \le l-1$, and let $1 \le j \le m-1$ be such that $[y_k, y_{k+1}] \subseteq [z_{j-1}, z_{j+1}]$, and so $y_{k+1} - y_k \ge \mathfrak{c}_5(z_{j+1} - z_{j-1})$. Since $\mathbf{x}_n$ is a δ_1-remesh of $\mathbf{z}_m$ and, clearly, $(x_i, x_{i+1}) \cap [z_{j-1}, z_j] \ne \emptyset$ or $(x_i, x_{i+1}) \cap [z_j, z_{j+1}] \ne \emptyset$, we also have

$$\begin{aligned} x_{i+1} - x_i &\le \delta_1(z_{j+1} - z_{j-1}) \le \frac{\delta_1}{\mathfrak{c}_5}(y_{k+1} - y_k) \\ &\le \frac{\delta_1\eta(\mathbf{y}_l)}{\mathfrak{c}_5} \min_{\nu=k-1,k,k+1}(y_{\nu+1} - y_\nu) \\ &\le \frac{8\delta_1\boldsymbol{\vartheta}^2}{\mathfrak{c}_5^2} \min_{\nu=k-1,k,k+1}(y_{\nu+1} - y_\nu) \\ &\le \delta \min_{\nu=k-1,k,k+1}(y_{\nu+1} - y_\nu). \end{aligned}$$

Step 3. It follows from Lemma 4.2 that, for $s \in \mathbb{S}_3(\mathbf{y}_l) \cap \Delta^3$, there is a ppf s_* of degree ≤ 3 with the same knots such that $s_* \in \Delta^3 \cap \mathbb{C}^2$ and

$$\begin{aligned}\|s - s_*\|_{\mathbb{L}_\infty} &\le c(\eta(\mathbf{y}_l), \mu(\mathbf{y}_l)) \max_{1\le j\le l-1} \omega_4(s, [y_{j-1}, y_{j+1}])_\infty \\ &\le c(r, \boldsymbol{\vartheta}) \max_{1\le j\le l-1} \omega_4(s, [y_{j-1}, y_{j+1}])_\infty.\end{aligned}$$

Step 4. It follows from Lemma 4.1 that there exists a spline $\tilde{s} \in \widetilde{\mathbb{S}}_r(\mathbf{x}_n) \cap \Delta^3$ (*i.e.*, $\tilde{s}$ is of minimal defect) satisfying, for each $0 \le j \le l$,

$$\|s_* - \tilde{s}\|_{\mathbb{L}_\infty[(y_{j-1}+y_j)/2,(y_j+y_{j+1})/2]} \le c(r)\omega_{r+1}(s_*, [(y_{j-1} + y_j)/2, (y_j + y_{j+1})/2])_\infty.$$

It remains to estimate the norm of $f - \tilde{s}$. Suppose that $\|f - \tilde{s}\|_{\mathbb{L}_\infty} = |f(x_*) - \tilde{s}(x_*)|$, and $x_* \in [y_\nu, y_{\nu+1})$, for some $0 \le \nu \le l - 1$. Also suppose that $1 \le \nu_1 \le m - 1$ is such that $[y_\nu, y_{\nu+1}] \subseteq [z_{\nu_1-1}, z_{\nu_1+1}]$ (and so $y_{\nu+1} - y_\nu \ge \mathfrak{c}_5(z_{\nu_1+1} - z_{\nu_1-1})$).

Using Lemma 5.2, and taking into account that the scales of $\mathbf{y}_l$ and $\mathbf{z}_m$ are bounded, we have (with $c = c(r, \boldsymbol{\vartheta})$)

$$\begin{aligned}\|f - \tilde{s}\|_{\mathbb{L}_\infty} &\le |f(x_*) - s(x_*)| + |s(x_*) - s_*(x_*)| + |s_*(x_*) - \tilde{s}(x_*)| \\ &\le \|f - s\|_{\mathbb{L}_\infty} + \|s - s_*\|_{\mathbb{L}_\infty} + \|s_* - \tilde{s}\|_{\mathbb{L}_\infty[(y_{\nu-1}+y_\nu)/2,(y_{\nu+1}+y_{\nu+2})/2]} \\ &\le \|f - s\|_{\mathbb{L}_\infty} + \|s - s_*\|_{\mathbb{L}_\infty} + c\omega_4\,(s_*, [y_{\nu-1}, y_{\nu+2}])_\infty \\ &\le c\,\|f - s\|_{\mathbb{L}_\infty} + c \max_{1\le j\le l-1} \omega_4(s, [y_{j-1}, y_{j+1}])_\infty + c\omega_4\,(f, [y_{\nu-1}, y_{\nu+2}])_\infty \\ &\le c\,\|f - s\|_{\mathbb{L}_\infty} + c \max_{1\le j\le l-1} \omega_4(f, [y_{j-1}, y_{j+1}])_\infty + c\omega_4\,(f, [y_{\nu-1}, y_{\nu+2}])_\infty \\ &\le c \max_{0\le j\le m-1} \omega_4(f, [z_{j-1}, z_{j+2}])_\infty + c \max_{1\le j\le l-1} \omega_4(f, [y_{j-1}, y_{j+1}])_\infty \\ &\le c \max_{1\le j\le m-1} \omega_4(f, [z_{j-1}, z_{j+1}])_\infty \\ &\le c \max_{1\le j\le n-1} \omega_4(f, [x_{j-1}, x_{j+1}])_\infty,\end{aligned}$$

where the last inequality follows from (18).

7. Counterexamples and "negative" results

The following result implies that the fourth modulus of smoothness in the statement of Theorem 1.2 (and its corollaries) cannot be replaced by any modulus of higher order.

Theorem 7.1. *For any $k \in \mathbb{N}$, $A > 0$, $0 < p \le \infty$, $r \in \mathbb{N}$, $n \in \mathbb{N}$ and $0 < \epsilon < 2$ there exists a function $f \in \mathbb{C}^k \cap \Delta^k$ such that*

$$\|f - q_r\|_{\mathbb{L}_p[1-\epsilon,1]} > A\omega_{k+2}(f, [-1, 1])_p \tag{19}$$

for any $q_r \in \Pi_r$ satisfying $q_r^{(k)}(1) \ge 0$.

Proof. The idea of the construction belongs to Shvedov [12], and the following proof is very similar to that of [5, Theorem 3.2]. Let f be such that $f^{(k)}(x) = (1-h-x)_+ := \max\{1-h-x, 0\}$, where $h > 0$ will be selected later. Let $Q \in \Pi_{k+1}$ be such that $Q^{(k)}(x) = 1 - h - x$, and $Q^{(i)}(-1) = f^{(i)}(-1)$ for all $0 \le i \le k - 1$. Since

$$f(x) - Q(x) = \frac{1}{(k-1)!}\int_{-1}^{x} (x-t)^{k-1}\Big(f^{(k)}(t) - Q^{(k)}(t)\Big)\,\mathrm{d}t,$$

we get

$$\|f - Q\|_{\mathbb{L}_\infty} \le \frac{1}{(k-1)!}\int_{-1}^{1}(1-t)^{k-1}\left|f^{(k)}(t) - Q^{(k)}(t)\right| \mathrm{d}t$$
$$\le \frac{h^{k-1}}{(k-1)!}\int_{1-h}^{1}(t-1+h)\,\mathrm{d}t = ch^{k+1}.$$

Consequently,

$$\|f - Q\|_{\mathbb{L}_p} \le 2^{1/p}\,\|f - Q\|_{\mathbb{L}_\infty} \le ch^{k+1}$$

and

$$\omega_{k+2}(f, [-1,1])_p = \omega_{k+2}(f - Q, [-1,1])_p \le c\,\|f - Q\|_{\mathbb{L}_p} \le ch^{k+1}.$$

Assuming that (19) is not true, for some polynomial $P \in \Pi_r$ such that $P^{(k)}(1) \ge 0$, we have $\|f - P\|_{\mathbb{L}_p[1-\epsilon,1]} \le A\omega_{k+2}(f, [-1,1])_p$. Then, using Lemma 3.1, we obtain

$$\left|P^{(k)}(1) - Q^{(k)}(1)\right| \le \left\|P^{(k)} - Q^{(k)}\right\|_{\mathbb{L}_\infty[1-\epsilon,1]} \le c\,\|P - Q\|_{\mathbb{L}_p[1-\epsilon,1]}$$
$$\le c\Big(\|P - f\|_{\mathbb{L}_p[1-\epsilon,1]} + \|f - Q\|_{\mathbb{L}_p[1-\epsilon,1]}\Big)$$
$$\le c\Big(A\omega_{k+2}(f, [-1,1])_p + \|f - Q\|_{\mathbb{L}_p}\Big) \le \tilde{\mathfrak{c}}h^{k+1},$$

where $\tilde{\mathfrak{c}}$ depends on k, r, p, ϵ, and A, but is independent of h. Hence,

$$P^{(k)}(1) \le Q^{(k)}(1) + \left|P^{(k)}(1) - Q^{(k)}(1)\right| \le -h + \tilde{\mathfrak{c}}h^{k+1} < 0,$$

for sufficiently small h, which is a contradiction. □

In order to prove a negative result for k-monotone approximation by a ppf with $k \ge 4$ we need to use several results from [2]. The next lemma follows immediately from [2, Theorem 1].

Lemma 7.2. *Suppose that $\xi \in \mathbb{R}$, $F \in \Delta^3[\xi - \frac{1}{2}, \xi + \frac{1}{2}]$ and*

$$d := \left\|F'(x) - (x-\xi)_+\right\|_{\mathbb{L}_\infty[\xi - \frac{1}{4}, \xi + \frac{1}{4}]}.$$

Then there exists an interval $I \subseteq [\xi - \frac{1}{2}, \xi + \frac{1}{2}]$ with $|I| \ge 1/64$ such that

$$|F(x) - (x-\xi)_+^2/2| \ge c\min\{d, d^2\} \quad \text{for all } x \in I,$$

where c is an absolute constant.

Lemma 7.3 (*[2, Lemma 5]*)**.** *Let $r \in \mathbb{N}$ and a function $G \in \mathbb{C}^r[a,b]$ be such that $|G^{(r)}(x)| \ge h$, for all $x \in [a,b]$. Then there exists an interval $I \subseteq [a,b]$, $|I| \ge 4^{-r}(b-a)$ such that*

$$|G(x)| \ge 2^{-r^2-r}h(b-a)^r, \quad \text{for all } x \in I.$$

Theorem 7.4. *For any $k \ge 4$, $r \in \mathbb{N}$, $0 < p \le \infty$ and $A > 0$, there is $n \in \mathbb{N}$ such that, for any partition $\mathbf{z}_n$ of $[-1,1]$ (into n subintervals), there exists a function $f \in \Delta^k \cap \mathbb{C}^{k-2}$ such that*

$$\|f - s\|_{\mathbb{L}_p} > A\omega_3(f, n^{-1}, [-1,1])_p, \tag{20}$$

for any $s \in \mathbb{S}_r(\mathbf{z}_n) \cap \Delta^k$.

Proof. Recall that $x_+ := \max\{x, 0\}$ and $x_+^0 := 1$ if $x > 0$, and $x_+^0 := 0$ if $x \le 0$. Given $n \ge 2$, let $\mathbf{z}_n$ be an arbitrary partition of $[-1, 1]$. Then, there exist $\xi \in [-1/2, 1/2]$ and ς, $0 \le \varsigma \le n-1$, such that $J := [\xi - (2n)^{-1}, \xi + (2n)^{-1}] \subseteq [z_\varsigma, z_{\varsigma+1}]$. Now,

$$\|t_+ - P(t)\|_{\mathbb{L}_\infty} \ge c(r), \quad \text{for any } P \in \Pi_r,$$

and after a linear change of variable (with $x = \xi + t(2n)^{-1}$ we have $(x-\xi)_+ = (2n)^{-1} t_+$ and $Q(x) = (2n)^{-1} P(t)$),

$$\|(x-\xi)_+ - Q(x)\|_{\mathbb{L}_\infty(J)} \ge c(r) n^{-1}, \quad \text{for any } Q \in \Pi_r.$$

With $f(x) := \frac{1}{(k-1)!}(x-\xi)_+^{k-1} \in \Delta^k \cap \mathbb{C}^{k-2}$, this means that, for any $s \in \mathbb{S}_r(\mathbf{z}_n) \cap \Delta^k$,

$$\left\| f^{(k-2)} - s^{(k-2)} \right\|_{L_\infty(J)} \ge c(k, r) n^{-1}.$$

Since $J \subseteq [\xi - \frac{1}{4}, \xi + \frac{1}{4}]$, Lemma 7.2 implies that there exists an interval $I \subseteq [\xi - \frac{1}{2}, \xi + \frac{1}{2}] \subset [-1, 1]$, $|I| \ge \frac{1}{64}$, such that

$$|f^{(k-3)}(x) - s^{(k-3)}(x)| \ge c(k, r) n^{-2}, \quad \text{for all } x \in I.$$

Hence, by Lemma 7.3, for some interval $\mathcal{J} \subseteq I$, $|\mathcal{J}| \ge c(k)$, we get

$$|f(x) - s(x)| \ge c(k, r) n^{-2}, \quad \text{for all } x \in \mathcal{J};$$

therefore

$$\|f - s\|_{\mathbb{L}_p} \ge c(k, r) n^{-2}.$$

Assuming that (20) is not true, for every $n \ge 2$, we can find $s \in \mathbb{S}_r(\mathbf{z}_n)$ so that

$$\begin{aligned} c(k, r) n^{-2} &\le \|f - s\|_{\mathbb{L}_p} \le A\omega_3(f, n^{-1}, [-1, 1])_p \le cAn^{-3} \left\| f^{(3)} \right\|_{\mathbb{L}_p} \\ &\le c(k) A n^{-3} \left\| x_+^{k-4} \right\|_{\mathbb{L}_p[-2,2]} \le c(k) A n^{-3}, \end{aligned}$$

which is a contradiction when n is large enough. $\square$

Acknowledgments

The second author was supported by the NSERC of Canada and the third author was supported in part by a PIMS Postdoctoral Fellowship.

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