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On copositive approximation by algebraic polynomials

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1. Introduction and main result

We are interested in how well one can approximate a function $f \in C[-1,1]$ with finitely many sign changes by polynomials p_n such that

$$f(x)p_n(x) \geq 0, \quad x \in [-1,1]$$

(we say that in this case f and p_n are copositive in $[-1,1]$).

Let Π_n denote the set of all algebraic polynomials of degree not exceeding n. Recall that the mth order Ditzian–Totik modulus of smoothness in the uniform metric is given by (see [3])

$$\omega_\varphi^m(f,\delta,[a,b]) = \sup_{0<h\leq\delta} \|\Delta_{h\varphi(x)}^m(f,x,[a,b])\|_{[a,b]}, \tag{1}$$

where $\|\cdot\|_{[a,b]}$ denotes the uniform norm on the interval $[a,b]$,

$$\varphi(x) := \sqrt{1-x^2}$$

and

$$\Delta_\eta^m(f,x,[a,b]) := \begin{cases} \sum_{k=0}^{m} \binom{m}{k}(-1)^{m-k} f\Big(x-\dfrac{m}{2}\eta+k\eta\Big) & \text{if } x \pm \frac{m}{2}\eta \in [a,b], \\ 0 & \text{otherwise.} \end{cases}$$

is the symmetric mth difference.

Note that if $\varphi = \varphi(x)$ is replaced by the constant 1, then (1) becomes the definition of the usual modulus of smoothness of order m.

For $I := [-1,1]$, for simplicity, we write

$$\|\cdot\| := \|\cdot\|_I, \quad \omega^m(f,\delta) := \omega^m(f,\delta,I)$$

and

$$\omega_\varphi^m(f,\delta) := \omega_\varphi^m(f,\delta,I).$$

One of recent results on copositive approximation is due to LEVIATAN [5] who proved the following theorem.

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Theorem A. *There exists an absolute constant $C = C(r)$ such that for every $f \in C[-1,1]$ which alternates in sign r times in $[-1,1]$, $0 < r < \infty$, and each $n \geq 1$, there is a polynomial $p_n \in \Pi_n$ which is copositive with f and satisfies*

$$\|f - p_n\| \leq C\,\omega(f, n^{-1}). \tag{2}$$

This result was later improved. In particular, the modulus ω was replaced by ω_φ, and the dependance of C on the set of points of sign changes was investigated (see [6], for example). However, it was not known for a long time whether ω in (2) can be replaced by the modulus of smoothness of higher order.

ZHOU (see [9] and [10]) showed that estimate (2) can not hold with ω^4 instead of ω. He also considered copositive approximation in L_p metric, $1 < p < \infty$, and proved that the estimate by the second integral modulus of smoothness $\omega^2(f, n^{-1})_p$ is not correct in this case. These results can be summarized in the following theorem.

Theorem B. *There are functions f_1 and f_2 in $C^1[-1,1]$ with $r \geq 1$ sign changes such that*

$$\limsup_{n\to\infty} \frac{E_n^{(0)}(f_1, r)_\infty}{\omega^4(f_1, n^{-1})} = \infty \quad \text{and} \quad \limsup_{n\to\infty} \frac{E_n^{(0)}(f_2, r)_p}{\omega^2(f_2, n^{-1})_p} = \infty, \quad 1 < p < \infty,$$

where $E_n^{(0)}(f, r)_p$ is the error of the best copositive L_p (C if $p = \infty$) approximation to f by polynomials from Π_n.

Recently, Y. HU, LEVIATAN and X. M. YU [4] showed that Theorem A can be considerably improved. They were able to replace ω by ω^2 in (2), thus, together with Theorem B, revealing an interesting and unexpected difference between the cases $p = \infty$ and $1 < p < \infty$ for copositive polynomial approximation. Their result is stated as follows.

Theorem C. *Let $f \in C[-1,1]$ change sign r times at $-1 < y_1 < \cdots < y_r < 1$, and let*

$$\delta := \min_{0 \leq i \leq r} |y_{i+1} - y_i|, \quad \text{where } y_0 := -1 \text{ and } y_{r+1} := 1.$$

Then there exists a constant $C = C(r, \delta)$ independent of f and n such that for each $n > 4\delta^{-1}$ there is a polynomial $p_n \in \Pi_{Cn}$, copositive with f, satisfying

$$\|f - p_n\| \leq C\omega^2(f, n^{-1}). \tag{3}$$

In fact, it is not difficult to show that ω^2 in (3) can be replaced by ω^3. However, there is still some room for improvement. It is well known that if one wants to characterize approximation properties of a function f in terms of its moduli of smoothness, then this characterization should involve either $\omega^m(f, \Delta_n(x))$ or $\omega_\varphi^m(f, n^{-1})$ (or equivalent quantities). Thus, in a sense, "exact" estimates for algebraic polynomial approximation are those in terms of the above mentioned quantities.

The following theorem is the main result of this note.

Theorem 1. *Let $f \in C[-1,1]$ change sign $r \geq 1$ times at $-1 < y_1 < \cdots < y_r < 1$, and let*

$$\delta := \min_{0 \leq i \leq r} |y_{i+1} - y_i|, \quad \text{where} \quad y_0 := -1 \text{ and } y_{r+1} := 1.$$

Then there exists a constant $C_1 = C_1(r, \delta)$ such that for each $n > C_1$ there is a polynomial $P_n \in \Pi_n$, copositive with f, satisfying

$$\|f - P_n\| \leq C(r)\omega_\varphi^3(f, n^{-1}). \tag{4}$$

Theorem 1 implies

Theorem 1′. *Let f be the same as in Theorem 1. Then for each $n \geq 0$ in the case $r \geq 3$, and $n \geq 2$ if $r = 1$ or 2, there is a polynomial $P_n \in \Pi_n$, copositive with f, such that*

$$\|f - P_n\| \leq C(r, \delta)\omega_\varphi^3(f, n^{-1}),$$

where $0^{-1} := 1$.

An immediate consequence of Theorems 1 and 1′ and converse theorems in terms of the Ditzian–Totik moduli (see [3] and [8]) is the following result.

Corollary 2. *Let $0 < \alpha < 3$, and let a function $f \in C[-1,1]$ change sign r times in $[-1,1]$. Then*

$$E_n(f) = O(n^{-\alpha}) \iff E_n^{(0)}(f, r) = O(n^{-\alpha}), \tag{5}$$

where

$$E_n(f) = \inf_{p_n \in \Pi_n} \|f - p_n\| \quad \text{and} \quad E_n^{(0)}(f, r) = \inf_{p_n \in \Pi_n, p_n f \geq 0} \|f - p_n\|.$$

The case $\alpha \geq 3$ in Corollary 2 remains open.

The next section contains auxiliary results. We also introduce some notations there. Theorems 1 and 1′ are proved in Section 3. Finally, some relevant remarks are given in Section 4.

2. Notations and auxiliary results

The following notations are used throughout this paper (cf. [8]):

$$x_j := \cos\frac{j\pi}{n}, \quad 0 \le j \le n,$$

$$I_j := [x_j, x_{j-1}], \quad h_j := |I_j| = x_{j-1} - x_j, \; 1 \le j \le n,$$

$$\Delta_n(x) := \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2} \quad \text{and} \quad \varphi(x) = \sqrt{1-x^2}.$$

It can be easily verified (see, for example, [8]) that

$$h_{j\pm 1} < 3h_j \quad \text{and} \quad \Delta_n(x) \le h_j \le 5\Delta_n(x) \text{ for } x \in I_j.$$

Also, let $L(x, f, t_1, \dots, t_\mu)$ denote the Lagrange polynomial of degree $\le \mu - 1$ interpolating the function f at $t_1, \dots, t_\mu$.

Suppose $f \in C[-1,1]$ satisfies the conditions of Theorem 1, i.e., it changes sign $1 \le r < \infty$ times at $-1 < y_1 < \cdots < y_r < 1$. Also, let n be fixed and sufficiently large. If $y_i \in (x_{j(i)}, x_{j(i)-1})$, $i = 1, 2, \dots, r$, then it is convenient to denote

$$y_i' := x_{j(i)+1}, \quad y_i'' := x_{j(i)-2},$$

$$\mathcal{I}_i := [y_i', y_i''] := I_{j(i)+1} \cup I_{j(i)} \cup I_{j(i)-1} = [x_{j(i)+1}, x_{j(i)-2}]$$

and

$$\mathcal{Y}_i := \left[\frac{y_i + y_i'}{2}, \frac{y_i + y_i''}{2}\right] \quad \text{for } i = 1, 2, \dots, r.$$

Then

$$\frac{5}{3}h_{j(i)} < |\mathcal{I}_i| = 2|\mathcal{Y}_i| < 7h_{j(i)}, \quad i = 1, \dots, r,$$

and, therefore,

$$|\mathcal{I}_i| \sim |\mathcal{Y}_i| \sim h_{j(i)} \sim \Delta_n(x) \quad \text{for } x \in \mathcal{I}_i.$$

Throughout the paper C denote constants which are independent of f and n, and are not necessarily the same even when they occur on the same line, and K_i, $i \ge 1$, denote constants which are independent of f and n and remain fixed everywhere in the proofs.

While proving Theorem 1 we will need to smooth the function f which is only assumed to be continuous on $[-1,1]$. The idea to consider a smooth approximation instead of the original function is very well known. It is frequently used in different areas of approximation theory. In particular, the construction of such an approximation is crucial in the proofs of theorems on the equivalence of K-functionals and the appropriate moduli of smoothness (see [1], [3] and [7], for example). There are numerous approaches to this problem. Thus, it is often convenient first to extend the function to a larger interval preserving some of its smoothness characteristics. In particular, this idea was employed in the proof of the main lemma in [4]. In our proof we will avoid the problems of smoothing and extending f (though, this approach is possible) simply by considering an algebraic polynomial which sufficiently approximates f and satisfies some extra conditions (in fact, the polynomial of best approximation to f in $C[-1,1]$ will do). Then, we will modify this polynomial near the points of sign changes obtaining a smooth piecewise polynomial approximation f_n with controlled first and third derivatives. The following lemma is crucial for the proof of Theorem 1.

Lemma 3. *Let f be the same as in Theorem 1. Then for each $n \geq 4\delta^{-1}$ there exists a function $f_n \in C^3[-1,1]$, copositive with f in $Y := \bigcup_{i=1}^r \mathcal{Y}_i$, such that*

$$\|f - f_n\| \leq C(r)\omega_\varphi^3(f, n^{-1}), \tag{6}$$

$$\|\varphi(x)^3 f_n'''(x)\| \leq K_1(r) n^3 \omega_\varphi^3(f, n^{-1}), \tag{7}$$

and

$$|\Delta_n(x) f_n'(x)| \geq \omega_\varphi^3(f, n^{-1}) \quad \text{for } x \in Y. \tag{8}$$

Proof. Let $n \geq 4\delta^{-1}$ and let the index $1 \leq i \leq r$ be fixed. For $x \in \mathcal{I}_i$ we set $\widetilde{f}_i$ to be the polynomial of degree ≤ 2 which vanishes at y_i:

$$\widetilde{f}_i(x) := \frac{x - y_i}{y_i'' - y_i'}\Big(\frac{x - y_i'}{y_i'' - y_i}\widetilde{f}_i(y_i'') + \frac{x - y_i''}{y_i - y_i'}\widetilde{f}_i(y_i')\Big),$$

where $\widetilde{f}_i(y_i')$ and $\widetilde{f}_i(y_i'')$ are chosen so that

$$\widetilde{f}_i(y_i') = \begin{cases} 60\omega_\varphi^3(f, n^{-1})\operatorname{sgn}(f(y_i')) & \text{if } |f(y_i')| \leq 60\omega_\varphi^3(f, n^{-1}) , \\ f(y_i') & \text{otherwise.} \end{cases}$$

and

$$\widetilde{f}_i(y_i'') = \begin{cases} 60\omega_\varphi^3(f, n^{-1})\operatorname{sgn}(f(y_i'')) & \text{if } |f(y_i'')| \leq 60\omega_\varphi^3(f, n^{-1}), \\ f(y_i'') & \text{otherwise.} \end{cases}$$

(If $f(y_i') = 0$, e.g., then $\operatorname{sgn}(f(y_i'))$ equals to the sign of f on (y_{i-1}, y_i).)

Since $\widetilde{f}_i \in \Pi_2$, and $\widetilde{f}_i(y_i')$ and $\widetilde{f}_i(y_i'')$ have opposite signs, then the only zero of $\widetilde{f}_i$ in $\mathcal{I}_i$ is y_i. Hence, $\widetilde{f}_i$ is copositive with f in $\mathcal{I}_i$. Also, the first derivative of $\widetilde{f}_i$:

$$\widetilde{f}_i'(x) = \frac{2x - y_i - y_i'}{(y_i'' - y_i')(y_i'' - y_i)}\widetilde{f}_i(y_i'') + \frac{2x - y_i - y_i''}{(y_i'' - y_i')(y_i - y_i')}\widetilde{f}_i(y_i')$$

is a linear function, and

$$\widetilde{f}_i'\Big(\frac{y_i + y_i'}{2}\Big) = -\frac{\widetilde{f}_i(y_i')}{y_i - y_i'} \quad \text{and} \quad \widetilde{f}_i'\Big(\frac{y_i + y_i''}{2}\Big) = \frac{\widetilde{f}_i(y_i'')}{y_i'' - y_i}$$

are of the same sign, which implies that $\widetilde{f}_i'$ does not change sign in $\mathcal{Y}_i$, and for any $x \in \mathcal{Y}_i$ we have

(9)

$$|\widetilde{f}_i'(x)| \geq \min\Big\{\Big|\widetilde{f}_i'\Big(\frac{y_i + y_i'}{2}\Big)\Big|, \Big|\widetilde{f}_i'\Big(\frac{y_i + y_i''}{2}\Big)\Big|\Big\} = \min\Big\{\frac{|\widetilde{f}_i(y_i')|}{y_i - y_i'}, \frac{|\widetilde{f}_i(y_i'')|}{y_i'' - y_i}\Big\} \geq$$

$$\geq \frac{1}{60\Delta_n(x)} \min\Big\{|\widetilde{f}_i(y_i')|, |\widetilde{f}_i(y_i'')|\Big\} \geq \Delta_n(x)^{-1}\omega_\varphi^3(f, n^{-1}).$$

Now we will show that

$$|\widetilde{f}_i(x) - f(x)| \leq C\omega_\varphi^3(f, n^{-1}), \quad x \in \mathcal{I}_i. \tag{10}$$

We use the fact that

$$|f(x) - L(x, f)| \leq C\omega_\varphi^3(f, n^{-1}), \quad x \in \mathcal{I}_i, \tag{11}$$

where

$$L(x, f) := L(x, f, y_i', y_i, y_i'') = \frac{x - y_i}{y_i'' - y_i'}\Big(\frac{x - y_i'}{y_i'' - y_i} f(y_i'') + \frac{x - y_i''}{y_i - y_i'} f(y_i')\Big)$$

is the Lagrange polynomial of degree ≤ 2, which interpolates f at y_i', y_i and y_i''. Inequality (11) is an analog of Whitney's inequality for Ditzian–Totik moduli and can be found in [8, Lemma 18.2] by SHEVCHUK, for example.

Using (11) and the above presentations of $\widetilde{f}_i(x)$ and $L(x, f)$, we write for $x \in \mathcal{I}_i$,

$$|\widetilde{f}_i(x) - f(x)| \leq |\widetilde{f}_i(x) - L(x, f)| + |L(x, f) - f(x)| \leq$$

$$\leq \Big|\frac{(x - y_i)(x - y_i')}{(y_i'' - y_i')(y_i'' - y_i)}\Big| \Big|\widetilde{f}_i(y_i'') - f(y_i'')\Big| +$$

$$+ \Big|\frac{(x - y_i)(x - y_i'')}{(y_i'' - y_i')(y_i - y_i')}\Big| \Big|\widetilde{f}_i(y_i') - f(y_i')\Big| + C\omega_\varphi^3(f, n^{-1}) \leq C\omega_\varphi^3(f, n^{-1}),$$

and (10) is proved.

At this stage it is worth mentioning that the ability to construct a function $\widetilde{f}_i$ for which (9) and (10) hold determines the possibility to obtain the estimates in terms of the third modulus of smoothness in Theorem 1. For instance, if we could find a polynomial $\widetilde{g}_i$ of degree ≤ 3, copositive with f in $\mathcal{I}_i$, and such that inequalities (9) and (10) held with ω_φ^4 instead of ω_φ^3, then we would be able to replace ω_φ^3 in Theorem 1 by ω_φ^4. However, because of Theorem B, the creation of such $\widetilde{g}_i$ is impossible in general. The function

$\widetilde{f}_i$ is the best of what one can construct. At the same time, if we add some conditions on the behavior of f near the points of sign change, then estimate (4) can be improved (see Remark 2 in Section 4).

Now, let us continue with the proof of the lemma.

It is well known (see [3, Theorems 7.2.1 and 7.3.1], for example) that there exists a polynomial $Q(x)$ of degree $\le n$ (the polynomial of best approximation to f in $C[-1,1]$ will do) satisfying

$$\|f - Q\| \le C\omega_\varphi^3(f, n^{-1}) \tag{12}$$

and

$$\|\varphi(x)^3 Q'''(x)\| \le Cn^3\omega_\varphi^3(f, n^{-1}). \tag{13}$$

Now we define the piecewise polynomial function $\mathcal{S}(x)$ as follows:

$$\mathcal{S}(x) := \begin{cases} 1 & \text{if } x \notin \bigcup_{i=1}^r \mathcal{I}_i, \\ 0 & \text{if } x \in \bigcup_{i=1}^r \mathcal{Y}_i, \\ \lambda_i \int_{(y_i+y_i')/2}^{x} (y - y_i')^3 \Big(\frac{y_i + y_i'}{2} - y\Big)^3 dy & \text{if } x \in [y_i', \frac{y_i+y_i'}{2}], \\ & \quad i = 1, \dots, r, \\ \widetilde{\lambda}_i \int_{(y_i+y_i'')/2}^{x} \Big(y - \frac{y_i + y_i''}{2}\Big)^3 (y_i'' - y)^3 dy & \text{if } x \in [\frac{y_i+y_i''}{2}, y_i''], \\ & \quad i = 1, \dots, r. \end{cases}$$

where the normalizing constants λ_i and $\widetilde{\lambda}_i$ are chosen so that $\mathcal{S}$ be a continuous function,

$$\lambda_i = \Big(\int_{(y_i+y_i')/2}^{y_i'} (y - y_i')^3 \Big(\frac{y_i + y_i'}{2} - y\Big)^3 dy\Big)^{-1}$$

and

$$\widetilde{\lambda}_i = \Big(\int_{(y_i+y_i'')/2}^{y_i''} \Big(y - \frac{y_i + y_i''}{2}\Big)^3 (y_i'' - y)^3 dy\Big)^{-1}$$

Moreover, it is easy to see that not only $\mathcal{S}$ is continuous, but also $\mathcal{S} \in C^3[-1,1]$.

Finally, the function $f_n(x)$ such that

$$f_n(x) := \begin{cases} (Q(x) - \widetilde{f}_i(x))\mathcal{S}(x) + \widetilde{f}_i(x) & \text{if } x \in \mathcal{I}_i, \\ Q(x) & \text{otherwise.} \end{cases}$$

is copositive with f in $Y = \bigcup_{i=1}^r \mathcal{Y}_i$, and inequalities (6)–(8) are satisfied.

Indeed, f_n coincides with $\widetilde{f}_i$ on $\mathcal{Y}_i$ and, hence, it is copositive with f in Y, and (8) holds. Also, $\mathcal{S} \in C^3[-1,1]$ and $\mathcal{S}(x) = 1$, $x \notin \bigcup_{i=1}^r \mathcal{I}_i$ imply that f_n is in $C^3[-1,1]$. Inequality (6) follows from (10), (12), and the observation that $f_n(x)$ is a convex combination of $Q(x)$ and $\widetilde{f}_i(x)$ for every fixed $x \in \mathcal{I}_i \setminus \mathcal{Y}_i$ (since $0 \le \mathcal{S}(x) \le 1$).

To prove the remaining inequality (7) we need the following well known Kolmogorov type inequality (see, e.g., [1] or [8]):

(14) $$\|g^{(\nu)}\|_{[a,b]} \le C((b-a)^{r-\nu}\|g^{(r)}\|_{[a,b]} + (b-a)^{-\nu}\|g\|_{[a,b]}),$$

where $g \in C^r[a,b]$ and $0 \le \nu \le r$.

For

$$x \in \Big[y_i', \frac{y_i + y_i'}{2}\Big], \quad i = 1, \dots, r$$

(for $x \in [\frac{y_i+y_i''}{2}, y_i'']$ considerations are similar, (7) follows from (13) for $x \notin \bigcup_{i=1}^r \mathcal{I}_i$, and for $x \in Y$ it is trivial), using the fact that

$$\varphi(x) \sim n\Delta_n(x) \sim n|\mathcal{I}_i| \quad \text{for} \quad x \in \mathcal{I}_i,$$

we have

$$|\varphi(x)^3 f_n'''(x)| \le Cn^3|\mathcal{I}_i|^3 \sum_{\nu=0}^{3} |Q^{(\nu)}(x) - \widetilde{f}_i^{(\nu)}(x)||\mathcal{S}^{(3-\nu)}(x)|.$$

Applying (14) for $|Q^{(\nu)}(x) - \widetilde{f}_i^{(\nu)}(x)|$ and Markov's inequality for $|\mathcal{S}^{(3-\nu)}(x)|$, together with (10), (12), and (13), we obtain

$$|\varphi(x)^3 f_n'''(x)| \le Cn^3|\mathcal{I}_i|^3 \sum_{\nu=0}^{3} \Big(|\mathcal{I}_i|^{3-\nu}\|Q'''\|_{\mathcal{I}_i} +$$

$$+ |\mathcal{I}_i|^{-\nu}\|Q - \widetilde{f}_i\|_{\mathcal{I}_i}\Big)|\mathcal{I}_i|^{\nu-3}\|\mathcal{S}\|_{\mathcal{I}_i} \le Cn^3\omega_\varphi^3(f, n^{-1}).$$

This completes the proof of (7).

The following lemma can be found in [8, Theorem 18.2] by SHEVCHUK. It will be used to construct a polynomial approximant to f_n established in Lemma 3.

Lemma D. *If $g \in C^s[-1,1]$ is such that*

$$|(1-x^2)^{s/2} g^{(s)}(x)| \le M, \quad x \in [-1,1],$$

then for every $n \ge s-1$ there exists a polynomial $q_n(g) \in \Pi_n$ satisfying

(15) $$\|g - q_n(g)\| \le CMn^{-s}$$

and

(16) $$|\Delta_n(x)^\nu (g^{(\nu)}(x) - q_n^{(\nu)}(g,x))| \le CMn^{-s}, \quad 0 < \nu < s/2.$$

Corollary 4. *If $g \in C^3[-1,1]$ is such that*

$$|(1-x^2)^{3/2} g'''(x)| \le M, \quad x \in [-1,1], \quad -1 < y_1 < \dots < y_r < 1,$$

and

$$\delta := \min_{1 \le i \le r-1} |y_{i+1} - y_i|,$$

then for every $n \geq C(r,\delta)$ there exists a polynomial $p_n(g) \in \Pi_n$, which interpolates g at $y_1, \dots, y_r$ and such that

$$\|g - p_n(g)\| \leq K_2(r,\delta) M n^{-3} \tag{17}$$

and

$$\|\Delta_n(x)\,(g'(x) - p_n'(g,x))\,\| \leq K_3(r) M n^{-3}. \tag{18}$$

Proof. Let $q_n \in \Pi_n$ satisfy (15) and (16) with $s = 3$ and $\nu = 1$. Then the polynomial $p_n(g,x)$ given by

$$p_n(g,x) := q_n(x) + L(x, g - q_n, y_1, \dots, y_r),$$

interpolates g at $y_1, \dots, y_r$ and satisfies (17) and (18). Indeed,

$$\|g - p_n(g)\| \leq \|g - q_n\| + \|L(x, g - q_n, y_1, \dots, y_r)\| \leq C\|g - q_n\| \leq CMn^{-3},$$

which is inequality (17). Inequality (18) is valid since

$$\begin{aligned} &\|\Delta_n(x)(g'(x) - p_n'(g,x))\| \leq \\ &\leq \|\Delta_n(x)(g'(x) - q_n'(x))\| + 2n^{-1}\|L'(x, g - q_n, y_1, \dots, y_r)\| \leq \\ &\leq CMn^{-3} + \frac{2^{r-1} r(r-1)}{\delta^{r-1}} n^{-1} \|g - q_n\| \leq CMn^{-3} \end{aligned}$$

for sufficiently large n ($n > C\delta^{1-r}$). The proof of the corollary is complete.

Proposition 5. *For every y_i, $i = 1, \dots, r$, there exists an increasing polynomial $T_n(y_i, x)$ of degree $\leq n$, copositive with* $\operatorname{sgn}(x - y_i)$ *in $[-1,1]$, satisfying*

$$T_n(y_i, -1) = -1, \quad T_n(y_i, 1) = 1,$$

and such that

$$|\operatorname{sgn}(x - y_i) - T_n(y_i, x)| \leq K_4 \Big(\frac{\Delta_n(y_i)}{|x - y_i| + \Delta_n(y_i)} \Big)^2. \tag{19}$$

Proof. Let the index $i = 1, \dots, r$ and the integer n be fixed. It is known (see, e.g., [8, Lemma 17.2]) that for every $N = Cn \in \mathbf{N}$ there exist increasing polynomials $\widetilde{T}_N(y_i', x)$ and $\widetilde{T}_N(y_i'', x)$ of degree $\leq N$ such that

$$\widetilde{T}_N(y_i', -1) = \widetilde{T}_N(y_i'', -1) = -1, \quad \widetilde{T}_N(y_i', 1) = \widetilde{T}_N(y_i'', 1) = 1,$$

and satisfying

$$|\operatorname{sgn}(x - y_i') - \widetilde{T}_N(y_i', x)| \leq K_5 \Big(\frac{\Delta_N(y_i')}{|x - y_i'| + \Delta_N(y_i')} \Big)^2 \tag{20}$$

and

$$|\operatorname{sgn}(x - y_i'') - \widetilde{T}_N(y_i'', x)| \leq K_5 \Big(\frac{\Delta_N(y_i'')}{|x - y_i''| + \Delta_N(y_i'')} \Big)^2. \tag{21}$$

Now, we choose N to be sufficiently large, say, $N := [2\sqrt{K_5}+1]n$. Then, the following inequalities hold:

$$\widetilde{T}_N(y_i', y_i) \geq 1 - K_5\Big(\frac{\Delta_N(y_i')}{y_i - y_i' + \Delta_N(y_i')}\Big)^2 \geq$$

$$\geq 1 - K_5\Big(\frac{\Delta_N(y_i')}{\Delta_n(y_i')}\Big)^2 \geq 1 - 4K_5\Big(\frac{n}{N}\Big)^2 > 0$$

and, similarly,

$$\widetilde{T}_N(y_i'', y_i) \leq -1 + K_5\Big(\frac{\Delta_N(y_i'')}{y_i'' - y_i + \Delta_N(y_i'')}\Big)^2 \leq$$

$$\leq -1 + K_5\Big(\frac{\Delta_N(y_i'')}{\Delta_n(y_i'')}\Big)^2 \leq -1 + 4K_5\Big(\frac{n}{N}\Big)^2 < 0.$$

Therefore, there exists $0 < \alpha_i < 1$ such that

$$\alpha_i\widetilde{T}_N(y_i', y_i) + (1-\alpha_i)\widetilde{T}_N(y_i'', y_i) = 0.$$

Now, let

$$T_n(y_i, x) := \alpha_i\widetilde{T}_N(y_i', x) + (1-\alpha_i)\widetilde{T}_N(y_i'', x).$$

Then T_n is an increasing polynomial of degree $\leq Cn$ such that $T_n(y_i, y_i) = 0$ (this implies that T_n is copositive with $\operatorname{sgn}(y_i - x)$), and the following inequalities hold:

$$|\operatorname{sgn}(x-y_i) - T_n(y_i, x)| \leq$$

$$\leq |\operatorname{sgn}(x-y_i) - \operatorname{sgn}(x-y_i')| + |\operatorname{sgn}(x-y_i') - \widetilde{T}_N(y_i', x)| +$$

$$+ |\operatorname{sgn}(x-y_i) - \operatorname{sgn}(x-y_i'')| + |\operatorname{sgn}(x-y_i'') - \widetilde{T}_N(y_i'', x)| \leq$$

$$\leq C\Big(\frac{\Delta_n(y_i)}{|x-y_i| + \Delta_n(y_i)}\Big)^2 + C\Big(\frac{\Delta_N(y_i')}{|x-y_i'| + \Delta_N(y_i')}\Big)^2 +$$

$$+ C\Big(\frac{\Delta_N(y_i'')}{|x-y_i''| + \Delta_N(y_i'')}\Big)^2 \leq C\Big(\frac{\Delta_n(y_i)}{|x-y_i| + \Delta_n(y_i)}\Big)^2.$$

The proof of the proposition is complete.

The following result is a generalization of Lemma 1 of [4].

Lemma 6. *Let f be as in Theorem 1. If for $n \geq 4\delta^{-1}$ there exists $p_n \in \Pi_n$, copositive with f in $Y = \bigcup_{i=1}^r \mathcal{Y}_i$, then there is a polynomial $P_n \in \Pi_{K_6 n}$, $K_6 = K_6(r)$, which is copositive with f in $[-1,1]$ and such that*

$$\|f - P_n\| \leq C(r)\|f - p_n\|. \tag{22}$$

Sketch of the proof. The polynomial

$$P_n(x) := p_n(x) + 2^r \|f - p_n\| \eta \prod_{i=1}^{r} T_N(y_i, x),$$

where N is sufficiently large ($N = ([18\sqrt{K_4}] + 1)n$ will do), and $\eta = \pm 1$ is such that

$$\operatorname{sgn}(f(x)) = \eta \prod_{i=1}^{r} \operatorname{sgn}(x - y_i),$$

satisfies the assertion of the lemma. The verification of this fact is similar to the proof of Lemma 1 of [4]. The only difference is that instead of $(y_i - 1/2n, y_i + 1/2n)$ the intervals $\mathcal{Y}_i$, $i = 1, \dots, r$, are considered. Namely,

$$P_n(x)f(x) \geq 0 \quad \text{in} \quad \bigcup_{i=1}^{r} \mathcal{Y}_i,$$

since both p_n and $\eta \prod_{i=1}^{r} T_N(y_i)$ are copositive with f in this set. Also,

$$P_n(x)f(x) \geq 0 \quad \text{in} \quad [-1,1] \setminus \bigcup_{i=1}^{r} \mathcal{Y}_i,$$

since

$$\eta \prod_{i=1}^{r} T_N(y_i, x) f(x) \geq 0 \quad \text{and} \quad \Big|\prod_{i=1}^{r} T_N(y_i, x)\Big| \geq 2^{-r}.$$

Finally, (22) holds since

$$\left|\prod_{i=1}^{r} T_N(y_i, x)\right| \leq 1.$$

Proofs of Theorems 1 and 1′

Proof of Theorem 1. The proof of Theorem 1 is based on a modification of the ideas used by Y. Hu, Leviatan and X. M. Yu in [4].

Let $n \geq 4\delta^{-1}$ be fixed, and let $N = N(n) \geq n$ be an integer (we will prescribe its exact value later). Also, let $f_n \in C^3[-1,1]$ be a function which was described in Lemma 3. Inequality (7) can be written as

$$\|(1 - x^2)^{3/2} f_n'''(x)\| \leq M \quad \text{with} \quad M := K_1 n^3 \omega_\varphi^3(f, n^{-1}).$$

It follows from Corollary 4 that there exists a polynomial $p_N(f_n, x) \in \Pi_N$, which interpolates f_n at $y_1, \dots, y_r$ (i.e., $p_N(f_n, y_i) = 0$, $i = 1, \dots, r$), and such that

$$\|f_n - p_N(f_n)\| \leq K_1 K_2 \Big(\frac{n}{N}\Big)^3 \omega_\varphi^3(f, n^{-1}) \tag{23}$$

and

$$\|\Delta_N(x)\Big(f_n'(x) - p_N'(f_n,x)\Big)\| \le K_1K_3\Big(\frac{n}{N}\Big)^3\omega_\varphi^3(f,n^{-1}). \tag{24}$$

We prescribe N to be such that

$$K_1K_2(n/N)^3 \le 1 \quad \text{and} \quad K_1K_3(n/N)^2 \le \frac{1}{4}.$$

For instance,

$$N := K_7 n := ([(K_1K_2)^{1/3}] + [2\sqrt{K_1K_3}] + 2)n.$$

It follows from (24) that for $x \in \mathcal{Y}_i$, $i = 1, \dots, r$, the following estimate is valid:

$$|f_n'(x) - p_N'(f_n,x)| \le K_1K_3\frac{n}{N\Delta_N(x)}\Big(\frac{n}{N}\Big)^2\omega_\varphi^3(f,n^{-1}) \le$$

$$\le K_1K_3\frac{n}{\sqrt{1-x^2}}\Big(\frac{n}{N}\Big)^2\omega_\varphi^3(f,n^{-1}) \le \frac{1}{2}\Delta_n(x)^{-1}\omega_\varphi^3(f,n^{-1}).$$

Together with (8) this implies that

$$\operatorname{sgn}(p_N(f_n,x)) = \operatorname{sgn}(f_n(x)), \quad x \in \bigcup_{i=1}^r \mathcal{Y}_i.$$

In turn, it follows that $p_N(f_n)$ is copositive with f in $\bigcup_{i=1}^r \mathcal{Y}_i$, and also by (6) and (23),

$$\|f - p_N(f_n)\| \le \|f - f_n\| + \|f_n - p_N(f_n)\| \le C\omega_\varphi^3(f,n^{-1}).$$

Together with Lemma 6, this yields the assertion of Theorem 1 for $n > K_8 := 4\delta^{-1}K_6K_7$, $K_8 = K_8(r,\delta)$.

Proof of Theorem 1′. Clearly, we only have to prove Theorem 1′ for $0 \le n \le K_8$. If $r \ge 3$, then it is sufficient to choose

$$P_n(x) := L(x,f,y_1,\dots,y_r) \equiv 0.$$

In this case denoting

$$\widetilde{L}(x) := L\Big(x,f,-1,-1+\frac{2}{r-1},\dots,-1+\frac{2(r-2)}{r-1},1\Big),$$

we have for any $x \in [-1,1]$,

$$|f(x) - P_n(x)| = |f(x) - L(x,f,y_1,\dots,y_r)| =$$

$$= |f(x) - \widetilde{L}(x) - L(x,f-\widetilde{L},y_1,\dots,y_r)| \le \Big(1 + \frac{2^{r-1}r}{\delta^{r-1}}\Big)\|f - \widetilde{L}\|.$$

Now using Whitney's inequality we conclude that

$$\|f - P_n\| \le C\omega^r(f,1) \le C\omega^3(f,1) \le$$

$$\leq C\omega_\varphi^3(f,1) \leq C\omega_\varphi^3(f,K_8^{-1}) \leq C\omega_\varphi^3(f,n^{-1}),$$

where $C = C(r,\delta)$.

In the cases $r = 1$ and $r = 2$ for $2 \leq n \leq K_8$, one should apply a similar consideration for the polynomials of the second degree

$$P_n(x) := L(x,f,-1,y_1,1) \quad \text{and} \quad P_n(x) := L(x,f,-1,y_1,y_2),$$

respectively.

The proof of Theorem 1′ is now complete.

4. Remarks

Remark 1. All the considerations will remain the same if f vanishes on some subinterval(s), say, $[\alpha_i,\beta_i] \subset [-1,1]$. In this case, if $[\alpha_i,\beta_i]$ is an interval of sign change (i.e., if $f(\alpha_i-\varepsilon)f(\beta_i+\varepsilon) < 0$ for small ε), then it is sufficient to fix any $x_0 \in [\alpha_i,\beta_i]$ as a *point of sign change*. Thus, Theorem 1 is valid for any $f \in C[-1,1]$ with finitely many changes of sign. In fact, if f vanishes in all the intervals of sign change, then Theorems 1 and 1′ can be considerably improved (see the next remark).

Remark 2. As we mentioned in the proof of Lemma 3, estimate (4) can be improved if f satisfies extra conditions near the points of sign change. For example, the following theorem is valid.

Theorem 7. *Suppose $f \in C[-1,1]$ changes sign $1 \leq r < \infty$ times in $[-1,1]$ and vanishes in the intervals of sign change, i.e., suppose that*

$$f(x) = 0, \quad x \in \bigcup_{i=1}^{r} [y_i-\delta_i, y_i+\delta_i]$$

and

$$f(y_i-\delta_i-\varepsilon)f(y_i+\delta_i+\varepsilon) < 0 \quad \textit{for } i = 1,\dots,r,$$

and all sufficiently small ε. Let $\delta := \min\{\delta_1,\dots,\delta_r\}$ and $m \in \mathbf{N}$. Then there exists a sequence of polynomials $P_n \in \Pi_n$, copositive with f, such that

$$\|f-P_n\| \leq C(r,m,\delta)\omega_\varphi^m(f,n^{-1}). \tag{25}$$

Proof. Theorem 7 can be proved by using considerations similar to those in the proof of Theorem 1. The most important difference which makes estimate (25) possible, is that $\widetilde{f}_i$ in the proof of Lemma 3 can be replaced by the linear polynomial

$$\widetilde{l}_i(x) := \frac{60\omega_\varphi^m(f,n^{-1})}{y_i''-y_i'}((x-y_i')\operatorname{sgn} f(y_i+\delta_i+\varepsilon)+(y_i''-x)\operatorname{sgn} f(y_i-\delta_i-\varepsilon)).$$

However, there is a trivial proof.

For sufficiently large $n \in \mathbf{N}$, let $\mathcal{P}_n \in \Pi_n$ be the best approximant to f in $C[-1,1]$. Then

$$\|f - \mathcal{P}_n\| \le C\omega_\varphi^m(f, n^{-1}) \quad \text{and} \quad f(x)\mathcal{P}_n(x) = 0, \quad x \in Y := \bigcup_{i=1}^{r} \mathcal{Y}_i,$$

i.e., by definition, $\mathcal{P}_n$ is copositive with f in Y. Now, Lemma 6 implies that there exists a polynomial P_n, copositive with f in $[-1,1]$, such that

$$\|f - P_n\| \le C\|f - \mathcal{P}_n\| \le C\omega_\varphi^m(f, n^{-1}).$$

Another example of how the behavior of f near the points of sign change determines the rate of copositive polynomial approximation, is presented in the following theorem.

Theorem 8. *Let $f \in C[-1,1]$ change sign $r \ge 1$ times at $-1 < y_1 < \cdots < y_r < 1$, and let $f(x) \in \Pi_1$ for $x \in [y_i - \delta_i, y_i + \delta_i]$, $i = 1, \ldots, r$. Then, for any $m \in \mathbf{N}$, a sequence of polynomials $P_n \in \Pi_n$, copositive with f in* $[-1,1]$, *exists such that* (25) *holds with* $\delta := \min\{\delta_1, \ldots, \delta_r\}$.

The proof of Theorem 8 is less trivial than that of Theorem 7. We omit the details, and only mention that the crucial idea is again to replace $\widetilde{f}_i$ in the proof of Lemma 3 by a rapidly increasing or decreasing linear polynomial.

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О коположительной аппроксимации алгебраическими многочленами

КИРИЛЛ КОПОТУН

Для функции $f \in C[-1,1]$ с ограниченным числом перемен знака строится последовательность многочленов p_n, коположительных с f (т.е. $f(x)p_n(x) \geq 0$, $-1 \leq x \leq 1$) и таких, что

$$\|f - p_n\|_\infty \leq C\omega_\varphi^3(f, n^{-1}),$$

где $\omega_\varphi^3(f,\delta)$ — модуль непрерывности Дитциана–Тотика третьего порядка. Известно, что ω_φ^3 нельзя заменить ни на ω_φ^4, ни на ω^4. Таким образом, приведенная оценка точна в некотором смысле. В качестве следствия установлена эквивалентность соотношений

$$E_n(f) = O(n^{-\alpha}) \quad \text{и} \quad E_n^{(0)}(f,r) = O(n^{-\alpha}) \quad \text{для} \quad 0 < \alpha < 3.$$

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