On k-monotone Interpolation

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Abstract. The errors of approximation of k-monotone functions by k-monotone interpolants are obtained. Suppose that a partition $P_N = \{x_i\}_{i=0}^N$, $N \ge k$, of an interval $[a,b]$ into $\le N$ subintervals is such that $a = x_0 \le x_1 \le \ldots \le x_{N-1} \le x_N = b$, and either

(i) $\|P_N\| < \frac{b-a}{3(k-1)}$

or

(ii) $\|P_N\| \ge \frac{b-a}{3(k-1)}$ and $b - a \le A \mathrm{dist}\{y_{k-1}, \{a,b\}\}$,

where $\|P_N\| := \max_{0 \le i \le N-1}\{x_{i+1} - x_i\}$, and $\{y_i\}_{i=1}^{N-1}$ is a permutation of $\{x_i\}_{i=1}^{N-1}$ ordered so that $\mathrm{dist}\{y_i, \{a,b\}\} \ge \mathrm{dist}\{y_{i+1}, \{a,b\}\}$ for all $1 \le i \le N-1$.

We show, in particular, that if f and g are k-monotone functions on $[a,b]$ such that $f(x_i) = g(x_i)$ for all $0 \le i \le N$, then

$$\|f - g\|_{C[a,b]} \le C \min\left\{\omega_k(f, \|P_N\|, [a,b]), \omega_k(g, \|P_N\|, [a,b])\right\},$$

where ω_k is the usual kth modulus of smoothness, and the constant C depends only on k in the case (i), and on k and A in the case (ii). Moreover, we show that dependence of C on A in the case (ii) is essential and cannot be removed.

§1. Introduction

In this paper, we discuss the errors of approximation of k-monotone functions by k-monotone interpolants. It turns out that any two k-monotone functions f and g whose graphs intersect each other at certain (sufficiently many) points in $[a,b]$ have to be "close" to each other in the sense that $\|f - g\|_{[a,b]}$ has to be small. The only requirement on the points is that there should be enough of them "in the middle" of the interval $[a,b]$, *i.e.*, they should not all be accumulated near the endpoints of $[a,b]$. These

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results are no longer valid for functions f and g which are not necessarily k-monotone (see Section 4 for more details).

A function $f : [a,b] \to \mathbb{R}$ is said to be k-monotone on $[a,b]$ if its kth divided differences $[x_0,\dots,x_k;f]$ are nonnegative for all selections of $k+1$ distinct points $x_0,\dots,x_k$ in $[a,b]$. We denote the class of all such functions by $\mathcal{M}^k := \mathcal{M}^k[a,b]$, and note that $\mathcal{M}^0$, $\mathcal{M}^1$, and $\mathcal{M}^2$ are convex cones of all nonnegative, monotone, and convex functions, respectively.

Let $\|\cdot\| := \|\cdot\|_I$ denote the uniform norm on an interval I, and let $\omega_k(f,\delta,[a,b])$ be the usual uniform kth modulus of smoothness:

$$\omega_k(f,\delta,[a,b]) := \sup_{0<h\le\delta} \left\| \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f(x+ih) \right\|_{[a,b-kh]},$$

and $\omega_k(f,I) := \omega_k(f,|I|,I)$, where $|I|$ is the length of I.

Also, let $L_{k-1}(f,x;t_1,\dots,t_k)$ denote the Lagrange polynomial of degree $\le k-1$ interpolating f at the points t_j, $1\le j\le k$. If some (or all) interpolation points coincide, we use the usual convention that interpolation of corresponding derivatives takes place which results in a Hermite-Taylor polynomial. More precisely, let l_j denote the number of points t_i such that $t_i = t_j$ with $i \le j$, *i.e.*,

$$l_j := l_j(\{t_i\}) := \#\{i | i \le j, t_i = t_j\}.$$

Then, $L_{k-1}(f,x;t_1,\dots,t_k)$ is the unique polynomial of degree $\le k-1$ such that, for all $1\le j\le k$,

$$L_{k-1}^{(l_j-1)}(f,t_j) = f^{(l_j-1)}(t_j).$$

Hence, in order for Hermite-Taylor polynomial to be properly defined the function f has to have (m_j-1)st derivative at t_j, where m_j is the multiplicity of t_j, *i.e.*, $m_j := \#\{i | t_i = t_j\}$. We now recall that any k-monotone function on $[a,b]$ has absolutely continuous $(k-2)$nd derivative on any closed subinterval of (a,b), and $f^{(k-2)}$ has left and right derivatives which are, respectively, left- and right-continuous and nondecreasing on (a,b). Thus, if $f\in\mathcal{M}^k$, the only problem with the notation $L_{k-1}(f,x;t_1,\dots,t_k)$ that we might encounter is when one has to deal with the $(k-1)$st derivative of f which may not be defined (this can happen only if all t_j coincide, *i.e.*, $t_j = t$ for all $1\le j\le k$). In this case, to avoid ambiguity, we define

$$L_{k-1}(f,x;\overbrace{t,\dots,t}^{k}) := \sum_{i=0}^{k-2} \frac{1}{i!} f^{(i)}(t)(x-t)^i + \frac{1}{(k-1)!} f_+^{(k-1)}(t)(x-t)^{k-1},$$

and note that all of the results below involving $L_{k-1}(f,x;t,\dots,t)$ are valid if $f_+^{(k-1)}(t)$ is replaced by $f_-^{(k-1)}(t)$ or any number between $f_-^{(k-1)}(t)$

and $f_+^{(k-1)}(t)$. This follows from the fact that, for any $f \in \mathbb{C}^{k-2}(a,b)$, $L_{k-1}(f,x;t_1,\dots,t_k)$ is a continuous function of $T=(t_1,\dots,t_k)$ at every point $T^*=(t_1^*,\dots,t_k^*) \in (a,b)^k$ such that not all t_j^*, $1 \le j \le k$, are the same (see [2]), and the existence of left and right $(k-1)$st derivatives at every point in (a,b).

If $I=[\alpha,\beta]$, denote

$$\mathcal{O}_\mu(I) := [\alpha-\mu(\beta-\alpha), \beta+\mu(\beta-\alpha)],$$

i.e., $\mathcal{O}_\mu(I)$ is the interval of length $(1+2\mu)|I|$ such that I is in its center.

The following lemma is an immediate corollary of [3, Theorem 1] (with A in that theorem defined by $A := \mu/(1+2\mu)$ and $J_A := I$).

Lemma 1. *Let $f \in \mathcal{M}^k[a,b]$, $k \ge 1$. If an interval $I \subset [a,b]$ is such that $\operatorname{dist}\{I,\{a,b\}\} > 0$, then, for any set $\{t_1,\dots,t_k\}$ of k points in I and any $\mu > 0$ such that $\mathcal{O}_\mu(I) \subset [a,b]$, we have*

$$\|f - L_{k-1}(f,\cdot;t_1,\dots,t_k)\|_{\mathcal{O}_\mu(I)} \le C(k,\mu)\omega_k(f,\mathcal{O}_\mu(I)), \tag{1}$$

where the constant C, which depends only on k and μ, can be chosen to be a non-increasing function of μ: $C(k,\mu_1) \le C(k,\mu_2)$ for $\mu_1 \ge \mu_2$.

Note that, in general, the constant C in (1) becomes unbounded as μ approaches 0 (except in the case $k=1$) since (see [3, Proposition 4])

$$\lim_{\mu\to 0^+}\left(\sup_{f\in\mathcal{M}^k[a,b]}\ \sup_{\{t_1,\dots,t_k\}\subset I} \frac{\|f-L_{k-1}(f,\cdot;t_1,\dots,t_k)\|_{\mathcal{O}_\mu(I)}}{\omega_k(f,\mathcal{O}_\mu(I))}\right) = \infty.$$

The paper is organized as follows. In Section 2, we generalize Lemma 1 to allow interpolation at the endpoints of an interval (Corollaries 3 and 4), discuss how exact these new estimates are (Lemmas 5 and 5′), and then obtain local estimates as a consequence (Theorem 6). Section 3 is devoted to global estimates which are the main result of this paper (Theorems 7 and 8). Finally, in Section 4, we discuss the validity of the estimates if the condition that f and g are k-monotone is removed.

§2. Local Estimates

First, we show that Lemma 1 can be slightly generalized to allow interpolation at the endpoints of the interval $[a,b]$ (see Corollaries 3 and 4). We need the following auxiliary lemma.

Lemma 2. *Let f be a bounded function on $[a,b]$. Suppose that an interval $I \subset [a,b]$ is such that $b-a \le \mathcal{A}\operatorname{dist}\{I,\{a,b\}\}$ for some $\mathcal{A} \in \mathbb{R}$. Also, let $\{t_1,\dots,t_{k-1}\}$ be a set of any (not necessarily distinct) $k-1$ points in I,* *and let q_{k-1} be a polynomial of degree $\le k-1$ which interpolates f at $\{t_1, \dots, t_{k-1}\}$. Then, the following inequality is valid:*

$$\|f - l_{k-1}(f)\|_{[a,b]} \le C\|f - q_{k-1}\|_{[a,b]}, \tag{2}$$

where $l_{k-1}(f)$ is $L_{k-1}(f, \cdot; a, t_1, \dots, t_{k-1})$ or $L_{k-1}(f, \cdot; t_1, \dots, t_{k-1}, b)$, and the constant C depends only on k and $\mathcal{A}$.

Proof: Since

$$\begin{aligned}\|f - l_{k-1}(f)\|_{[a,b]} &= \|f - q_{k-1} - l_{k-1}(f - q_{k-1})\|_{[a,b]} \\ &\le \|f - q_{k-1}\|_{[a,b]} + \|l_{k-1}(f - q_{k-1})\|_{[a,b]},\end{aligned}$$

the proof of (2) will be complete if we show that

$$\|l_{k-1}(f - q_{k-1})\|_{[a,b]} \le C\|f - q_{k-1}\|_{[a,b]}. \tag{3}$$

In order to prove (3), we recall that $(f - q_{k-1})^{(l_j - 1)}(t_j) = 0$ for all $1 \le j \le k-1$, and, hence,

$$l_{k-1}(f - q_{k-1}, x) = (f(c) - q_{k-1}(c)) \prod_{i=1}^{k-1} \frac{x - t_i}{c - t_i},$$

where c is either a or b.

This immediately implies

$$\begin{aligned}\|l_{k-1}(f - q_{k-1})\|_{[a,b]} &\le \left(\frac{b-a}{\operatorname{dist}\{I, \{a,b\}\}}\right)^{k-1} |f(c) - q_{k-1}(c)| \\ &\le \mathcal{A}^{k-1}\|f - q_{k-1}\|_{[a,b]},\end{aligned}$$

and the proof of the lemma is complete. □

Corollary 3. *Let $f \in \mathcal{M}^k[a,b]$, $k \ge 2$. Suppose that an interval $I \subset [a,b]$ is such that $b - a \le \mathcal{A} \operatorname{dist}\{I, \{a,b\}\}$ for some $\mathcal{A} \in \mathbb{R}$, and let $\{t_1, \dots, t_{k-1}\}$ be a set of any (not necessarily distinct) $k-1$ points in I. Then,*

$$\|f - l_{k-1}(f)\|_{[a,b]} \le C\omega_k(f, [a,b]), \tag{4}$$

where $l_{k-1}(f)$ is $L_{k-1}(f, \cdot; a, t_1, \dots, t_{k-1})$ or $L_{k-1}(f, \cdot; t_1, \dots, t_{k-1}, b)$, and the constant C depends only on k and $\mathcal{A}$.

Proof: Lemma 2 implies that

$$\|f - l_{k-1}(f)\|_{[a,b]} \le C\|f - L_{k-1}(f, \cdot; t_1, \dots, t_{k-1}, \widetilde{t})\|_{[a,b]}, \tag{5}$$

for any $\widetilde{t} \in [a,b]$ and, in particular, for any $\widetilde{t} \in I$. Let $I =: [\alpha, \beta]$, and suppose that $\operatorname{dist}\{I, a\} \ge \operatorname{dist}\{I, b\}$, *i.e.*, $\operatorname{dist}\{I, \{a,b\}\} = \operatorname{dist}\{I, b\} = b - \beta$ (the other case is treated similarly), and let $\widetilde{\alpha} = a + (b - \beta)$. Since $I \subset [\widetilde{\alpha}, \beta]$, we have $\{t_1, \dots, t_{k-1}, \widetilde{t}\} \subset [\widetilde{\alpha}, \beta]$. Now, let $\mu := (b - \beta)/(\beta - \widetilde{\alpha})$. Then, $\mathcal{O}_\mu[\widetilde{\alpha}, \beta] = [a,b]$, and (4) follows from (5), Lemma 1 and the observation that $\mu \ge \operatorname{dist}\{I, \{a,b\}\}/(b-a) \ge \mathcal{A}^{-1}$. □

The following statement can be proved using Corollary 3 and the method of the proof of Lemma 2. We omit details.

Corollary 4. *Let $f \in \mathcal{M}^k[a,b]$, $k \geq 2$. Suppose that an interval $I \subset [a,b]$ is such that $b-a \leq \mathcal{A}\operatorname{dist}\{I, \{a,b\}\}$ for some $\mathcal{A} \in \mathbb{R}$, and let $\{t_1, \dots, t_{k-2}\}$ be a set of any $k-2$ points in I. Then,*

$$\|f - L_{k-1}(f, \cdot; a, t_1, \dots, t_{k-2}, b)\|_{[a,b]} \leq C\omega_k(f, [a,b]), \tag{6}$$

where the constant C depends only on k and $\mathcal{A}$.

Remark. In the case $k = 2$ ($k = 2$ or 3) Corollary 3 (4) is valid for all bounded functions f and not only for those in $\mathcal{M}^k$. This immediately follows from the classical Whitney theorem.

The following Lemmas 5 and 5′ imply that constants C in (4) depend, in general, not only on k but also on the ratios $\frac{b-a}{b-\max\{t_1,\dots,t_{k-1}\}}$ (if $l_{k-1}(f) = L_{k-1}(f, \cdot; t_1, \dots, t_{k-1}, b)$) and $\frac{b-a}{\min\{t_1,\dots,t_{k-1}\}-a}$ (if $l_{k-1}(f) = L_{k-1}(f, \cdot; a, t_1, \dots, t_{k-1})$). This shows that the dependence of constants C in (4) and (6) on $\mathcal{A}$ is essential and cannot be removed in general.

Lemma 5. *Let $k \geq 2$. For any set $\{t_1, \dots, t_{k-1}\}$ of $k-1$ points in $[a,b)$ such that $a \leq t_1 \leq \dots \leq t_{k-1} < b$ there exists a function $f \in \mathcal{M}^k[a,b]$ satisfying*

$$\frac{\|L_{k-1}(f, \cdot; t_1, \dots, t_{k-1}, b)\|_{[a,b]}}{\|f\|_{[a,b]}} \geq C(k)\frac{b-a}{b-t_{k-1}}.$$

Lemma 5′. *Let $k \geq 2$. For any set $\{t_1, \dots, t_{k-1}\}$ of $k-1$ points in $(a,b]$ such that $a < t_1 \leq \dots \leq t_{k-1} \leq b$ there exists a function $f \in \mathcal{M}^k[a,b]$ satisfying*

$$\frac{\|L_{k-1}(f, \cdot; a, t_1, \dots, t_{k-1})\|_{[a,b]}}{\|f\|_{[a,b]}} \geq C(k)\frac{b-a}{t_1-a}.$$

Proof of Lemma 5: We only prove Lemma 5 since the proof of Lemma 5′ is similar.

Let $t_0 := a$ and $t_k := b$, and define $I_i := [t_i, t_{i+1}]$ for $0 \leq i \leq k-1$. Now, let $f(x) := (b-t_{k-1})^{1-k}(x-t_{k-1})_+^{k-1}$, and note that $f \in \mathcal{M}^k[a,b]$, and $\|f\|_{[a,b]} = 1$. Also, since $f(x) = 0$ for $x \leq t_{k-1}$, and $f(b) = 1$, we have

$$L_{k-1}(f, \cdot; t_1, \dots, t_{k-1}, b) = \prod_{i=1}^{k-1} \frac{x-t_i}{b-t_i}.$$

Suppose now that $\widetilde{I}$ is the largest of the intervals I_i, $0 \leq i \leq k-1$, and that ξ is its midpoint. Then, $k|\widetilde{I}| \geq \sum_{i=0}^{k-1}|I_i| = b-a$ and, hence, for all $1 \leq i \leq k-1$, we have $|\xi - t_i| \geq |\widetilde{I}|/2 \geq (b-a)/(2k)$. Therefore, we have

$$\frac{\|L_{k-1}(f, \cdot; t_1, \dots, t_{k-1}, b)\|_{[a,b]}}{\|f\|_{[a,b]}} \geq |L_{k-1}(f, \xi; t_1, \dots, t_{k-1}, b)|$$

$$\geq \frac{((b-a)/(2k))^{k-1}}{(b-a)^{k-2}(b-t_{k-1})} = (2k)^{1-k}\frac{b-a}{b-t_{k-1}},$$

and the proof is complete. □

Theorem 6. *Let $k \geq 2$, and an interval $I \subset [a,b]$ be such that $b-a \leq \mathcal{A}\,\mathrm{dist}\{I,\{a,b\}\}$ for some $\mathcal{A} \in \mathbb{R}$, and $\{t_1,\ldots,t_{k-1}\}$ be a set of any $k-1$ points in I. If $f,g \in \mathcal{M}^k[a,b]$ are such that $f^{(l_j-1)}(t_j) = g^{(l_j-1)}(t_j)$ for all $0 \leq j \leq k$ (where $t_0 := a$, $t_k := b$, and $l_j := l_j\left(\{t_i\}_{i=0}^k\right)$), then*

$$\|f-g\|_{[a,b]} \leq C\min\left\{\omega_k(f,[a,b]),\omega_k(g,[a,b])\right\}, \tag{7}$$

where the constant C depends only on k and $\mathcal{A}$.

Proof: Without loss of generality, assume that $\omega_k(f,[a,b]) \leq \omega_k(g,[a,b])$. It was shown by Bullen [1] (see also [4, Lemma 8.3]) that, if f is k-monotone, then $f - L_{k-1}(f,\cdot;x_1,\ldots,x_k)$ changes sign at $x_1,\ldots,x_k$. More precisely, let $k \in \mathbb{N}$, $f \in \mathcal{M}^k(a,b)$, and recall that $L_{k-1}(f,x;x_1,\ldots,x_k)$ is the Lagrange (Hermite-Taylor) polynomial of degree $\leq k-1$ interpolating f (or f together with its derivatives) at the points x_i, $1 \leq i \leq k$, where $a =: x_0 \leq x_1 \leq \ldots \leq x_k \leq x_{k+1} := b$. Then, for $0 \leq i \leq k$,

$$(-1)^{k-i}\left(f(x) - L_{k-1}(f,x;x_1,\ldots,x_k)\right) \geq 0, \quad x \in (x_i,x_{i+1}). \tag{8}$$

Now, let

$$q_{k-1}(x) := L_{k-1}(f,x;t_0,t_1,\ldots,t_{k-1})\,(= L_{k-1}(g,x;t_0,t_1,\ldots,t_{k-1}))$$

and

$$\widetilde{q}_{k-1}(x) := L_{k-1}(f,x;t_1,\ldots,t_{k-1},t_k)\,(= L_{k-1}(g,x;t_1,\ldots,t_{k-1},t_k)).$$

Inequalities (8) imply that, for every $0 \leq i \leq k-1$ and $x \in [t_i,t_{i+1}]$, the following inequalities are valid:

$$\begin{aligned} (-1)^{k-1-i}\left(f(x) - q_{k-1}(x)\right) &\geq 0, \qquad (-1)^{k-1-i}\left(g(x) - q_{k-1}(x)\right) \geq 0,\\ (-1)^{k-i}\left(f(x) - \widetilde{q}_{k-1}(x)\right) &\geq 0, \qquad (-1)^{k-i}\left(g(x) - \widetilde{q}_{k-1}(x)\right) \geq 0, \end{aligned}$$

and, therefore, for every $x \in [a,b]$, the values $f(x)$ and $g(x)$ lie between $q_{k-1}(x)$ and $\widetilde{q}_{k-1}(x)$. This implies that

$$|f(x) - g(x)| \leq |q_{k-1}(x) - \widetilde{q}_{k-1}(x)|, \; x \in [a,b].$$

Now, using Corollary 3 we have

$$\begin{aligned} \|f-g\|_{[a,b]} &\leq \|q_{k-1} - \widetilde{q}_{k-1}\|_{[a,b]} \leq \|f - q_{k-1}\|_{[a,b]} + \|f - \widetilde{q}_{k-1}\|_{[a,b]}\\ &\leq C\omega_k(f,[a,b]), \end{aligned}$$

where C depends only on k and $\mathcal{A}$. □

§3. Global Estimates

Let $\mathbb{P}_N[a,b] = \{x_i\}_{i=0}^N = \{a = x_0 \le x_1 \le \ldots \le x_{N-1} \le x_N = b\}$ be a partition of $[a,b]$ into $\le N$ subintervals.

We denote

$$\|\mathbb{P}_N\| := \|\mathbb{P}_N[a,b]\| := \max_{0\le i\le N-1} \{x_{i+1} - x_i\}$$

the length of the largest interval in that partition (the norm of the partition), and denote the length of the smallest interval by $\langle \mathbb{P}_N \rangle$:

$$\langle \mathbb{P}_N \rangle := \langle \mathbb{P}_N[a,b] \rangle := \min_{0\le i\le N-1} \{x_{i+1} - x_i\}.$$

Notation. *Following Oswald [5] we call a partition* $\mathbb{P}_N[a,b]$ almost uniform *if* $\|\mathbb{P}_N[a,b]\| \le 3\langle \mathbb{P}_N[a,b]\rangle$.

It was shown in [5, Lemma 1.1] that any partition $\mathbb{P}_N[a,b]$ of $[a,b]$ can be made almost uniform by deleting some of the partition points:

> For any partition $\mathbb{P}[a,b]$ there exists a superpartition $\widetilde{\mathbb{P}}[a,b]$ (*i.e.*, partition $\widetilde{\mathbb{P}}$ is obtained from $\mathbb{P}$ by deleting some of the points of $\mathbb{P}$) which is *almost uniform* and such that
>
> $$\|\mathbb{P}\| \le \langle \widetilde{\mathbb{P}} \rangle \le \|\widetilde{\mathbb{P}}\| \le 3\|\mathbb{P}\|. \tag{9}$$

Theorem 7. *Suppose that $N \ge k \ge 2$, and let $\mathbb{P}_N := \mathbb{P}_N[a,b] = \{a = x_0 \le x_1 \le \ldots \le x_{N-1} \le x_N = b\}$ be a partition of $[a,b]$ into $\le N$ subintervals such that $\|\mathbb{P}_N\| < \frac{b-a}{3(k-1)}$. Also, let $f, g \in \mathcal{M}^k[a,b]$ be such that $f(x_i) = g(x_i)$, $0 \le i \le N$. Then*

$$\|f-g\|_{[a,b]} \le C \min\{\omega_k(f, \|\mathbb{P}_N\|, [a,b]), \omega_k(g, \|\mathbb{P}_N\|, [a,b])\}, \tag{10}$$

where the constant C depends only on k.

Proof: Without loss of generality, assume that $\omega_k(f, \|\mathbb{P}_N\|, [a,b]) \le \omega_k(g, \|\mathbb{P}_N\|, [a,b])$. According to (9) there exists an almost uniform partition of $[a,b]$, $\widetilde{\mathbb{P}}$ (which is a superpartition of $\mathbb{P}_N[a,b]$) such that $\|\mathbb{P}_N\| \le \langle \widetilde{\mathbb{P}} \rangle \le \|\widetilde{\mathbb{P}}\| \le 3\|\mathbb{P}_N\|$. Since $\|\mathbb{P}_N\| < \frac{b-a}{3(k-1)}$, this implies that $\|\widetilde{\mathbb{P}}\| < \frac{b-a}{k-1}$, and therefore $\widetilde{\mathbb{P}}$ consists of at least k intervals. Now, it is sufficient to prove (10) for the partition $\widetilde{\mathbb{P}}$ instead of $\mathbb{P}_N$. Equivalently, we can assume that the original partition $\mathbb{P}_N$ is almost uniform. Hence, we finish the proof of the theorem assuming that $\|\mathbb{P}_N\| \le 3\langle \mathbb{P}_N \rangle$ and that $\mathbb{P}_N$ consists of at least k intervals.

Let i, $0 \le i \le N-1$, be fixed, and denote $\alpha(i) := \max\{0, i-k+1\}$ and $J_i := [x_{\alpha(i)}, x_{\alpha(i)+k}]$. Since $\mathbb{P}_N[a,b]$ consists of at least k intervals, then $[x_i, x_{i+1}] \subset J_i \subset [a,b]$. Taking into account that $|J_i| \sim \|\mathbb{P}_N\|$, we can now apply Theorem 6 (with $[a,b] := J_i$, $t_j := x_{\alpha(i)+j}$, $0 \le j \le k$, and $I := [t_1, t_{k-1}]$) to conclude that

$$\|f-g\|_{[x_i,x_{i+1}]} \le \|f-g\|_{J_i} \le C\omega_k(f, J_i) \le C\omega_k(f, \|\mathbb{P}_N\|, [a,b]),$$

where C depends only on k, because we can choose a constant $\mathcal{A}$ in the statement of Theorem 6 to be $3k$, since

$$\frac{|J_i|}{\operatorname{dist}\{[t_1,t_{k-1}], \{x_{\alpha(i)}, x_{\alpha(i)+k}\}\}} \le \frac{k\|\mathbb{P}_N\|}{\langle \mathbb{P}_N \rangle} \le 3k.$$

Since there exists i, $0 \le i \le N-1$, such that $\|f-g\|_{[a,b]} = \|f-g\|_{[x_i,x_{i+1}]}$, the proof of the theorem is complete. □

Theorem 8. *Suppose that* $N \ge k \ge 2$, $\mathbb{P}_N = \mathbb{P}_N[a,b] = \{a = x_0 \le x_1 \le \dots \le x_{N-1} \le x_N = b\}$ *is a partition of* $[a,b]$ *such that* $\|\mathbb{P}_N\| \ge \frac{b-a}{3(k-1)}$, *and each* x_j*'s multiplicity is at most* k. *Let* $f, g \in \mathcal{M}^k[a,b]$ *be such that* $f^{(l_j-1)}(x_j) = g^{(l_j-1)}(x_j)$, $0 \le j \le N$, *where* $l_j := l_j\left(\{x_i\}_{i=0}^N\right)$. *Also, let* $\{y_i\}_{i=1}^{N-1}$ *be a permutation of the set* $\{x_i\}_{i=1}^{N-1}$ *such that* $\operatorname{dist}\{y_i, \{a,b\}\} \ge \operatorname{dist}\{y_{i+1}, \{a,b\}\}$ *for all* $1 \le i \le N-1$, *and suppose that* $b - a \le \mathcal{A}\operatorname{dist}\{y_{k-1}, \{a,b\}\}$. *Then*

$$\|f-g\|_{[a,b]} \le C \min\left\{\omega_k(f, \|\mathbb{P}_N\|, [a,b]), \omega_k(g, \|\mathbb{P}_N\|, [a,b])\right\},$$

where the constant C *depends only on* k *and* $\mathcal{A}$.

Proof: First, note that $\|\mathbb{P}_N\| \ge \frac{b-a}{3(k-1)}$ implies $\|\mathbb{P}_N\| \sim b-a$ and, hence, $\omega_k(f, \|\mathbb{P}_N\|, [a,b]) \sim \omega_k(f, [a,b])$ and $\omega_k(g, \|\mathbb{P}_N\|, [a,b]) \sim \omega_k(g, [a,b])$. Now, the statement of the theorem immediately follows from Theorem 6 with $t_j = y_j$, $1 \le j \le k-1$, taking into account that

$$\begin{aligned} &\operatorname{dist}\{[\min\{y_1, \dots, y_{k-1}\}, \max\{y_1, \dots, y_{k-1}\}], \{a,b\}\} \\ &\qquad = \operatorname{dist}\{y_{k-1}, \{a,b\}\} \ge (b-a)/\mathcal{A}. \quad \square \end{aligned}$$

The following example shows that dependence on $\mathcal{A}$ in Theorem 8 is essential and cannot be removed.

Example. Let $k \ge 2$. Suppose for simplicity that $[a,b] = [0,1]$, and let

$$f(x) := (1-\xi)^{1-k}(x-\xi)_+^{k-1}$$

and

$$g(x) := (1-\zeta)^{1-k}(x-\zeta)_+^{k-1},$$

where $\xi, \zeta \in (0,1)$ and $\xi + 1/2 < \zeta$. Then,

$$\begin{aligned}\omega_k(f,[0,1]) &= \omega_k(f-(1-\xi)^{1-k}(\cdot-\xi)^{k-1},[0,1])\\ &\le 2^k\|f-(1-\xi)^{1-k}(\cdot-\xi)^{k-1}\|_{[0,1]}\\ &= 2^k(1-\xi)^{1-k}\|(\cdot-\xi)^{k-1}\|_{[0,\xi]}\\ &= 2^k(1-\xi)^{1-k}\xi^{k-1},\end{aligned}$$

and, at the same time,

$$\|f-g\| \ge |f(\zeta)-g(\zeta)| = (1-\xi)^{1-k}(\zeta-\xi)^{k-1} \ge (1-\xi)^{1-k}2^{1-k}.$$

Hence,

$$\begin{aligned}&\frac{\|f-g\|}{\min\{\omega_k(f,[0,1]),\omega_k(g,[0,1])\}} \ge \frac{\|f-g\|}{\omega_k(f,[0,1])} \ge \frac{(1-\xi)^{1-k}2^{1-k}}{2^k(1-\xi)^{1-k}\xi^{k-1}}\\ &\quad = 2^{1-2k}\left(\operatorname{dist}\{\xi,\{0,1\}\}\right)^{1-k} \to \infty \quad \text{as} \quad \xi \to 0.\end{aligned}$$

Finally, recall that f and g interpolate each other at the endpoints of $[0,1]$ and are identical on $[0,\xi]$.

At the same time, as was mentioned above the case $k=1$ is somewhat different and simpler (see also Remarks after Lemma 10) . The proof of the following statement is trivial and will be omitted.

Lemma 9 **($k=1$).** *Let $N \ge 1$, $\mathbb{P}_N[a,b] = \{a = x_0 < x_1 < \ldots < x_{N-1} < x_N = b\}$ be a partition of $[a,b]$, $I_i := [x_i, x_{i+1}]$, and let f and g be monotone functions on $[a,b]$ such that $f(x_i) = g(x_i)$, $0 \le i \le N$. Then, for any $0 \le i \le N-1$, the following estimate holds*

$$\|f-g\|_{I_i} \le f(x_{i+1}) - f(x_i) = g(x_{i+1}) - g(x_i)\, (= \omega(f, I_i) = \omega(g, I_i))\,.$$

Therefore,

$$\|f-g\|_{[a,b]} \le \min\{\omega(f,\|\mathbb{P}_N\|,[a,b]),\omega(g,\|\mathbb{P}_N\|,[a,b])\}\,.$$

We finally remark that ω_k in Theorems 7 and 8 cannot be replaced by ω_m with $m \ge k+1$. To see this, it is sufficient to let f be a polynomial of degree $\le k$, and g be any k-monotone function (different from f) interpolating f at the points in $\mathbb{P}_N$ (the existence of such g follows from Zwick [10], for example).

§4. Appendix: Interpolation Without Constraints

All statements in this section are simple and can be easily proved by anyone familiar with the classical Whitney theorem. The only reason for including them here is for a quick comparison with corresponding results for k-monotone interpolation.

Lemma 10. *Let $m \in \mathbb{N}$, $N \geq m-1$, and suppose that $\mathbb{P}_N := \mathbb{P}_N[a,b] = \{a = x_0 < x_1 < \ldots < x_{N-1} < x_N = b\}$ is a partition of $[a,b]$ into N subintervals which is quasi-uniform, i.e.,*

$$\|\mathbb{P}_N\| \leq \mathcal{B}\langle \mathbb{P}_N \rangle.$$

If $f, g \in \mathbb{C}[a,b]$ are such that $f(x_i) = g(x_i)$, $0 \leq i \leq N$, then

$$\|f - g\|_{[a,b]} \leq C \max\left\{\omega_m(f, \|\mathbb{P}_N\|, [a,b]), \omega_m(g, \|\mathbb{P}_N\|, [a,b])\right\}, \tag{11}$$

where the constant C depends only on m and $\mathcal{B}$.

Proof: The statement of the lemma immediately follows from the observation that, for $0 \leq i \leq N - m + 1$,

$$L_{m-1}(f, \cdot; x_i, x_{i+1}, \ldots, x_{i+m-1}) = L_{m-1}(g, \cdot; x_i, x_{i+1}, \ldots, x_{i+m-1}),$$

and the classical Whitney theorem of interpolatory type: for $f \in \mathbb{C}[\alpha, \beta]$ and $\mathbb{P}_{m-1}[\alpha,\beta] = \{\alpha = \xi_0 < \xi_1 < \ldots < \xi_{m-1} = \beta\}$ such that $\|\mathbb{P}_{m-1}\| \leq \mathcal{B}\langle \mathbb{P}_{m-1}\rangle$ the following estimate holds:

$$\|f - L_{m-1}(f, \cdot; \xi_0, \xi_1, \ldots, \xi_{m-1})\|_{[\alpha,\beta]} \leq C\omega_m(f, [\alpha,\beta]),$$

where the constant C depends only on m and $\mathcal{B}$. $\square$

Remark 1. Obviously, "max" in Lemma 10 cannot be replaced by "min" (consider f to be a polynomial of degree $m-1$, for example).

Remark 2 ($m = 1$). It immediately follows from the definition of the modulus of continuity that, if $f(\xi) = g(\xi)$ for some $\xi \in [a,b]$, then

$$\|f - g\|_{[a,b]} \leq 2\max\left\{\omega(f, [a,b]), \omega(g, [a,b])\right\}.$$

Hence, for $m = 1$, (11) is valid without the requirement that $\mathbb{P}_N$ is quasi-uniform.

Remark 3 ($m = 2$). It immediately follows from Lemma 10 that if $f(a) = g(a)$ and $f(b) = g(b)$, then

$$\|f - g\|_{[a,b]} \leq C \max\left\{\omega_2(f, [a,b]), \omega_2(g, [a,b])\right\},$$

and therefore (11) is valid without the requirement that $\mathbb{P}_N$ is quasi-uniform in the case $m = 2$ as well.

If interpolation at the endpoints is not required, it is easy to show that ω_2 is impossible in (11) for non-quasi-uniform partitions (consider, for example, $f \equiv 0$, and $g_\varepsilon := x - (a+b)/2$, $a \leq x \leq (a+b)/2$; $g_\varepsilon(x) := x - \varepsilon - (a+b)/2$, $\varepsilon + (a+b)/2 \leq x \leq b$, and $g_\varepsilon(x) := 0$, $(a+b)/2 < x < \varepsilon + (a+b)/2$).

Remark 4 ($m \geq 3$). In the case $m \geq 3$, the statement of Lemma 10 is no longer true if the condition that a partition $\mathbb{P}_N$ is quasi-uniform is removed (in other words, the constant C in (11) cannot be made independent of $\mathcal{B}$). For example, for small $\varepsilon > 0$, define $f \equiv 0$, and $g_\varepsilon(x) := (b-x)(x-a-\varepsilon)$ if $a+\varepsilon \leq x \leq b$, and $g_\varepsilon(x) := 0$ if $a \leq x < a+\varepsilon$. Then, f and g coincide on $[a, a+\varepsilon]$ and have the same value at $x = b$. Therefore, they interpolate each other at all points in $\mathbb{P}_N = \{x_i\}_{i=0}^N$ such that $a = x_0 < x_1 < \ldots < x_{N-1} \leq a+\varepsilon < x_N = b$ and, at the same time, $\|f-g\|_{[a,b]} \sim (b-a)^2$ and $\omega_3(g, [a,b]) \sim (b-a)\varepsilon$.

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