Moduli of Smoothness of Splines and Applications in Constrained Approximation

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Abstract

In this paper, we generalize [8, Theorem 4.1] to be applicable to the classes of (almost) weakly/nearly (co)-$k$-monotone functions and discuss some applications and open problems.

1 Introduction and Main Results

Let $S_r(z_n)$ be the space of all piecewise polynomial functions (ppf’s) of degree $r$ (order $r+1$) with the knots $z_n := (z_i)_i^{n}$, $-1 := z_0 < z_1 < \ldots < z_{n-1} < z_n := 1$. In other words, we say that $s \in S_r(z_n)$ if, on each interval $(z_i, z_{i+1})$, $0 \leq i \leq n-1$, $s$ is in $\Pi_r$, where $\Pi_r$ denotes the space of algebraic polynomials of degree $\leq r$.

For a partition $z_n := \{z_0, \ldots, z_n\} =: z_{-1} < z_1 < \ldots < z_{n-1} < z_n := 1$, let $J_j := [z_j, z_{j+1}]$ with $z_j := -1$, $j < 0$, and $z_j := 1$, $j > n$, and $|J| := \text{meas } J$.

Given an absolute constant $\Delta$ we say that $z_n$ is “$\Delta$-quasi-uniform” if $\Delta(z_n) := \max_{0 \leq j \leq n-1} |J_j| / \min_{0 \leq j \leq n-1} |J_j| \leq \Delta$, and denote by $U^\Delta_n$ the class of all such partitions. (Note that $U^1_n$ consists of only one partition which is the uniform partition of $[-1, 1]$ into $n$ subintervals of equal lengths.)

We also use the notation $t_n = (t_i)_i^{n}$, where $t_i := -\cos (\pi i/n)$, $0 \leq i \leq n$, for the Chebyshev partition of $[-1, 1]$.

As usual, $L_p(J)$, $0 < p \leq \infty$, denotes the space of all measurable functions $f$ on $J$ such that $\|f\|_{L_p(J)} < \infty$, where $\|f\|_{L_p(J)} := \left( \int_J |f(x)|^p \, dx \right)^{1/p}$ if $0 < p < \infty$, and $\|f\|_{L_\infty(J)} := \text{ess sup}_{x \in J} |f(x)|$, and write $L_p := L_p([-1, 1])$ and $\| \cdot \|_p := \| \cdot \|_{L_p([-1, 1])}$. We say that a function $f$ is in the Sobolev Space $W^\nu p(L_p)$ if it has an absolutely continuous $(\nu - 1)$st derivative such that $f^{(\nu)} \in L_p$.

The $k$th symmetric difference is $\Delta^k_h(f, x, J) := \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f(x - kh/2 + ih)$, if $x + kh/2 \in J$, and $\Delta^k_h(f, x, J) := 0$, otherwise. The $k$th modulus of smoothness of a

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function $f \in \mathbb{L}_p(J)$ is defined by

$$
\omega_k(f, t, J)_p := \sup_{0 < h \leq t} \| \Delta^k_{f, \cdot}(f) \|_{\mathbb{L}_p(J)},
$$

and $\omega_k(f, t)_p := \omega_k(f, t, [-1, 1])_p$.

The (usual) Ditzian-Totik modulus $k$th modulus of smoothness (see [3]) is

$$
\omega^\varphi_k(f, t)_p := \sup_{0 < h \leq t} \| \Delta^k_{\varphi, \cdot}(f, \cdot) \|_p,
$$

where $\varphi(x) := \sqrt{1 - x^2}$.

We also need the weighted Ditzian-Totik $k$th modulus of smoothness of a function $f \in \mathbb{L}_p[-1, 1]$, $0 < p \leq \infty$, which we define as

$$
\omega^\varphi_k(f, t)_{W,p} := \sup_{0 < h \leq t} \| W(\cdot, kh/2) \Delta^k_{\varphi, \cdot}(f, \cdot) \|_p.
$$

where $W$ is some weight function. Note that $\Delta^k_{\varphi(x)}(f, x, [-1, 1])$ is defined to be identically 0 if $x \notin \mathcal{D}_{kh/2}$, where

$$
(1.1) \quad \mathcal{D}_\delta := \{ x \mid 1 - \delta \varphi(x) \geq |x| \} \setminus \{ \pm 1 \} = \left\{ x \mid |x| \leq \frac{1 - \delta^2}{1 + \delta^2} \right\},
$$

and so $W$ should only be defined on $\mathcal{D}_{kh/2}$.

In this paper, we use weighted moduli with the weights (see [8, 12])

$$
W_1(x, kh/2) := \varphi'(x) \quad \text{and} \quad W_2(x, \mu) := \varphi'(|x| + \mu \varphi(x)),
$$

and denote

$$
\omega^\varphi_{k,\nu}(f, t)_p := \omega^\varphi_k(f, t)_{W_1,p} = \sup_{0 < h \leq t} \| \varphi'(\cdot) \Delta^k_{h\varphi,\cdot}(f, \cdot) \|_p
$$

and

$$
\bar{\omega}^\varphi_{k,\nu}(f, t)_p := \omega^\varphi_k(f, t)_{W_2,p} = \sup_{0 < h \leq t} \| \varphi'(\cdot) + kh\varphi(\cdot)/2 \Delta^k_{h\varphi,\cdot}(f, \cdot) \|_p.
$$

Also, note that

$$
\omega^\varphi_{k,0}(f, t)_p = \bar{\omega}^\varphi_{k,0}(f, t)_p = \| \varphi' f \|_p \quad \text{and} \quad \omega^\varphi_{k,0}(f, t)_p = \bar{\omega}^\varphi_{k,0}(f, t)_p = \omega^\varphi_k(f, t)_p.
$$

Clearly,

$$
(1.2) \quad \bar{\omega}^\varphi_{k,\nu}(f, t)_p \leq \omega^\varphi_{k,\nu}(f, t)_p.
$$

We also emphasize that $\varphi' f \in \mathbb{L}_p$ does NOT imply that $\omega^\varphi_{k,\nu}(f, t)_p < \infty$ (consider $f(x) = (1 + x)^{-1/p}$, $k = \nu = 1$ for $0 < p < \infty$, and $f(x) = (1 + x)^{-1}$, $\nu = 2$, $k = 1$ for $p = \infty$). At the same time, if $\varphi' f \in \mathbb{L}_p$, then $\bar{\omega}^\varphi_{k,\nu}(f, t)_p < \infty$.

Given a set (usually, an interval) $J$, let $\mathcal{M}(J)$ be a “constraints class” of functions defined on $J$. For example, $\mathcal{M}(J)$ could be the class of all monotone or convex functions on $J$, or the class of functions changing their $k$-monotonicity the given number of times, or a class of functions satisfying some interpolation conditions, or having their range
restricted on $J$, or some class of functions having certain other shape characteristics on various subsets of $J$, etc. If $J$ is an interval we abuse the notation and omit parantheses in this notation, i.e., $\mathcal{M}[a,b] := \mathcal{M}([a,b])$.

Let $J_\lambda := \{x/\lambda \mid x \in J\}$ and, in particular, $[a,b]_\lambda := [a/\lambda,b/\lambda]$. Given a constraints class $\mathcal{M}(J)$ and a parameter $\lambda > 0$, we denote by $[\mathcal{M}(J)]_\lambda$ the class of all functions which are defined on $J_\lambda$ and such that

$$f \in \mathcal{M}(J) \quad \text{if and only if} \quad f_\lambda \in [\mathcal{M}(J)]_\lambda,$$

where $f_\lambda := f(\lambda \cdot)$. Hence, since $(f_\lambda)_{1/\lambda} \equiv f$, the above is equivalent to:

$$f \in [\mathcal{M}(J)]_\lambda \quad \text{if and only if} \quad f_{1/\lambda} \in \mathcal{M}(J).$$

For example, if $\mathcal{M}^2(J)$ denotes the class of all convex functions on $J$, then $[\mathcal{M}^2(J)]_\lambda = \mathcal{M}^2(J_\lambda)$ is the class of all convex functions on $J_\lambda$. Note, however, that, in general, $[\mathcal{M}(J)]_\lambda \neq \mathcal{M}(J_\lambda)$. For example, if $\tilde{\mathcal{M}}_{1/4}[-1,1]$ is the set of all functions which are defined on $[-1,1]$ and have an inflection point at $1/4$, then functions in $\tilde{\mathcal{M}}_{1/4}([-1,1]_{1/2}) = \tilde{\mathcal{M}}_{1/4}[-2,2]$ have their inflection points at $1/2$ while functions from $\tilde{\mathcal{M}}_{1/4}(([-1,1]_{1/2}) = \tilde{\mathcal{M}}_{1/4}[-2,2]$ still have their inflection points at $1/4$.

In order to unify the statements for different partitions, following [8, (4.1)], we define

$$w_m(g^{(\nu)}, t)_p := \begin{cases} \omega_m(g^{(\nu)}, t)_p, & \text{if } z_n \in U_n^\Delta, \\ \omega_{m,\nu}(g^{(\nu)}, t)_p, & \text{if } z_n = t_n. \end{cases} \quad (1.3)$$

The following theorem is our main result.

**Theorem 1.1** Let $n, m, \nu \in \mathbb{N}$, $m' \in \mathbb{N}_0$, $m' + \nu \geq m$, $n \geq m' + \nu - 1$, $f \in L^p_{-1,1}$, $0 < p \leq \infty$, and let $W^\nu \subset W^\nu(L^p)$ be some class of functions such that $f(\mu \cdot) =: f_\mu \in W^\nu$ whenever $f \in W^\nu$ and $\mu > 0$. Also, for $\tau \geq 0$, let $\lambda := 1 - \tau/\nu^2$ be such that $1/2 \leq \lambda \leq 1$.

Additionally, let $\Delta \in \mathbb{R}$ and $z_n$ be either a $\Delta$-quasi-uniform or the Chebyshev partition of $[-1,1]$ into $n$ intervals, i.e., $z_n \in U_n^\Delta$ or $z_n = t_n$. Suppose that the following assumptions are satisfied.

**Assumption 1:** for some $r \in \mathbb{N}$, there exists $s \in S_r(z_n) \cap \mathcal{M}[-\lambda, \lambda] \cap W^\nu$ such that

$$\|f - s\|_{L^p_{-1,1}} \leq c_1 w_m(f, n^{-1})_p.$$

**Assumption 2:** for any function $g \in W^\nu \cap [\mathcal{M}[-\lambda, \lambda]]_\lambda$ there exists a polynomial $q_n \in \Pi_n \cap [\mathcal{M}[-\lambda, \lambda]]_\lambda$, such that

$$\|g - q_n\|_{L^p_{-1,1}} \leq c_2 n^{-\nu} w_m(g^{(\nu)}, n^{-1})_p.$$

Then there exists a polynomial $p_n \in \Pi_n \cap \mathcal{M}[-\lambda, \lambda]$ such that

$$\|f - p_n\|_{L^p_{-1,1}} \leq c w_m(f, n^{-1})_p,$$

where $c$ depends on $c_1, c_2, r, m, m', \nu, p$, and also on $\Delta$ if $z_n \in U_n^\Delta$. 

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Lemma 2.2 (Remez inequality) and A.4.10, for example.

Throughout this paper, $c(\gamma_1, \gamma_2, \ldots)$ denote positive constants which depend only on the parameters $\gamma_1, \gamma_2, \ldots$ (note that $c(p, \ldots)$ depends on $p$ only as $p \to 0$) and which may be different on different occurrences. At the same time, $c_\mu$ denote positive constants which are fixed throughout the paper.

2 Auxiliary Results

Theorem 2.1 ([8,9]) Let $z_n$ be either a $\Delta$-quasi-uniform or the Chebyshev partition of $[-1,1]$ into $n$ intervals, i.e., $z_n \in U^\Delta_n$ or $z_n = t_n$, and let $s \in S_r(z_n) \cap \mathbb{C}^m[-1,1]$, $r \in \mathbb{N}$, $0 \leq m \leq r - 1$. Then, for any $1 \leq k \leq r + 1$, $1 \leq \nu \leq \min\{k, m + 1\}$ and $0 < p \leq \infty$, we have

$$ n^{-\nu}w_{k-\nu}(s^{(\nu)}, n^{-1})_p \sim w_k(s, n^{-1})_p, $$

with equivalence constants depending only on $r$, $\Delta$ (if $z_n \in U^\Delta_n$) and $p$ as $p \to 0$.

The following lemma is rather well known and can be found in [1, Theorems A.4.1 and A.4.10], for example.

Lemma 2.2 (Remez inequality) For any $q \in \Pi_n$ and a set $A$ such that $\text{meas}\{[-1,1] \setminus A\} \leq s \leq 1/2$ the following inequalities hold:

$$ \|q\|_{C[-1,1]} \leq e^{5n\sqrt{s}} \|q\|_{C(A)} $$

and

$$ \|q\|_{L^p[-1,1]} \leq \left(1 + e^{8pn\sqrt{s}}\right)^{1/p} \|q\|_{L^p(A)}, \quad 0 < p < \infty. $$

Corollary 2.3 For any $q \in \Pi_n$, a set $A$ such that $\text{meas}\{[-1,1] \setminus A\} \leq s \leq 1/2$, and $0 < p \leq \infty$,

$$ \|q\|_{L^p[-1,1]} \leq 2^{1+1/p}e^{8n\sqrt{s}} \|q\|_{L^p(A)}. $$

In particular, for any $0 < \tau \leq n^2/4$,

$$ \|q\|_{L^p[-1,1]} \leq c(\tau, p) \|q\|_{L^p[-1+\tau/n^2,1-\tau/n^2]}. $$

(2.5)

Lemma 2.4 Suppose that $f \in L^p[-1,1]$, $0 < p \leq \infty$, $\tau \geq 0$, $\lambda := 1 - \tau t^2$ is such that $\lambda \geq 1/2$, and $f_\lambda(x) := f(\lambda x)$. Then,

$$ \omega^\Delta_k(f_\lambda, t)_p \leq c(\tau, p) \omega^\Delta_k(f, t)_p. $$

(2.6)

We remark that the condition $\lambda \geq 1/2$ is not essential and $1/2$ can be replaced by any positive constant $c_0$ (the constant $c$ in (2.6) will then depend on $c_0$ as well). It also immediately follows from the definition that the statement of this lemma holds for the usual $k$th modulus, i.e., $\omega_k(f_\lambda, t)_p \leq c(p)\omega_k(f, t)_p$. 

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Proof. In the proof, it is more convenient to work with the Ivanov moduli which are equivalent to the Ditzian-Totik moduli. Recall that, the Ivanov modulus of smoothness is defined by (see [5, 6])

\[ \tau_k(f, \psi(t))_{p,p} := \|\omega_k(f, \cdot, \psi(t, \cdot))\|_p, \]

where

\[ \omega_k(f, x, \psi(t, x))^p := \frac{1}{2\psi(t, x)} \int_{\psi(t, x)}^{\psi(t, x)} |\Delta^k_h(f, x + kh/2, [-1, 1])|^p dh \]

with \( \psi(t, x) := t^2 + xt \). It is known (see [15] and [2]) that, for all \( 0 < p \leq \infty \), \( \tau_k(f, \psi(t))_{p,p} \sim \omega^p_k(f, t, p) \) with equivalence constants depending only on \( k \) and \( p \).

Changing variables we have

\[
\tau_k(f, \psi(t))_{p,p} = \int_{-1}^{1} \int_{\psi(t,x)}^{\psi(t,x)} \frac{1}{2\psi(t, x)} |\Delta^k_h(f, \lambda x + k\lambda h/2, [-1, 1])|^p dh dx \\
= \int_{-\lambda}^{\lambda} \int_{-\lambda\psi(t, x/\lambda)}^{\lambda\psi(t, x/\lambda)} \frac{1}{2\lambda^2 \psi(t, x/\lambda)} |\Delta^k_h(f, x + kh/2, [-1, 1])|^p dh dx.
\]

Now, \( \lambda \psi(t, x/\lambda) = t^2 + x^2 + \lambda t^2 \leq \psi(t, x) \). It is also a simple exercise to show that, for \( \lambda = 1 - \tau t^2 \) such that \( \lambda \geq 1/2 \), \( \lambda^2 \psi(t, x/\lambda) \geq c(\tau) \psi(t, x) \) for all \( x \in [-\lambda, \lambda] \). Therefore,

\[
\tau_k(f, \psi(t))_{p,p} \leq c(\tau) \int_{-1}^{1} \int_{\psi(t,x)}^{\psi(t,x)} \frac{1}{2\psi(t, x)} |\Delta^k_h(f, x + kh/2, [-1, 1])|^p dh dx \\
= c(\tau) \tau_k(f, \psi(t))_{p,p},
\]

which completes the proof. \( \square \)

3 Proof of Theorem 1.1

In the case \( \tau = 0 \), the statement of the theorem immediately follows from [8, Theorem 4.1], since \( [\mathcal{M}(J)]_1 = \mathcal{M}(J) \). Hence, we only need to consider the case \( \tau > 0 \).

Assumption 1 guarantees that there exists \( s \in S_r(z_n) \cap \mathcal{M}[-\lambda, \lambda] \cap W^\nu \) is such that

\[ \|f - s\|_{L^p_{[-1,1]}} \leq c_1 \omega_m(f, n^{-1})_p. \]

Consider the function \( s_\lambda := s(\lambda) \). The same proof as in [8, Section 3.3] shows that (2.4) is valid with \( s \) replaced by \( s_\lambda \).

Now, since \( s_\lambda \in W^\nu \cap [\mathcal{M}[-\lambda, \lambda]], \) Assumption 2 implies that there exists a polynomial \( q_n \in \Pi_n \cap [\mathcal{M}[-\lambda, \lambda]]_\lambda \) such that

\[ \|s_\lambda - q_n\|_{L^p_{[-1,1]}} \leq c_2 n^{-\nu} \omega_m(s_\lambda(n^{-1}))_p \leq c_3 \omega_m(s_\lambda(n^{-1}))_p \leq c_4 \omega_m(s, n^{-1})_p, \]

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where the last inequality follows from Lemma 2.4 and the remark after its statement. Hence, recalling that \( m' + \nu \geq m \), we have

\[
\| s_\lambda - q_n \|_{L^p[-1,1]} \leq cw_m(s, n^{-1})_p.
\]

Therefore, for \( p_n := q_n(\cdot / \lambda) \), using the fact that \( [[\mathcal{M}(J)]_{\lambda}]_{1/\lambda} = \mathcal{M}(J) \) we have \( p_n \in \Pi_n \cap \mathcal{M}[-\lambda, \lambda] \), and

\[
\| s - p_n \|_{L^p[-\lambda,\lambda]} = \lambda^{1/p} \| s_\lambda - q_n \|_{L^p[-1,1]} \leq cw_m(s, n^{-1})_p.
\]

Suppose now that a polynomial \( P_n \in \Pi_n \) is such that

\[
\| s - P_n \|_{L^p[-1,1]} \leq cw_m(s, n^{-1})_p
\]

(a polynomial of best approximation to \( s \) will do). Then, by the Remez inequality (2.5),

\[
\| P_n - p_n \|_{L^p[-1,1]} \leq c(\tau, p) \| P_n - p_n \|_{L^p[-\lambda,\lambda]},
\]

and hence

\[
\| s - p_n \|_{L^p[-1,1]} \leq c \| s - P_n \|_{L^p[-1,1]} + c \| P_n - p_n \|_{L^p[-1,1]}
\leq c \| s - P_n \|_{L^p[-1,1]} + c \| P_n - s \|_{L^p[-\lambda,\lambda]} + c \| s - p_n \|_{L^p[-\lambda,\lambda]}
\leq c \| s - P_n \|_{L^p[-1,1]} + c \| s - p_n \|_{L^p[-\lambda,\lambda]}
\leq cw_m(s, n^{-1})_p \leq cw_m(f, n^{-1})_p,
\]

where in the last inequality we used Assumption 1 and standard inequalities for moduli of smoothness. Finally,

\[
\| f - p_n \|_{L^p[-1,1]} \leq c \| f - s \|_{L^p[-1,1]} + c \| s - p_n \|_{L^p[-1,1]} \leq cw_m(f, n^{-1})_p,
\]

which completes the proof of the theorem.

## 4 Weak Co-\( k \)-monotone Polynomial Approximation

Given \( k \geq 0 \) and an interval \( I \), a function \( f \) is said to be \( k \)-monotone on \( I \) if its \( k \)th divided differences \([x_0, \ldots, x_k]f\) are nonnegative for all choices of \((k + 1)\) distinct points \( x_0, \ldots, x_k \) in \( I \). We denote the class of all \( k \)-monotone functions on \( I \) by \( \mathcal{M}^k(I) \).

Let \( \mathbb{Y}_\sigma, \sigma \geq 1 \), be the set of all collections \( Y_\sigma := \{y_i\}_{i=1}^\sigma \), such that \( y_{\sigma+1} := -1 < y_\sigma < \ldots < y_1 < 1 =: y_0 \), and \( Y_0 := \{\emptyset\} \). Let \( \mathcal{M}^k(Y_\sigma) \) denote the collection of all functions \( f \) that change \( k \)-monotonicity at the points in \( Y_\sigma \), and are \( k \)-monotone in \( [y_1, 1] \), i.e.,

\[
\mathcal{M}^k(Y_\sigma) := \{ f \mid (-1)^i f \in \mathcal{M}^k[y_{i+1}, y_i], \ 0 \leq i \leq \sigma \}.
\]

(Note that \( \mathcal{M}^k(Y_0) = \mathcal{M}^k[-1,1] \). If \( f \in \mathcal{C}^k(-1,1) \), then \( f \in \mathcal{M}^k(Y_\sigma) \) if and only if \( f^{(k)}(x) \Pi(x) \geq 0, \ x \in (-1,1) \),
where \( \Pi(x) := \prod_{i=1}^{k} (x - y_i) \). We say that functions \( f \) and \( g \) are “co-\( k \)-monotone” if they both belong to the same class \( \mathcal{M}^{k}(Y_{\sigma}) \) (note that it is possible for a function to belong to more than one class \( \mathcal{M}^{k}(Y_{\sigma}) \), for example, \( f \equiv 0 \) is in \( \mathcal{M}^{k}(Y_{\sigma}) \) for all sets \( Y_{\sigma} \).

For an interval \([a, b]\), we also denote

\[
\mathcal{M}^{k}(Y_{\sigma})[a, b] := \{ f \mid (-1)^{i} f \in \mathcal{M}^{k}([y_{i+1}, y_{i}] \cap [a, b]), \ 0 \leq i \leq \sigma \},
\]

and note that \( \mathcal{M}^{k}(Y_{\sigma}) = \mathcal{M}^{k}(Y_{\sigma})[-1, 1] \).

We now introduce the notions of “(weakly) almost”, “(weakly) nearly” and “weakly co-\( k \)-monotone” functions (see also [4] where somewhat similar notions were introduced for (co)positive and intertwining approximation).

Let \( \rho_{n}(x) := \psi(1/n, x) = n^{-1} \varphi(x) + n^{-2}, \beta \geq 0, \tau \geq 0, \) and denote

\[
J_{i}(n, \beta) := (y_{i} - \beta \rho_{n}(y_{i}), y_{i} + \beta \rho_{n}(y_{i})) \cap [-1, 1], \ 1 \leq i \leq \sigma,
\]

\[
J_{0}(n, \tau) := (1 - \tau n^{-2}, 1], \quad J_{\sigma+1}(n, \tau) := [-1, -1 + \tau n^{-2})
\]

and

\[
O(n, \tau, \beta, Y_{\sigma}) := \bigcup_{i=1}^{\sigma} J_{i}(n, \beta) \cup J_{0}(n, \tau) \cup J_{\sigma+1}(n, \tau).
\]

We say that functions \( f \) and \( g \) are “almost co-\( k \)-monotone” or “weakly almost co-\( k \)-monotone” if they have the same \( k \)-monotonicity on \([-1, 1] \setminus O(n, 0, \beta, Y_{\sigma}) \) or on \([-1, 1] \setminus O(n, \tau, \beta, Y_{\sigma}) \), respectively.

Given \( f \in \mathcal{M}^{k}(Y_{\sigma}) \) we denote the class of all “weakly almost co-\( k \)-monotone” functions with \( f \) by \( \mathcal{M}_{wa}^{k}(n, \tau, \beta, Y_{\sigma}) \), i.e.,

\[
\mathcal{M}_{wa}^{k}(n, \tau, \beta, Y_{\sigma}) := \{ f \mid (-1)^{i} f \in \mathcal{M}^{k}([y_{i+1}, y_{i}] \setminus O(n, \tau, \beta, Y_{\sigma})), \ 0 \leq i \leq \sigma \},
\]

and the class of all “almost co-\( k \)-monotone” functions with \( f \) by

\[
\mathcal{M}_{a}^{k}(n, \beta, Y_{\sigma}) := \mathcal{M}_{wa}^{k}(n, 0, \beta, Y_{\sigma}).
\]

In particular, we say that a function \( g \) is “weakly co-\( k \)-monotone” with \( f \in \mathcal{M}^{k}(Y_{\sigma}) \) if \( f \) and \( g \) are co-\( k \)-monotone on \([-1 + \tau n^{-2}, 1 - \tau n^{-2}] \), and denote

\[
\mathcal{M}_{w}^{k}(n, \tau, Y_{\sigma}) := \mathcal{M}_{wa}^{k}(n, 0, Y_{\sigma}) = \mathcal{M}^{k}(Y_{\sigma})[-1 + \tau n^{-2}, 1 - \tau n^{-2}].
\]

We say that a function \( g \) is “nearly co-\( k \)-monotone” with \( f \in \mathcal{M}^{k}(Y_{\sigma}) \) if there exists \( \bar{Y}_{\sigma} = \{ \bar{y}_{i} \}_{i=1}^{\sigma} \in \mathcal{Y}_{\sigma} \) such that

\[
\text{(4.7)} \quad |\bar{y}_{i} - y_{i}| \leq \beta \rho_{n}(y_{i}), \quad 1 \leq i \leq \sigma,
\]

and \( g \in \mathcal{M}^{k}(\bar{Y}_{\sigma}) \). Given \( f \in \mathcal{M}^{k}(Y_{\sigma}) \) we denote the class of all nearly co-\( k \)-monotone functions with \( f \) by \( \mathcal{M}_{n}^{k}(n, \beta, Y_{\sigma}) \), i.e.,

\[
\mathcal{M}_{n}^{k}(n, \beta, Y_{\sigma}) := \left\{ f \mid f \in \mathcal{M}^{k}(\bar{Y}_{\sigma}) \text{ for some } \bar{Y}_{\sigma} = \{ \bar{y}_{i} \}_{i=1}^{\sigma} \text{ s.t. (4.7) holds} \right\}.
\]
Finally, we say that \( g \) is “weakly nearly co-k-monotone” with \( f \in M^k(Y_\sigma) \) if there exists \( \tilde{Y}_\sigma = \{\tilde{y}_i\}_{i=1}^\sigma \subset Y_\sigma \) such that (4.7) is satisfied and \( g \in M^k(\tilde{Y}_\sigma)[-1+\tau n^{-2}, 1-\tau n^{-2}] \), i.e.,

\[
\mathcal{M}_{wn}^k(n, \tau, \beta, Y_\sigma) := \left\{ f \mid f \in \mathcal{M}^k(\tilde{Y}_\sigma)[-1+\tau n^{-2}, 1-\tau n^{-2}] \text{ for some } \tilde{Y}_\sigma = \{\tilde{y}_i\}_{i=1}^\sigma \text{ s.t. } (4.7) \text{ holds} \right\}.
\]

Here are some of the properties of the above classes:

- \( \mathcal{M}_n^k(n, \beta, Y_\sigma) \subset \mathcal{M}_{wa}^k(n, \tau, \beta, Y_\sigma) \) and \( \mathcal{M}_n^k(n, \beta, Y_\sigma) \subset \mathcal{M}_{wn}^k(n, \tau, \beta, Y_\sigma) \) if \( \tau > 0 \), with “\( \subset \)” becoming “\( = \)” if \( \tau = 0 \);
- \( \mathcal{M}_n^k(n, 0, Y_\sigma) = \mathcal{M}_w^k(n, 0, Y_\sigma) = \mathcal{M}_n^k(n, 0, Y_\sigma) = \mathcal{M}_w^k(Y_\sigma) \);
- \( \mathcal{M}^k(Y_\sigma) \subset \mathcal{M}_n^k(n, \beta, Y_\sigma) \subset \mathcal{M}_w^k(n, \beta, Y_\sigma) \) for \( \beta > 0 \) and \( \sigma > 0 \);
- \( \mathcal{M}^k(Y_\sigma) \subset \mathcal{M}_{wn}^k(n, \tau, \beta, Y_\sigma) \subset \mathcal{M}_{wa}^k(n, \tau, \beta, Y_\sigma) \subset \mathcal{M}_{wn}^k(n, \tau, \beta, Y_\sigma) \) for \( \tau > 0 \), \( \beta > 0 \) and \( \sigma > 0 \);
- if \( \sigma = 0 \), then \( \mathcal{M}_n^k(n, \beta, Y_0) = \mathcal{M}_w^k(n, \beta, Y_0) = \mathcal{M}^k[-1, 1] \) and \( \mathcal{M}_{wa}^k(n, \tau, \beta, Y_0) = \mathcal{M}_{wn}^k(n, \tau, \beta, Y_0) = \mathcal{M}_w^k(n, \tau, Y_0) \).

**Open Problem** Let \( 0 < \rho \leq \infty, Y_\sigma \in Y_\sigma, \sigma \geq 1, \) and let \( f \in \mathcal{M}^k(Y_\sigma) \). Suppose that a spline or a polynomial \( p \) is such that \( p \in \mathcal{M}_{wa}^k(n, \tau, \beta, Y_\sigma) \) for some \( \tau \geq 0 \) and \( \beta > 0 \).

Does there exist a spline or a polynomial \( q \) such that \( q \in \mathcal{M}_{wn}^k(n, \tau, \beta, Y_\sigma) \) and

\[
\| f - q \|_{L_\rho[-1, 1]} \leq c \| f - p \|_{L_\rho[-1, 1]} ?
\]

Note that if \( \tau = 0 \), the above becomes an open problem involving \( \mathcal{M}_n^k(n, \beta, Y_\sigma) \) and \( \mathcal{M}_w^k(n, \beta, Y_\sigma) \), i.e., the classes of almost co-k-monotone and nearly co-k-monotone functions, respectively.

## 5 Corollaries and applications

For \( f \in L_\rho, 0 < \rho \leq \infty \), let

\[
E(f, \mathcal{F})_\rho := \inf_{s \in \mathcal{F}} \| f - s \|_{L_\rho[-1, 1]},
\]

be the error of \( L_\rho \)-approximation of \( f \) by elements from the set \( \mathcal{F} \subset L_\rho \) on \([-1, 1]\).

In particular,

\[
\tilde{E}_r^{(k)}(f, z_n, J)_\rho := E(f, s_r(z_n) \cap M^k(J) \cap C^{r-1})_\rho
\]

and

\[
E_n^{(k)}(f, J)_\rho := E(f, \Pi_n \cap M^k(J))_\rho
\]
are, respectively, the errors of $L_p$-approximation of $f$ on $[-1,1]$ by splines from $S_r(z_n) \cap C^{r-1}$ (i.e., having maximum smoothness without becoming polynomials) and by polynomials of degree $\leq n$, which are $k$-monotone with $f$ on $J \subseteq [-1,1]$.

We now state a corollary of Theorem 1.1. For simplicity, we only state it for the classes defined in Section 4 with $\sigma = 0$ (i.e., for “weakly $k$-monotone” classes $M^k_w(n, \tau, Y_0) = M^k[-1 + \tau n^{-2}, 1 - \tau n^{-2}]$) and note that similar results hold for other classes introduced in the previous section. Additionally, we let $W^\nu := \overline{W^\nu}(L_p)$ and $z_n = t_n$ (and so $w_m(g^{(\nu)}, t)_p := \omega_{m,\nu}^\phi(g^{(\nu)}, t)_p$) and emphasize that these restrictions are only used in order to simplify the statement.

Taking into account that $[M^k(J)]_{\lambda} = M^k(J, \lambda)$ and $[-\lambda, \lambda]_{\lambda} = [-1,1]$, the following is an immediate consequence of Theorem 1.1.

**Corollary 5.1 (Weak $k$-monotone approximation)** Let $n, m, \nu \in \mathbb{N}$, $m' \in \mathbb{N}_0$, $m' + \nu \geq m$, $n \geq m' + \nu - 1$, $f \in L_p[-1,1]$, $0 < p \leq \infty$. Also, for $\tau \geq 0$, let $\lambda := 1 - \tau/n^2$ be such that $1/2 \leq \lambda \leq 1$.

Suppose that the following assumptions are satisfied.

**Assumption 1:** for some $r \in \mathbb{N}$, there exists $s \in S_r(t_n) \cap M^k[-\lambda, \lambda] \cap W^\nu(L_p)$ such that

$$
\|f - s\|_{L_p[-1,1]} \leq c_1 \omega_m^\phi(f, n^{-1})_p.
$$

**Assumption 2:** for any function $g \in W^\nu(L_p) \cap M^k[-1,1],$

$$
E_n^{(k)}(g, [-1,1])_p \leq c_2 n^{-\nu} \omega_{m',\nu}^\phi(g^{(\nu)}, n^{-1})_p.
$$

Then

$$
E_n^{(k)}(f, [-\lambda, \lambda])_p \leq c \omega_m^\phi(f, n^{-1})_p,
$$

where $c = c(c_1, c_2, \tau, r, m, m', \nu, p)$.

The following theorems follows from [10, Theorem 1.1] and [7, Theorems 3-4].

**Theorem 5.2 ([10])** Let $f \in M^k[-1,1] \cap L_p$, $0 < p \leq \infty$, $k = 1, 2$, and $r \geq k + 1$. Then, there exists a constant $\tau = \tau(r) > 0$, such that for every $n \in \mathbb{N}$,

$$
\overline{E}_n^{(k)}(f, t_n, [-1 + \tau n^{-2}, 1 - \tau n^{-2}])_p \leq c \omega_{k+2}^\phi(f, 1/n)_p,
$$

where $c$ are constants independent of $f$ and $n$ which may depend on $r$ and on $p$ as $p \to 0$.

**Theorem 5.3 ([7])** Let $m \in \mathbb{N}$, $k = 1, 2$, $\nu \in \mathbb{N}$, $\nu \geq 2k + 1$, and $f \in M^k[-1,1] \cap C[-1,1] \cap C^\nu(-1,1)$ be such that $\|\phi^{(\nu)}f^{(\nu)}\|_\infty < \infty$. Then for every $n \geq m + \nu - 1$,

$$
E(f, \Pi_n \cap M^k[-1,1])_{\nu [-1,1]} \leq c(m, \nu) n^{-\nu} \omega_{m,\nu}^\phi(f^{(\nu)}, n^{-1})_\infty.
$$

**Corollary 5.4 (see [13,14])** Let $f \in M^k[-1,1] \cap C$, $k = 1, 2$. Then, there exists an absolute constant $\tau > 0$, such that for every $n \geq k + 1$,

$$
E_n^{(k)}(f, [-1 + \tau n^{-2}, 1 - \tau n^{-2}])_\infty \leq c \omega_{k+2}^\phi(f, 1/n)_\infty.
$$
For \( n \geq 2k + 1 \), Corollary 5.4 immediately follows from Theorems 5.2 and 5.3 and Corollary 5.1 with \( m' = 1, \nu = 2k + 1, m = k + 2 \) and \( r = 2k + 2 \). For \( k + 1 \leq n < 2k + 1 \), Corollary 5.4 becomes the usual unconstrained Jackson type estimate if one requires that \( \tau \) is sufficiently large (\( \tau \geq 16 \) will do).

We also note that it is possible to use this approach to obtain similar results for (weakly) almost/nearly co-\( k \)-comonotone polynomial approximation (with \( \sigma \geq 1 \) and \( k = 1,2 \)) in \( L_\infty \) by constructing piecewise polynomial functions (having minimal order and smoothness), smoothing them preserving needed constrains using the approach from [11] (and hence verifying Assumption 1 in Theorem 1.1), and then applying Theorem 1.1 (with Assumption 2 verified using known results on (pure) co-\( k \)-comonotone polynomial approximation of smooth functions).

References


