A NOTE ON SIMULTANEOUS APPROXIMATION IN $L_p[-1,1]$ $(1 \le p < \infty)$

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Abstract

We prove, in particular, that for a function f such that $f^{(r-1)} \in AC$ and $f^{(r)} \in L_p[-1,1]$, $1 \le p < \infty$, and $m, n \in N$, there exists a polynomial P_n of degree $\le n$ such that

$$\|f^{(k)} - P_n^{(k)}\|_p \le C(m,r)\, \omega_\varphi^{m+r-k}(f^{(k)}, 1/n)_p, \quad 0 \le k \le r\,,$$

where $\omega_\varphi^s(g,\delta)_p$ is the Ditzian-Totik s-th order modulus of smoothness of g in L_p .

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1. Introduction

Let $L_p[a,b]$ be the set of all functions, which are measurable on $[a,b]$, and such that $\|f\|_{L_p[a,b]} := \left\{\int_a^b |f(x)|^p\,dx\right\}^{1/p} < \infty$. Also, let Π_n denote the set of all algebraic polynomials of degree not exceeding n and $\Delta_n(x) := \sqrt{1-x^2}\,n^{-1} + n^{-2}$.

Recall that the usual m-th order modulus of smoothness of a function $f \in L_p[a,b]$ is given by

$$\omega^m(f,\delta,[a,b])_p = \sup_{0<h\le\delta} \left\{ \int_a^b |\Delta_h^m(f,x,[a,b])|^p\,dx \right\}^{1/p},$$

where

$$\Delta_\eta^m(f,x,[a,b]) := \begin{cases} \sum_{k=0}^m \binom{m}{k}(-1)^{m-k} f(x - \frac{m}{2}\eta + k\eta), & \text{if } x \pm \frac{m}{2}\eta \in [a,b], \\ 0, & \text{otherwise.} \end{cases}$$

is the symmetric m-th difference.

If $\varphi(x) = \sqrt{1-x^2}$, then the Ditzian-Totik modulus is (see [4])

$$\omega_\varphi^m(f, \delta, [a,b])_p = \sup_{0<h\le\delta} \left\{ \int_a^b |\Delta_{h\varphi(x)}^m (f, x, [a,b])|^p \, dx \right\}^{1/p} .$$

For $I := [-1,1]$ for simplicity we write $\|\cdot\|_p := \|\cdot\|_{L_p(I)}$, $\omega^m(f,\delta)_p := \omega^m(f,\delta,I)_p$, $\omega_\varphi^m(f,\delta)_p := \omega_\varphi^m(f,\delta,I)_p$ and $\Delta_\eta^m(f,x) := \Delta_\eta^m(f,x,I)$.

In this note we prove the following theorem.

<u>Theorem 1.</u> *Let $m, r \in N$ and $f = f(x)$ be such that $f^{(r-1)}$ is absolutely continuous on $[-1,1]$ and $f^{(r)} \in L_p[-1,1]$ $(1 \le p < \infty)$. Then for any $n \ge m+r-1$ there exists a linear operator $P_n(f) : L_p[-1,1] \mapsto \Pi_n$ such that*

$$(1) \qquad \left\| \frac{f^{(k)}(x) - P_n^{(k)}(f,x)}{\Delta_n(x)^{r_k - k}} \right\|_p \le C(m,r)\, \omega_\varphi^{m+r-r_k} \left(f^{(r_k)}, 1/n \right)_p ,$$

for $k = 0, 1, \ldots, r$ and any integer r_k satisfying $k \le r_k \le r$.
Moreover, for $k \ge m+r$ and any integer $\tilde r$, $0 \le \tilde r \le r$ the following inequality holds:

$$(2) \qquad \left\| \Delta_n(x)^{k-\tilde r}\, P_n^{(k)}(f,x) \right\|_p \le C(k)\, \omega_\varphi^{m+r-\tilde r} \left(f^{(\tilde r)}, 1/n \right)_p .$$

For $p = \infty$ better estimates than those in Theorem 1 were proved in [8]. Choosing $r_k = k$ for $0 \le k \le r$ and $\tilde r = 0$ in Theorem 1 we obtain the following result.

<u>Corollary 2.</u> *Let $m, r \in N$ and $f = f(x)$ be the same as in Theorem 1. Then for any $n \ge m+r-1$ there exists $P_n(x) \in \Pi_n$ satisfying*

$$\left\| f^{(k)} - P_n^{(k)} \right\|_p \le C(m,r)\, \omega_\varphi^{m+r-k} \left(f^{(k)}, 1/n \right)_p \quad \text{for} \quad 0 \le k \le r$$

and

$$\left\| \Delta_n(x)^k P_n^{(k)}(x) \right\|_p \le C(k)\, \omega_\varphi^{m+r} (f, 1/n)_p \quad \text{for} \quad k \ge m+r .$$

By the same argument as was used by D. Leviatan in [9], employing the inequality $\omega_\varphi \left(f^{(r)}, 1/n \right)_p \le C \left\| f^{(r)} \right\|_p$, one can also show the validity of the following corollary of Theorem 1 (choose $r_k = r$ for $0 \le k \le r$ and $m = 1$).

Corollary 3. *Let $r \in N$ and $f = f(x)$ be such that $f^{(r-1)}$ is an absolutely continuous function on $[-1,1]$ and $f^{(r)} \in L_p[-1,1]$ $(1 \le p < \infty)$. Then for any $n \ge r$ there exists $P_n(x) \in \Pi_n$ such that*

$$(3) \qquad \left\| \frac{f^{(k)}(x) - P_n^{(k)}(x)}{\Delta_n(x)^{r-k}} \right\|_p \le C(r)\, E_{n-r}\left(f^{(r)}\right)_p, \quad 0 \le k \le r,$$

where $E_s(g)_p := \inf_{P_s \in \Pi_s} \|g - P_s\|_p$.

Corollary 3 was recently proved by D. Jiang [6]. In the case $p = \infty$ a better inequality than (3) is valid (see T. Kilgore [7]).

The next section contains auxiliary results. We also introduce some notations there. Finally, the proof of Theorem 1 is given in Section 3.

2. Notations and auxiliary results

The following notations are used throughout this paper (cf. [12]):

$$x_j := \cos\frac{j\pi}{n}, \; 0 \le j \le n,$$

$$I_j := [x_j, x_{j-1}], \; h_j := |I_j| = x_{j-1} - x_j, \; 1 \le j \le n,$$

$$\mathcal{I}_j := \begin{cases} I_j \cup I_{j-1}, & \text{if } 2 \le j \le n, \\ I_1, & \text{if } j = 1. \end{cases}$$

$$\Delta_n(x) := \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}.$$

It can be easily verified (see [12], for example) that $h_{j\pm1} < 3h_j$ and $\Delta_n(x) \le h_j \le 5\Delta_n(x)$ for $x \in I_j$.

As usual, C denote constants which are independent of f and n, and are not necessarily the same even when they occur on the same line.

For simplicity we also denote $\psi_j := \dfrac{h_j}{|x - x_j| + h_j}$.

The following lemma, which was proved by E. A. Storozhenko [13, Theorems 1 and 2], is an analog of the well known Whitney's theorem for the spaces $L_p[a,b]$, $1 \le p < \infty$.

Lemma A (E. A. Storozhenko). *Let $r \in N$ and $f = f(x)$ be such that $f^{(r-1)}$ is absolutely continuous on $[a,b]$ and $f^{(r)} \in L_p[a,b]$, $1 \le p < \infty$. Then for every $m \in N$ a linear operator $Q_{m+r-1}(f,[a,b]) : L_p[a,b] \mapsto \Pi_{m+r-1}$ exists satisfying*

$$(4) \qquad \left\| f^{(k)} - Q_{m+r-1}^{(k)}(f,[a,b]) \right\|_{L_p[a,b]} \le C(m,r)\, \omega^{m+r-k}\left(f^{(k)}, \frac{b-a}{2(m+r)}, [a,b]\right)_p$$

for $k = 0, 1, \ldots, r$.

Also, we need the following lemma which is a simple consequence of the results obtained in [8].

<u>Lemma B.</u> *Let $\mu, \xi \in N$ be such that $\mu \geq 7\xi$, and let $1 \leq j \leq n-1$ be a fixed index. Then there exists a polynomial $T_j(x)$ of degree $\leq 4\mu n$ such that the following inequalities hold for $x \in [-1, 1]$:*

$$(5) \qquad |T_j(x) - \chi_j(x)| \leq C(\mu)\, \psi_j^\mu$$

and

$$(6) \qquad \left|T_j^{(k)}(x)\right| \leq C(\mu)\, \psi_j^\mu\, h_j^{-k}, \ \ 1 \leq k \leq \xi\,,$$

where $\chi_j(x) := \begin{cases} 1, & \text{if } x \geq x_j, \\ 0, & \text{otherwise.} \end{cases}$

<u>Lemma C.</u> *For a function $g \in L_p[-1, 1]$ $(1 \leq p < \infty)$ and $s \in N$ the following inequality holds:*

$$(7) \qquad \sum_{j=1}^{n} \omega^s(g, h_j, \mathcal{I}_j)_p^p \leq C^p\, \omega_\varphi^s(g, n^{-1})_p^p\,.$$

<u>*Proof.*</u> In one form or another, the lemma is known. In fact, for $0 < p < 1$ the inequality, similar to (7), was proved by R. DeVore, D. Leviatan and X. M. Yu [1, inequalities (4.1), (4.5) and (4.6)]. The same proof works in the case $1 \leq p < \infty$. The main its idea is the employment of the inequality

$$(8) \qquad \omega^s(g, \delta, [a, b])_p^p \leq C\, \frac{1}{\delta} \int_0^\delta \int_a^b |\Delta_h^s(g, x, [a, b])|^p \, dx\, dh\,,$$

which appeared in [10, Lemma 7.2] (see also [3]). Namely, using (8) we have

$$\begin{aligned} \omega^s(g, h_j, \mathcal{I}_j)_p^p &\leq C^p\, h_j^{-1} \int_0^{h_j} \int_{\mathcal{I}_j} |\Delta_h^s(g, x, \mathcal{I}_j)|^p \, dx\, dh \\ &\leq C^p \int_{\mathcal{I}_j} \int_0^{h_j/\varphi(x)} \frac{\varphi(x)}{h_j} |\Delta_{h\varphi(x)}^s(g, x)|^p \, dh\, dx\,. \end{aligned}$$

Since $h_j/\varphi(x) \sim n^{-1}$ for $x \in \mathcal{I}_j$, $j = 3, \ldots, n-1$ we conclude that for these j

$$(9) \qquad \omega^s(g, h_j, \mathcal{I}_j)_p^p \leq C^p\, n \int_{\mathcal{I}_j} \int_0^{Cn^{-1}} |\Delta_{h\varphi(x)}^s(g, x)|^p \, dh\, dx\,.$$

Now, since $\Delta_{h\varphi(x)}^s(g, x) = 0$ if $\frac{s}{2} h\varphi(x) > 1 \pm x$, then it equals zero for $x \in \mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_n$ and $h > 30n^{-1}$. Hence, (9) also holds for $j = 1, 2$ and n.

Finally, inequality (9) yields

$$
\begin{aligned}
\sum_{j=1}^{n} \omega^s(g, h_j, \mathcal{I}_j)_p^p &\le C^p\, n \sum_{j=1}^{n} \int_{\mathcal{I}_j} \int_0^{Cn^{-1}} |\Delta^s_{h\varphi(x)}(g,x)|^p\, dh\, dx \\
&\le C^p\, n \int_{-1}^{1} \int_0^{Cn^{-1}} |\Delta^s_{h\varphi(x)}(g,x)|^p\, dh\, dx \\
&\le C^p\, n \int_0^{Cn^{-1}} \|\Delta^s_{h\varphi(x)}(g,x)\|_p^p\, dh \\
&\le C^p\, \omega^s_\varphi(g, n^{-1})_p^p\,.
\end{aligned}
$$

□

3. Proof of Theorem 1

The idea behind the proof of Theorem 1 is rather well known (see [1], [2], [8] and [12], for example). Namely, first, we approximate a function f (and its derivatives $f^{(k)}$) by the spline $L_n(f,x)$ (and $L_n^{(k)}(f,x)$) given by

$$
L_n(f,x) := p_n(f,x) + \sum_{j=1}^{n-1} [\, p_j(f,x) - p_{j+1}(f,x)]\ \chi_j(x)
$$

where $p_j(f) := Q_{m+r-1}(f, \mathcal{I}_j)$ is a polynomial of degree $\le m+r-1$ (with Q_{m+r-1} defined in Lemma A). Now, the polynomial $P_n(f,x)$ (in fact, it is a linear operator $L_p[-1,1] \mapsto \Pi_{Cn}$ with $C = C(m,r)$) such that

$$
P_n(f,x) := p_n(f,x) + \sum_{j=1}^{n-1} [\, p_j(f,x) - p_{j+1}(f,x)]\ T_j(x)
$$

(where T_j is defined in Lemma B with $\xi = m+r$ and $\mu = 7(m+r)$) satisfies (1) and (2).

To justify the above claim we show that

$$
J_1 := \left\| \frac{f^{(k)}(x) - L_n^{(k)}(f,x)}{\Delta_n(x)^{r_k - k}} \right\|_p \le C\, \omega_\varphi^{m+r-r_k}(f^{(r_k)}, n^{-1})_p
$$

and

$$
J_2 := \left\| \frac{L_n^{(k)}(f,x) - P_n^{(k)}(f,x)}{\Delta_n(x)^{r_k - k}} \right\|_p \le C\, \omega_\varphi^{m+r-r_k}(f^{(r_k)}, n^{-1})_p\,.
$$

This will prove inequality (1).

To estimate J_1, keeping in mind that $L_n(f,x) = p_j(f,x)$ if $x \in I_j$, we write for every $k = 0, \dots, r$ and $k \le r_k \le r$:

$$
\begin{aligned}
J_1^p &\le C^p \sum_{j=1}^{n} \int_{I_j} \left| \frac{f^{(k)}(x) - p_j^{(k)}(f,x)}{h_j^{r_k - k}} \right|^p dx \\
&\le C^p \sum_{j=1}^{n} h_j^{(k-r_k)p}\, \|f^{(k)} - p_j^{(k)}(f)\|_{L_p(I_j)}^p\,.
\end{aligned}
$$

Together with Lemmas A and C this implies

$$\begin{aligned} J_1^p &\le C^p \sum_{j=1}^n h_j^{(k-r_k)p}\, \omega^{m+r-k}(f^{(k)}, h_j, I_j)_p^p \\ &\le C^p \sum_{j=1}^n \omega^{m+r-r_k}(f^{(r_k)}, h_j, \mathcal{I}_j)_p^p \\ &\le C^p\, \omega_\varphi^{m+r-r_k}(f^{(r_k)}, n^{-1})_p^p \,. \end{aligned}$$

In order to estimate J_2 we recall that for a polynomial $P_\nu(x)$ of degree $\le \nu$ the inequality

$$(10)\quad |P_\nu(x)| \le C\, \psi_j^{-\nu}\, \|P_\nu\|_{C(I_j)}\,,\ x \in [-1,1]$$

holds (this follows, for example, from the Lagrange interpolation formula), and that

$$(11)\quad \|P_\nu\|_{C(I_j)} \le C h_j^{-1/p} \|P_\nu\|_{L_p(I_j)}$$

(see [10, Lemma 7.3], for example). Using (10) we have

$$\begin{aligned} J_2^p &\le \int_{-1}^1 \Delta_n(x)^{(k-r_k)p} \left\{ \sum_{j=1}^{n-1} \Big(|p_j^{(k)}(f,x) - p_{j+1}^{(k)}(f,x)|\, |\chi_j(x) - T_j(x)| \right. \\ &\quad \left. + \sum_{\nu=0}^{k-1} \binom{k}{\nu} |p_j^{(\nu)}(f,x) - p_{j+1}^{(\nu)}(f,x)|\, |T_j^{(k-\nu)}(x)| \Big) \right\}^p dx \\ &\le C^p \int_{-1}^1 \Delta_n(x)^{(k-r_k)p} \left\{ \sum_{j=1}^{n-1} \Big(\|p_j^{(k)}(f) - p_{j+1}^{(k)}(f)\|_{C(I_j)}\, \psi_j^{\mu-m-r+k+1} \right. \\ &\quad \left. + \sum_{\nu=0}^{k-1} \|p_j^{(\nu)}(f) - p_{j+1}^{(\nu)}(f)\|_{C(I_j)}\, h_j^{\nu-k}\, \psi_j^{\mu-m-r+\nu+1} \Big) \right\}^p dx\,. \end{aligned}$$

Now applying Markov's inequality first and then Jensen's inequality (for the latter the inequality $\sum_{j=1}^n \psi_j^\alpha \le C$, $\alpha \ge 2$ is used) we write

$$\begin{aligned} (12)\quad J_2^p &\le C^p \int_{-1}^1 \Delta_n(x)^{(k-r_k)p} \left\{ \sum_{j=1}^{n-1} \|p_j(f) - p_{j+1}(f)\|_{C(I_j)}\, h_j^{-k}\, \psi_j^{\mu/2} \right\}^p dx \\ &\le C^p \int_{-1}^1 \Delta_n(x)^{(k-r_k)p} \sum_{j=1}^{n-1} \|p_j(f) - p_{j+1}(f)\|_{C(I_j)}^p\, h_j^{-kp}\, \psi_j^{rp}\, \psi_j^{\mu/2-r}\, dx \\ &\le C^p \sum_{j=1}^{n-1} \|p_j(f) - p_{j+1}(f)\|_{C(I_j)}^p\, h_j^{-r_k p} \int_{-1}^1 \psi_j^{\mu/2-r}\, dx\,, \end{aligned}$$

since $\Delta_n(x) \ge C h_j \psi_j$ (see [8]) and $\psi_j \le 1$ for all $x \in [-1,1]$ and, therefore,

$$\Delta_n(x)^{(k-r_k)p}\, \psi_j^{rp} \le C^p\, h_j^{(k-r_k)p}\, \psi_j^{(r+k-r_k)p} \le C^p\, h_j^{(k-r_k)p}.$$

Since $\int_{-1}^1 \psi_j^\alpha dx \le C(\alpha) h_j$ for any $\alpha \ge 2$ (this is verified by straightforward computations), we conclude that

$$J_2^p \le C^p \sum_{j=1}^{n-1} h_j^{1-r_k p}\, \|p_j(f) - p_{j+1}(f)\|_{C(I_j)}^p\,,$$

and, therefore, using (11)

$$\begin{aligned} J_2^p &\leq C^p \sum_{j=1}^{n-1} h_j^{-r_k p} \int_{I_j} |p_j(f,x) - p_{j+1}(f,x)|^p \, dx \\ &\leq C^p \sum_{j=1}^{n} h_j^{-r_k p} \int_{\mathcal{I}_j} |f(x) - p_j(f,x)|^p \, dx \,. \end{aligned}$$

Using Lemma C and the fact that $\|f - p_j(f)\|_{L_p(\mathcal{I}_j)} \leq C\,\omega^{m+r}(f, h_j, \mathcal{I}_j)_p$ (see Lemma A) we conclude that

$$\begin{aligned} J_2^p &\leq C^p \sum_{j=1}^{n-1} h_j^{-r_k p}\, \omega^{m+r}(f, h_j, \mathcal{I}_j)_p^p \\ &\leq C^p \sum_{j=1}^{n} \omega^{m+r-r_k}(f^{(r_k)}, h_j, \mathcal{I}_j)_p^p \\ &\leq C^p \,\omega_\varphi^{m+r-r_k}(f^{(r_k)}, n^{-1})_p^p \,. \end{aligned}$$

Thus, the proof of (1) is complete, and it remains to show the validity of (2).

For any $\tilde{r}$, $0 \leq \tilde{r} \leq r$ and $k = m + r$ we write

$$\begin{aligned} \text{(13)}\quad J_3^p &:= \left\| \Delta_n(x)^{k-\tilde{r}} P_n^{(k)}(f,x) \right\|_p^p \\ &\leq \int_{-1}^{1} \Delta_n(x)^{(k-\tilde{r})p} \left\{ \sum_{j=1}^{n-1} \sum_{\nu=0}^{m+r-1} \binom{k}{\nu} |p_j^{(\nu)}(f,x) - p_{j+1}^{(\nu)}(f,x)|\, |T_j^{(k-\nu)}(x)| \right\}^p dx \\ &\leq C^p \int_{-1}^{1} \Delta_n(x)^{(k-\tilde{r})p} \left\{ \sum_{j=1}^{n-1} \|p_j(f) - p_{j+1}(f)\|_{C(I_j)}\, h_j^{-k}\, \psi_j^{\mu/2} \right\}^p dx \,. \end{aligned}$$

The only difference between the last quantity and the one in the first inequality of (12) is that it contains $\tilde{r}$ instead of r_k. Thus, it follows from the above estimates that

$$J_3 \leq C\,\omega_\varphi^{m+r-\tilde{r}}(f^{(\tilde{r})}, n^{-1})_p \,,$$

which proves (2) for $k = m + r$. Finally, for $k > m + r$ the inequality (2) follows from the case $k = m + r$ and the estimate

$$\|\Delta_n(x)^{\rho+1} P_n'(x)\|_p \leq C\, \|\Delta_n(x)^\rho P_n(x)\|_p \,, \; P_n \in \Pi_n \,, \; \rho \in R \,,$$

which is due to M. K. Potapov [11] (see also K. G. Ivanov [5, ineq. (4.11)]).

The proof of Theorem 1 is now complete.

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