Moduli of Smoothness of Splines and Applications in Constrained Approximation

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Abstract

In this paper, we generalize [8, Theorem 4.1] to be applicable to the classes of (almost) weakly/nearly (co)-k-monotone functions and discuss some applications and open problems.

1 Introduction and Main Results

Let $\mathbb{S}_r(\mathbf{z}_n)$ be the space of all piecewise polynomial functions (ppf's) of degree r (order $r+1$) with the knots $\mathbf{z}_n := (z_i)_0^n$, $-1 =: z_0 < z_1 < \ldots < z_{n-1} < z_n := 1$. In other words, we say that $s \in \mathbb{S}_r(\mathbf{z}_n)$ if, on each interval (z_i, z_{i+1}), $0 \le i \le n-1$, s is in Π_r, where Π_r denotes the space of algebraic polynomials of degree $\le r$.

For a partition $\mathbf{z}_n := \{z_0, \ldots, z_n \mid -1 =: z_0 < z_1 < \cdots < z_n := 1\}$, let $J_j := [z_j, z_{j+1}]$ with $z_j := -1$, $j < 0$, and $z_j := 1$, $j > n$, and $|J| := \operatorname{meas} J$.

Given an absolute constant Δ we say that $\mathbf{z}_n$ is "Δ-*quasi-uniform*" if $\Delta(\mathbf{z}_n) := \max_{0\le j\le n-1} |J_j| / \min_{0\le j\le n-1} |J_j| \le \Delta$, and denote by $\mathbf{U}_n^\Delta$ the class of all such partitions. (Note that $\mathbf{U}_n^1$ consists of only one partition which is the uniform partition of $[-1,1]$ into n subintervals of equal lengths.)

We also use the notation $\mathbf{t}_n = (t_i)_0^n$, where $t_i := -\cos(\pi i/n)$, $0 \le i \le n$, for the Chebyshev partition of $[-1,1]$.

As usual, $\mathbb{L}_p(J)$, $0 < p \le \infty$, denotes the space of all measurable functions f on J such that $\|f\|_{\mathbb{L}_p(J)} < \infty$, where $\|f\|_{\mathbb{L}_p(J)} := \left(\int_J |f(x)|^p\, dx\right)^{1/p}$ if $0 < p < \infty$, and $\|f\|_{\mathbb{L}_\infty(J)} := \operatorname{ess\,sup}_{x\in J} |f(x)|$, and write $\mathbb{L}_p := \mathbb{L}_p[-1,1]$ and $\|\cdot\|_p := \|\cdot\|_{\mathbb{L}_p[-1,1]}$. We say that a function f is in the Sobolev Space $\mathbb{W}^\nu(\mathbb{L}_p)$ if it has an absolutely continuous $(\nu-1)$st derivative such that $f^{(\nu)} \in \mathbb{L}_p$.

The kth symmetric difference is $\Delta_h^k(f, x, J) := \sum_{i=0}^k \binom{k}{i} (-1)^{k-i} f(x - kh/2 + ih)$, if $x \pm kh/2 \in J$, and $\Delta_h^k(f, x, J) := 0$, otherwise. The kth modulus of smoothness of a

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function $f \in \mathbb{L}_p(J)$ is defined by

$$\omega_k(f,t,J)_p := \sup_{0<h\le t} \|\Delta_h^k(f,\cdot,J)\|_{\mathbb{L}_p(J)} ,$$

and $\omega_k(f,t)_p := \omega_k(f,t,[-1,1])_p$.

The (usual) Ditzian-Totik modulus kth modulus of smoothness (see [3]) is

$$\omega_k^\varphi(f,t)_p := \sup_{0<h\le t} \|\Delta_{h\varphi(\cdot)}^k(f,\cdot)\|_p ,$$

where $\varphi(x) := \sqrt{1-x^2}$.

We also need the weighted Ditzian-Totik kth modulus of smoothness of a function $f \in \mathbb{L}_p[-1,1]$, $0<p\le\infty$, which we define as

$$\omega_k^\varphi(f,t)_{W,p} := \sup_{0<h\le t} \left\| W(\cdot, kh/2)\Delta_{h\varphi(\cdot)}^k(f,\cdot)\right\|_p .$$

where W is some weight function. Note that $\Delta_{h\varphi(x)}^k(f,x,[-1,1])$ is defined to be identically 0 if $x \notin \mathfrak{D}_{kh/2}$, where

$$\mathfrak{D}_\delta := \{x \mid 1-\delta\varphi(x) \ge |x|\} \setminus \{\pm 1\} = \left\{ x \mid |x| \le \frac{1-\delta^2}{1+\delta^2} \right\} , \tag{1.1}$$

and so W should only be defined on $\mathfrak{D}_{kh/2}$.

In this paper, we use weighted moduli with the weights (see [8, 12])

$$W_1(x,kh/2) := \varphi^\nu(x) \quad \text{and} \quad W_2(x,\mu) := \varphi^\nu(|x|+\mu\varphi(x)) ,$$

and denote

$$\omega_{k,\nu}^\varphi(f,t)_p := \omega_k^\varphi(f,t)_{W_1,p} = \sup_{0<h\le t} \left\| \varphi^\nu(\cdot)\Delta_{h\varphi(\cdot)}^k(f,\cdot)\right\|_p$$

and

$$\bar{\omega}_{k,\nu}^\varphi(f,t)_p := \omega_k^\varphi(f,t)_{W_2,p} = \sup_{0<h\le t} \left\| \varphi^\nu(|\cdot| + kh\varphi(\cdot)/2)\Delta_{h\varphi(\cdot)}^k(f,\cdot)\right\|_p .$$

Also, note that

$$\omega_{0,\nu}^\varphi(f,t)_p = \bar{\omega}_{0,\nu}^\varphi(f,t)_p = \|\varphi^\nu f\|_p \quad \text{and} \quad \omega_{k,0}^\varphi(f,t)_p = \bar{\omega}_{k,0}^\varphi(f,t)_p = \omega_k^\varphi(f,t)_p .$$

Clearly,

$$\bar{\omega}_{k,\nu}^\varphi(f,t)_p \le \omega_{k,\nu}^\varphi(f,t)_p . \tag{1.2}$$

We also emphasize that $\varphi^\nu f \in \mathbb{L}_p$ does NOT imply that $\omega_{k,\nu}^\varphi(f,t)_p < \infty$ (consider $f(x) = (1+x)^{-1/p}$, $k=\nu=1$ for $0<p<\infty$, and $f(x) = (1+x)^{-1}$, $\nu = 2$, $k=1$ for $p=\infty$). At the same time, if $\varphi^\nu f \in \mathbb{L}_p$, then $\bar{\omega}_{k,\nu}^\varphi(f,t)_p < \infty$.

Given a set (usually, an interval) J, let $\mathcal{M}(J)$ be a "constraints class" of functions defined on J. For example, $\mathcal{M}(J)$ could be the class of all monotone or convex functions on J, or the class of functions changing their k-monotonicity the given number of times, or a class of functions satisfying some interpolation conditions, or having their range restricted on J, or some class of functions having certain other shape characteristics on various subsets of J, etc. If J is an interval we abuse the notation and omit parantheses in this notation, *i.e.,* $\mathcal{M}[a,b] := \mathcal{M}([a,b])$.

Let $J_\lambda := \{x/\lambda \mid x \in J\}$ and, in particular, $[a,b]_\lambda := [a/\lambda, b/\lambda]$. Given a constraints class $\mathcal{M}(J)$ and a parameter $\lambda > 0$, we denote by $[\mathcal{M}(J)]_\lambda$ the class of all functions which are defined on J_λ and such that

$$f \in \mathcal{M}(J) \quad \text{if and only if} \quad f_\lambda \in [\mathcal{M}(J)]_\lambda ,$$

where $f_\lambda := f(\lambda \cdot)$. Hence, since $(f_\lambda)_{1/\lambda} \equiv f$, the above is equivalent to:

$$f \in [\mathcal{M}(J)]_\lambda \quad \text{if and only if} \quad f_{1/\lambda} \in \mathcal{M}(J) .$$

For example, if $\mathcal{M}^2(J)$ denotes the class of all convex functions on J, then $[\mathcal{M}^2(J)]_\lambda = \mathcal{M}^2(J_\lambda)$ is the class of all convex functions on J_λ. Note, however, that, in general, $[\mathcal{M}(J)]_\lambda \neq \mathcal{M}(J_\lambda)$. For example, if $\tilde{\mathcal{M}}_{1/4}[-1,1]$ is the set of all functions which are defined on $[-1,1]$ and have an inflection point at $1/4$, then functions in $\left[\tilde{\mathcal{M}}_{1/4}[-1,1]\right]_{1/2}$ have their inflection points at $1/2$ while functions from $\tilde{\mathcal{M}}_{1/4}\left([-1,1]_{1/2}\right) = \tilde{\mathcal{M}}_{1/4}[-2,2]$ still have their inflection points at $1/4$.

In order to unify the statements for different partitions, following [8, (4.1)], we define

$$\text{(1.3)} \qquad \mathrm{w}_m(g^{(\nu)}, t)_p := \begin{cases} \omega_m(g^{(\nu)}, t)_p , & \text{if } \mathbf{z}_n \in \mathbf{U}_n^\Delta , \\ \omega_{m,\nu}^{\varphi}(g^{(\nu)}, t)_p , & \text{if } \mathbf{z}_n = \mathbf{t}_n . \end{cases}$$

The following theorem is our main result.

Theorem 1.1 *Let $n, m, \nu \in \mathbb{N}$, $m' \in \mathbb{N}_0$, $m' + \nu \geq m$, $n \geq m' + \nu - 1$, $f \in \mathbb{L}_p[-1,1]$, $0 < p \leq \infty$, and let $\mathcal{W}^\nu \subset \mathbb{W}^\nu(\mathbb{L}_p)$ be some class of functions such that $f(\mu \cdot) =: f_\mu \in \mathcal{W}^\nu$ whenever $f \in \mathcal{W}^\nu$ and $\mu > 0$. Also, for $\tau \geq 0$, let $\lambda := 1 - \tau/n^2$ be such that $1/2 \leq \lambda \leq 1$. Additionally, let $\Delta \in \mathbb{R}$ and $\mathbf{z}_n$ be either a Δ-quasi-uniform or the Chebyshev partition of $[-1,1]$ into n intervals,* i.e., *$\mathbf{z}_n \in \mathbf{U}_n^\Delta$ or $\mathbf{z}_n = \mathbf{t}_n$.*

Suppose that the following assumptions are satisfied.

Assumption 1: *for some $r \in \mathbb{N}$, there exists $s \in \mathcal{S}_r(\mathbf{z}_n) \cap \mathcal{M}[-\lambda, \lambda] \cap \mathcal{W}^\nu$ such that*

$$\|f - s\|_{\mathbb{L}_p[-1,1]} \leq c_1 \mathrm{w}_m(f, n^{-1})_p .$$

Assumption 2: *for any function $g \in \mathcal{W}^\nu \cap [\mathcal{M}[-\lambda, \lambda]]_\lambda$ there exists a polynomial $q_n \in \Pi_n \cap [\mathcal{M}[-\lambda, \lambda]]_\lambda$, such that*

$$\|g - q_n\|_{\mathbb{L}_p[-1,1]} \leq c_2 n^{-\nu} \mathrm{w}_{m'}(g^{(\nu)}, n^{-1})_p .$$

Then there exists a polynomial $p_n \in \Pi_n \cap \mathcal{M}[-\lambda, \lambda]$ such that

$$\|f - p_n\|_{\mathbb{L}_p[-1,1]} \leq c \mathrm{w}_m(f, n^{-1})_p ,$$

where c depends on $c_1, c_2, \tau, r, m, m', \nu, p$, and also on Δ if $\mathbf{z}_n \in \mathbf{U}_n^\Delta$.

Remark. *Because of (1.2), Theorem 1.1 in the case* $\mathbf{z}_n = \mathbf{t}_n$ *also holds for* $\mathrm{w}_m(g^{(\nu)}, t)_p := \bar{\omega}^{\varphi}_{m,\nu}(g^{(\nu)}, t)_p$.

Throughout this paper, $c(\gamma_1, \gamma_2, \ldots)$ denote positive constants which depend only on the parameters $\gamma_1, \gamma_2, \ldots$ (note that $c(p, \ldots)$ depends on p only as $p \to 0$) and which may be different on different occurrences. At the same time, c_μ denote positive constants which are fixed throughout the paper.

2 Auxiliary Results

Theorem 2.1 ([8,9]) *Let* $\mathbf{z}_n$ *be either a* Δ*-quasi-uniform or the Chebyshev partition of* $[-1,1]$ *into* n *intervals,* i.e., $\mathbf{z}_n \in \mathbf{U}_n^\Delta$ *or* $\mathbf{z}_n = \mathbf{t}_n$*, and let* $s \in \mathbb{S}_r(\mathbf{z}_n) \cap \mathbb{C}^m[-1,1]$, $r \in \mathbb{N}$, $0 \le m \le r-1$. *Then, for any* $1 \le k \le r+1$, $1 \le \nu \le \min\{k, m+1\}$ *and* $0 < p \le \infty$, *we have*

$$n^{-\nu} \mathrm{w}_{k-\nu}(s^{(\nu)}, n^{-1})_p \sim \mathrm{w}_k(s, n^{-1})_p \,, \tag{2.4}$$

with equivalence constants depending only on r, Δ *(if* $\mathbf{z}_n \in \mathbf{U}_n^\Delta$*) and* p *as* $p \to 0$.

The following lemma is rather well known and can be found in [1, Theorems A.4.1 and A.4.10], for example.

Lemma 2.2 (Remez inequality) *For any* $q \in \Pi_n$ *and a set* A *such that* $\operatorname{meas}\{[-1,1] \setminus A\} \le s \le 1/2$ *the following inequalities hold:*

$$\|q\|_{\mathbb{C}[-1,1]} \le e^{5n\sqrt{s}} \|q\|_{\mathbb{C}(A)}$$

and

$$\|q\|_{\mathbb{L}_p[-1,1]} \le \left(1 + e^{8pn\sqrt{s}}\right)^{1/p} \|q\|_{\mathbb{L}_p(A)} \,, \quad 0 < p < \infty \,.$$

Corollary 2.3 *For any* $q \in \Pi_n$, *a set* A *such that* $\operatorname{meas}\{[-1,1] \setminus A\} \le s \le 1/2$, *and* $0 < p \le \infty$,

$$\|q\|_{\mathbb{L}_p[-1,1]} \le 2^{1+1/p} e^{8n\sqrt{s}} \|q\|_{\mathbb{L}_p(A)} \,.$$

In particular, for any $0 < \tau \le n^2/4$,

$$\|q\|_{\mathbb{L}_p[-1,1]} \le c(\tau, p) \, \|q\|_{\mathbb{L}_p[-1+\tau/n^2, 1-\tau/n^2]} \,. \tag{2.5}$$

Lemma 2.4 *Suppose that* $f \in \mathbb{L}_p[-1,1]$, $0 < p \le \infty$, $\tau \ge 0$, $\lambda := 1 - \tau t^2$ *is such that* $\lambda \ge 1/2$, *and* $f_\lambda(x) := f(\lambda x)$. *Then,*

$$\omega_k^\varphi(f_\lambda, t)_p \le c(\tau, p) \, \omega_k^\varphi(f, t)_p \,. \tag{2.6}$$

We remark that the condition $\lambda \ge 1/2$ is not essential and $1/2$ can be replaced by any positive constant c_0 (the constant c in (2.6) will then depend on c_0 as well). It also immediately follows from the definition that the statement of this lemma holds for the usual kth modulus, *i.e.*, $\omega_k(f_\lambda, t)_p \le c(p) \omega_k(f, t)_p$.

Proof. In the proof, it is more convenient to work with the Ivanov moduli which are equivalent to the Ditzian-Totik moduli. Recall that, the Ivanov modulus of smoothness is defined by (see [5, 6])

$$\tau_k(f,\psi(t))_{p,p} := \left\|\omega_k(f,\cdot,\psi(t,\cdot))_p\right\|_p ,$$

where

$$\omega_k(f,x,\psi(t,x))_p^p := \frac{1}{2\psi(t,x)}\int_{-\psi(t,x)}^{\psi(t,x)} |\Delta_h^k(f,x+kh/2,[-1,1])|^p\,dh$$

with $\psi(t,x) := t\varphi(x)+t^2$. It is known (see [15] and [2]) that, for all $0<p\le\infty$, $\tau_k(f,\psi(t))_{p,p}\sim\omega_k^\varphi(f,t)_p$ with equivalence constants depending only on k and p.

Changing variables we have

$$\begin{aligned}
\tau_k(f_\lambda,\psi(t))_{p,p}^p &= \int_{-1}^{1}\int_{-\psi(t,x)}^{\psi(t,x)} \frac{1}{2\psi(t,x)}|\Delta_h^k(f_\lambda,x+kh/2,[-1,1])|^p\,dh\,dx\\
&= \int_{-1}^{1}\int_{-\psi(t,x)}^{\psi(t,x)} \frac{1}{2\psi(t,x)}|\Delta_{\lambda h}^k(f,\lambda x+k\lambda h/2,[-1,1])|^p\,dh\,dx\\
&= \int_{-\lambda}^{\lambda}\int_{-\lambda\psi(t,x/\lambda)}^{\lambda\psi(t,x/\lambda)} \frac{1}{2\lambda^2\psi(t,x/\lambda)}|\Delta_h^k(f,x+kh/2,[-1,1])|^p\,dh\,dx\,.
\end{aligned}$$

Now, $\lambda\psi(t,x/\lambda)=t\sqrt{\lambda^2-x^2}+\lambda t^2\le\psi(t,x)$. It is also a simple exercise to show that, for $\lambda=1-\tau t^2$ such that $\lambda\ge 1/2$, $\lambda^2\psi(t,x/\lambda)\ge c(\tau)\psi(t,x)$ for all $x\in[-\lambda,\lambda]$. Therefore,

$$\begin{aligned}
\tau_k(f_\lambda,\psi(t))_{p,p}^p &\le c(\tau)\int_{-1}^{1}\int_{-\psi(t,x)}^{\psi(t,x)} \frac{1}{2\psi(t,x)}|\Delta_h^k(f,x+kh/2,[-1,1])|^p\,dh\,dx\\
&= c(\tau)\tau_k(f,\psi(t))_{p,p}^p\,,
\end{aligned}$$

which completes the proof. $\square$

3 Proof of Theorem 1.1

In the case $\tau=0$, the statement of the theorem immediately follows from [8, Theorem 4.1], since $[\mathcal{M}(J)]_1=\mathcal{M}(J)$. Hence, we only need to consider the case $\tau>0$.

Assumption 1 guarantees that there exists $s\in\mathcal{S}_r(\mathbf{z}_n)\cap\mathcal{M}[-\lambda,\lambda]\cap\mathcal{W}^\nu$ is such that

$$\|f-s\|_{\mathbb{L}_p[-1,1]}\le c_1\mathrm{w}_m(f,n^{-1})_p\,.$$

Consider the function $s_\lambda:=s\,(\lambda\cdot)$. The same proof as in [8, Section 3.3] shows that (2.4) is valid with s replaced by s_λ.

Now, since $s_\lambda\in\mathcal{W}^\nu\cap[\mathcal{M}[-\lambda,\lambda]]_\lambda$, Assumption 2 implies that there exists a polynomial $q_n\in\Pi_n\cap[\mathcal{M}[-\lambda,\lambda]]_\lambda$ such that

$$\|s_\lambda-q_n\|_{\mathbb{L}_p[-1,1]} \le c_2n^{-\nu}\mathrm{w}_{m'}(s_\lambda^{(\nu)},n^{-1})_p\le c\mathrm{w}_{m'+\nu}(s_\lambda,n^{-1})_p\le c\mathrm{w}_{m'+\nu}(s,n^{-1})_p\,,$$

where the last inequality follows from Lemma 2.4 and the remark after its statement. Hence, recalling that $m' + \nu \geq m$, we have

$$\|s_\lambda - q_n\|_{\mathbb{L}_p[-1,1]} \leq c\mathrm{w}_m(s, n^{-1})_p \,.$$

Therefore, for $p_n := q_n(\cdot/\lambda)$, using the fact that $[[\mathcal{M}(J)]_\lambda]_{1/\lambda} = \mathcal{M}(J)$ we have $p_n \in \Pi_n \cap \mathcal{M}[-\lambda, \lambda]$, and

$$\|s - p_n\|_{\mathbb{L}_p[-\lambda,\lambda]} = \lambda^{1/p} \|s_\lambda - q_n\|_{\mathbb{L}_p[-1,1]} \leq c\mathrm{w}_m(s, n^{-1})_p \,.$$

Suppose now that a polynomial $P_n \in \Pi_n$ is such that

$$\|s - P_n\|_{\mathbb{L}_p[-1,1]} \leq c\mathrm{w}_m(s, n^{-1})_p$$

(a polynomial of best approximation to s will do). Then, by the Remez inequality (2.5),

$$\|P_n - p_n\|_{\mathbb{L}_p[-1,1]} \leq c(\tau, p) \|P_n - p_n\|_{\mathbb{L}_p[-\lambda,\lambda]} \,,$$

and hence

$$\begin{aligned}
\|s - p_n\|_{\mathbb{L}_p[-1,1]} &\leq c\|s - P_n\|_{\mathbb{L}_p[-1,1]} + c\|P_n - p_n\|_{\mathbb{L}_p[-1,1]} \\
&\leq c\|s - P_n\|_{\mathbb{L}_p[-1,1]} + c\|P_n - p_n\|_{\mathbb{L}_p[-\lambda,\lambda]} \\
&\leq c\|s - P_n\|_{\mathbb{L}_p[-1,1]} + c\|P_n - s\|_{\mathbb{L}_p[-\lambda,\lambda]} + c\|s - p_n\|_{\mathbb{L}_p[-\lambda,\lambda]} \\
&\leq c\|s - P_n\|_{\mathbb{L}_p[-1,1]} + c\|s - p_n\|_{\mathbb{L}_p[-\lambda,\lambda]} \\
&\leq c\mathrm{w}_m(s, n^{-1})_p \leq c\mathrm{w}_m(f, n^{-1})_p \,,
\end{aligned}$$

where in the last inequality we used Assumption 1 and standard inequalities for moduli of smoothness. Finally,

$$\|f - p_n\|_{\mathbb{L}_p[-1,1]} \leq c\|f - s\|_{\mathbb{L}_p[-1,1]} + c\|s - p_n\|_{\mathbb{L}_p[-1,1]} \leq c\mathrm{w}_m(f, n^{-1})_p \,,$$

which completes the proof of the theorem.

4 Weak Co-k-monotone Polynomial Approximation

Given $k \geq 0$ and an interval I, a function f is said to be k-monotone on I if its kth divided differences $[x_0, \ldots, x_k]f$ are nonnegative for all choices of $(k+1)$ distinct points $x_0, \ldots, x_k$ in I. We denote the class of all k-monotone functions on I by $\mathcal{M}^k(I)$.

Let $\mathbb{Y}_\sigma$, $\sigma \geq 1$, be the set of all collections $Y_\sigma := \{y_i\}_{i=1}^\sigma$, such that $y_{\sigma+1} := -1 < y_\sigma < \ldots < y_1 < 1 =: y_0$, and $Y_0 := \{\emptyset\}$. Let $\mathcal{M}^k(Y_\sigma)$ denote the collection of all functions f that change k-monotonicity at the points in Y_σ, and are k-monotone in $[y_1, 1]$, *i.e.*,

$$\mathcal{M}^k(Y_\sigma) := \left\{ f \mid (-1)^i f \in \mathcal{M}^k[y_{i+1}, y_i],\ 0 \leq i \leq \sigma \right\} .$$

(Note that $\mathcal{M}^k(Y_0) = \mathcal{M}^k[-1,1]$.) If $f \in \mathbb{C}^k(-1,1)$, then $f \in \mathcal{M}^k(Y_\sigma)$ if and only if

$$f^{(k)}(x)\Pi(x) \geq 0, \quad x \in (-1,1) \,,$$

where $\Pi(x) := \prod_{i=1}^{\sigma}(x-y_i)$. We say that functions f and g are "co-k-monotone" if they both belong to the same class $\mathcal{M}^k(Y_s)$ (note that it is possible for a function to belong to more than one class $\mathcal{M}^k(Y_s)$, for example, $f \equiv 0$ is in $\mathcal{M}^k(Y_s)$ for *all* sets Y_s).

For an interval $[a,b]$, we also denote

$$\mathcal{M}^k(Y_\sigma)[a,b] := \left\{ f \mid (-1)^i f \in \mathcal{M}^k\left([y_{i+1}, y_i] \cap [a,b]\right),\ 0 \le i \le \sigma \right\} ,$$

and note that $\mathcal{M}^k(Y_\sigma) = \mathcal{M}^k(Y_\sigma)[-1,1]$.

We now introduce the notions of "(weakly) almost", "(weakly) nearly" and "weakly co-k-monotone" functions (see also [4] where somewhat similar notions were introduced for (co)positive and intertwining approximation).

Let $\rho_n(x) := \psi(1/n, x) = n^{-1}\varphi(x) + n^{-2}$, $\beta \ge 0$, $\tau \ge 0$, and denote

$$J_i(n,\beta) := (y_i - \beta\rho_n(y_i), y_i + \beta\rho_n(y_i)) \cap [-1,1], \quad 1 \le i \le \sigma ,$$

$$J_0(n,\tau) := \left(1 - \tau n^{-2}, 1\right] , \quad J_{\sigma+1}(n,\tau) := \left[-1, -1 + \tau n^{-2}\right) ,$$

and

$$O(n,\tau,\beta,Y_\sigma) := \cup_{i=1}^{\sigma} J_i(n,\beta) \cup J_0(n,\tau) \cup J_{\sigma+1}(n,\tau) .$$

We say that functions f and g are **"almost co-k-monotone"** or **"weakly almost co-k-monotone"** if they have the same k-monotonicity on $[-1,1] \setminus O(n,0,\beta,Y_\sigma)$ or on $[-1,1] \setminus O(n,\tau,\beta,Y_\sigma)$, respectively.

Given $f \in \mathcal{M}^k(Y_\sigma)$ we denote the class of all **"weakly almost co-k-monotone"** functions with f by $\mathcal{M}^k_{\mathrm{wa}}(n,\tau,\beta,Y_\sigma)$, *i.e.*,

$$\mathcal{M}^k_{\mathrm{wa}}(n,\tau,\beta,Y_\sigma) := \left\{ f \mid (-1)^i f \in \mathcal{M}^k\left([y_{i+1}, y_i] \setminus O(n,\tau,\beta,Y_\sigma)\right),\ 0 \le i \le \sigma \right\} ,$$

and the class of all **"almost co-k-monotone"** functions with f by

$$\mathcal{M}^k_{\mathrm{a}}(n,\beta,Y_\sigma) := \mathcal{M}^k_{\mathrm{wa}}(n,0,\beta,Y_\sigma) .$$

In particular, we say that a function g is **"weakly co-k-monotone"** with $f \in \mathcal{M}^k(Y_\sigma)$ if f and g are co-k-monotone on $[-1+\tau n^{-2}, 1-\tau n^{-2}]$, and denote

$$\mathcal{M}^k_{\mathrm{w}}(n,\tau,Y_\sigma) := \mathcal{M}^k_{\mathrm{wa}}(n,\tau,0,Y_\sigma) = \mathcal{M}^k(Y_\sigma)[-1+\tau n^{-2}, 1-\tau n^{-2}] .$$

We say that a function g is **"nearly co-k-monotone"** with $f \in \mathcal{M}^k(Y_\sigma)$ if there exists $\widetilde{Y}_\sigma = \{\tilde{y}_i\}_{i=1}^{\sigma} \in \mathbb{Y}_\sigma$ such that

$$|\tilde{y}_i - y_i| \le \beta\rho_n(y_i), \quad 1 \le i \le \sigma , \tag{4.7}$$

and $g \in \mathcal{M}^k(\widetilde{Y}_\sigma)$. Given $f \in \mathcal{M}^k(Y_\sigma)$ we denote the class of all nearly co-k-monotone functions with f by $\mathcal{M}^k_{\mathrm{n}}(n,\beta,Y_\sigma)$, *i.e.*,

$$\mathcal{M}^k_{\mathrm{n}}(n,\beta,Y_\sigma) := \left\{ f \mid f \in \mathcal{M}^k(\widetilde{Y}_\sigma) \text{ for some } \widetilde{Y}_\sigma = \{\tilde{y}_i\}_{i=1}^{\sigma} \text{ s.t. (4.7) holds} \right\} .$$

Finally, we say that g is **"weakly nearly co-k-monotone"** with $f \in \mathcal{M}^k(Y_\sigma)$ if there exists $\widetilde{Y}_\sigma = \{\tilde{y}_i\}_{i=1}^{\sigma} \in \mathbb{Y}_\sigma$ such that (4.7) is satisfied and $g \in \mathcal{M}^k(\widetilde{Y}_\sigma)[-1+\tau n^{-2}, 1-\tau n^{-2}]$, *i.e.*,

$$\mathcal{M}^k_{\mathrm{wn}}(n,\tau,\beta,Y_\sigma) \\ := \left\{ f \mid f \in \mathcal{M}^k(\widetilde{Y}_\sigma)[-1+\tau n^{-2}, 1-\tau n^{-2}] \text{ for some } \widetilde{Y}_\sigma = \{\tilde{y}_i\}_{i=1}^{\sigma} \text{ s.t. (4.7) holds} \right\} .$$

Here are some of the properties of the above classes:

- $\mathcal{M}^k_{\mathrm{a}}(n,\beta,Y_\sigma) \subsetneq \mathcal{M}^k_{\mathrm{wa}}(n,\tau,\beta,Y_\sigma)$ and $\mathcal{M}^k_{\mathrm{n}}(n,\beta,Y_\sigma) \subsetneq \mathcal{M}^k_{\mathrm{wn}}(n,\tau,\beta,Y_\sigma)$ if $\tau > 0$, with "$\subsetneq$" becoming "$=$" if $\tau = 0$;
- $\mathcal{M}^k_{\mathrm{a}}(n,0,Y_\sigma) = \mathcal{M}^k_{\mathrm{n}}(n,0,Y_\sigma) = \mathcal{M}^k_{\mathrm{w}}(n,0,Y_\sigma) = \mathcal{M}^k(Y_\sigma)$;
- $\mathcal{M}^k(Y_\sigma) \subsetneq \mathcal{M}^k_{\mathrm{n}}(n,\beta,Y_\sigma) \subsetneq \mathcal{M}^k_{\mathrm{a}}(n,\beta,Y_\sigma)$ for $\beta > 0$ and $\sigma > 0$;
- $\mathcal{M}^k(Y_\sigma) \subsetneq \mathcal{M}^k_{\mathrm{w}}(n,\tau,Y_\sigma) \subsetneq \mathcal{M}^k_{\mathrm{wn}}(n,\tau,\beta,Y_\sigma) \subsetneq \mathcal{M}^k_{\mathrm{wa}}(n,\tau,\beta,Y_\sigma)$ for $\tau > 0$, $\beta > 0$ and $\sigma > 0$;
- if $\sigma = 0$, then $\mathcal{M}^k_{\mathrm{a}}(n,\beta,Y_0) = \mathcal{M}^k_{\mathrm{n}}(n,\beta,Y_0) = \mathcal{M}^k[-1,1]$ and $\mathcal{M}^k_{\mathrm{wa}}(n,\tau,\beta,Y_0) = \mathcal{M}^k_{\mathrm{wn}}(n,\tau,\beta,Y_0) = \mathcal{M}^k_{\mathrm{w}}(n,\tau,Y_0)$.

Open Problem *Let $0 < p \le \infty$, $Y_\sigma \in \mathbb{Y}_\sigma$, $\sigma \ge 1$, and let $f \in \mathcal{M}^k(Y_\sigma)$. Suppose that a spline or a polynomial p is such that $p \in \mathcal{M}^k_{\mathrm{wa}}(n,\tau,\beta,Y_\sigma)$ for some $\tau \ge 0$ and $\beta > 0$.*

Does there exist a spline or a polynomial q such that $q \in \mathcal{M}^k_{\mathrm{wn}}(n,\tau,\beta,Y_\sigma)$ and

$$\|f-q\|_{\mathbb{L}_p[-1,1]} \le c\,\|f-p\|_{\mathbb{L}_p[-1,1]}\,?$$

Note that if $\tau = 0$, the above becomes an open problem involving $\mathcal{M}^k_{\mathrm{a}}(n,\beta,Y_\sigma)$ and $\mathcal{M}^k_{\mathrm{n}}(n,\beta,Y_\sigma)$, *i.e.*, the classes of almost co-k-monotone and nearly co-k-monotone functions, respectively.

5 Corollaries and applications

For $f \in \mathbb{L}_p$, $0 < p \le \infty$, let

$$E(f,\mathcal{F})_p := \inf_{s\in\mathcal{F}} \|f-s\|_{\mathbb{L}_p[-1,1]} \,,$$

be the error of $\mathbb{L}_p$-approximation of f by elements from the set $\mathcal{F} \subset \mathbb{L}_p$ on $[-1,1]$.

In particular,

$$\widetilde{\mathcal{E}}^{(k)}_r(f,\mathbf{z}_n,J)_p := E(f, \mathcal{S}_r(\mathbf{z}_n) \cap \mathcal{M}^k(J) \cap \mathbb{C}^{r-1})_p$$

and

$$E^{(k)}_n(f,J)_p := E(f, \Pi_n \cap \mathcal{M}^k(J))_p$$

are, respectively, the errors of $\mathbb{L}_p$-approximation of f on $[-1,1]$ by splines from $\mathcal{S}_r(\mathbf{z}_n)\cap \mathbb{C}^{r-1}$ (*i.e.,* having maximum smoothness without becoming polynomials) and by polynomials of degree $\le n$, which are k-monotone with f on $J\subseteq[-1,1]$.

We now state a corollary of Theorem 1.1. For simplicity, we only state it for the classes defined in Section 4 with $\sigma=0$ (*i.e.,* for "weakly k-monotone" classes $\mathcal{M}^k_{\mathrm{w}}(n,\tau,Y_0)=\mathcal{M}^k[-1+\tau n^{-2},1-\tau n^{-2}]$) and note that similar results hold for other classes introduced in the previous section. Additionally, we let $\mathcal{W}^\nu := \mathbb{W}^\nu(\mathbb{L}_p)$ and $\mathbf{z}_n=\mathbf{t}_n$ (and so $\mathrm{w}_m(g^{(\nu)},t)_p := \omega^\varphi_{m,\nu}(g^{(\nu)},t)_p$) and emphasize that these restrictions are only used in order to simplify the statement.

Taking into account that $\left[\mathcal{M}^k(J)\right]_\lambda=\mathcal{M}^k(J_\lambda)$ and $[-\lambda,\lambda]_\lambda=[-1,1]$, the following is an immediate consequence of Theorem 1.1.

Corollary 5.1 (Weak k-monotone approximation) *Let $n,m,\nu\in\mathbb{N}$, $m'\in\mathbb{N}_0$, $m'+\nu\ge m$, $n\ge m'+\nu-1$, $f\in\mathbb{L}_p[-1,1]$, $0<p\le\infty$. Also, for $\tau\ge 0$, let $\lambda:=1-\tau/n^2$ be such that $1/2\le\lambda\le 1$.*

Suppose that the following assumptions are satisfied.

Assumption 1: *for some $r\in\mathbb{N}$, there exists $s\in\mathcal{S}_r(\mathbf{t}_n)\cap\mathcal{M}^k[-\lambda,\lambda]\cap\mathbb{W}^\nu(\mathbb{L}_p)$ such that*

$$\|f-s\|_{\mathbb{L}_p[-1,1]}\le c_1\omega^\varphi_m(f,n^{-1})_p\,.$$

Assumption 2: *for any function $g\in\mathbb{W}^\nu(\mathbb{L}_p)\cap\mathcal{M}^k[-1,1]$,*

$$E^{(k)}_n(g,[-1,1])_p\le c_2 n^{-\nu}\omega^\varphi_{m',\nu}(g^{(\nu)},n^{-1})_p\,.$$

Then

$$E^{(k)}_n(f,[-\lambda,\lambda])_p\le c\,\omega^\varphi_m(f,n^{-1})_p\,,$$

where $c=c(c_1,c_2,\tau,r,m,m',\nu,p)$.

The following theorems follows from [10, Theorem 1.1] and [7, Theorems 3-4].

Theorem 5.2 ([10]) *Let $f\in\mathcal{M}^k[-1,1]\cap\mathbb{L}_p$, $0<p\le\infty$, $k=1,2$, and $r\ge k+1$. Then, there exists a constant $\tau=\tau(r)>0$, such that for every $n\in\mathbb{N}$,*

$$\widetilde{\mathcal{E}}^{(k)}_r(f,\mathbf{t}_n,[-1+\tau n^{-2},1-\tau n^{-2}])_p\le c\,\omega^\varphi_{k+2}(f,1/n)_p, \tag{5.8}$$

where c are constants independent of f and n which may depend on r and on p as $p\to 0$.

Theorem 5.3 ([7]) *Let $m\in\mathbb{N}$, $k=1,2$, $\nu\in\mathbb{N}$, $\nu\ge 2k+1$, and $f\in\mathcal{M}^k[-1,1]\cap\mathbb{C}[-1,1]\cap\mathbb{C}^\nu(-1,1)$ be such that $\left\|\varphi^\nu f^{(\nu)}\right\|_\infty<\infty$. Then for every $n\ge m+\nu-1$,*

$$E(f,\Pi_n\cap\mathcal{M}^k[-1,1])_{\mathbb{C}[-1,1]}\le c(m,\nu)n^{-\nu}\bar{\omega}^\varphi_{m,\nu}(f^{(\nu)},n^{-1})_\infty\,.$$

Corollary 5.4 (see [13,14]**)** *Let $f\in\mathcal{M}^k[-1,1]\cap\mathbb{C}$, $k=1,2$. Then, there exists an absolute constant $\tau>0$, such that for every $n\ge k+1$,*

$$E^{(k)}_n(f,[-1+\tau n^{-2},1-\tau n^{-2}])_\infty\le c\,\omega^\varphi_{k+2}(f,1/n)_\infty. \tag{5.9}$$

For $n \geq 2k+1$, Corollary 5.4 immediately follows from Theorems 5.2 and 5.3 and Corollary 5.1 with $m' = 1$, $\nu = 2k+1$, $m = k+2$ and $r = 2k+2$. For $k+1 \leq n < 2k+1$, Corollary 5.4 becomes the usual unconstained Jackson type estimate if one requires that τ is sufficiently large ($\tau \geq 16$ will do).

We also note that it is possible to use this approach to obtain similar results for (weakly) almost/nearly co-k-comonotone polynomial approximation (with $\sigma \geq 1$ and $k = 1, 2$) in $\mathbb{L}_\infty$ by constructing piecewise polynomial functions (having minimal order and smoothness), smoothing them preserving needed constrains using the approach from [11] (and hence verifying Assumption 1 in Theorem 1.1), and then applying Theorem 1.1 (with Assumption 2 verified using known results on (pure) co-k-comonotone polynomial approximation of smooth functions).

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