On K-monotone Polynomial and Spline Approximation in L_p, $0 < p < \infty$ (Quasi)norm

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Abstract. Negative results for k-monotone polynomial and spline approximation in $L_p(0 < p < \infty)$ metric and unconstrained polynomial approximation in $L_p(0 < p < 1)$ quasi-norm are obtained. In particular, it is shown that the estimates $E_n^{(k)}(f)_p \le C\omega_2(f^{(k)}, 1)_p$ and $\mathcal{E}_{r,n}^{(k)}(f)_p \le C\omega_2(f^{(k)}, 1)_p$ are not true in general for $0 < p < \infty$, and the estimate $E_n(f)_p \le Cn^{-1}\omega_m(f', n^{-1})_p$ is true for not all $f \in AC$, if $0 < p < 1$, where $E_n^{(k)}(f)_p$, $\mathcal{E}_{r,n}^{(k)}(f)_p$ and $E_n(f)_p$ denote the rates of best k-monotone polynomial, k-monotone spline and unconstrained polynomial approximation in L_p, respectively.

§1 Introduction

Let $L_p[0,1]$, $0 < p \le \infty$ denote the space of all measurable functions on $[0,1]$ such that

$$\|f\|_p := \|f\|_{L_p[0,1]} := \begin{cases} \left(\int_0^1 |f(x)|^p\, dx\right)^{1/p}, & 0 < p < \infty\,, \\ \operatorname{ess\,sup}_{x\in[0,1]} |f(x)|\,, & p = \infty \end{cases}$$

is finite, and let $L_p^j[0,\,1]$ be the space of functions which are j-fold integrals of $L_p[0,\,1]$ functions. As usual, the integral modulus of smoothness of order m is given by $\omega_m(f,\delta)_p := \sup_{0<h\le\delta} \|\Delta_h^m(f,\cdot)\|_p$, where $\Delta_h^m(f,x) := \sum_{i=0}^m \binom{m}{i}(-1)^{m-i} f(x+ih)$, if $[x, x+mh] \subset [0,1]$ and $\Delta_h^m(f,x) := 0$, if $[x, x+mh] \not\subset [0,1]$, is the m-th (forward) difference (if $m = 0$, we set $\omega_0(f,\delta)_p := \|f\|_p$).

Also, let Π_n denote the set of all algebraic polynomials of degree $\le n$, and $\mathcal{S}(r,N)$ be the space of all splines of order r with knots $\{i/N\}_{i=0}^N$ (*i.e.*, $s \in \mathcal{S}(r,N)$ if and only if s is a polynomial of degree $\le r-1$ in each

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interval $[i/N,(i+1)/N]$, and $s \in C^{r-2}[0,1]$). We also denote by Δ^k the set of all functions f such that $\Delta_h^k(f,x) \ge 0$ for all $h \ge 0$ and $x \in [0,1]$. In particular, Δ^1 is the set of all nondecreasing functions. It is easy to see that, if $f \in C^k[0,1]$, then $f \in \Delta^k$ if and only if $f^{(k)}(x) \ge 0$, $x \in [0,1]$. We are interested in approximation of such functions by polynomials and splines from Δ^k, *i.e.*, in the so called "k-monotone approximation". Recall that the rates of best unconstrained and k-monotone polynomial approximation are given, respectively, by $E_n(f)_p := \inf_{P_n\in\Pi_n} \|f-P_n\|_p$ and $E_n^{(k)}(f)_p := \inf_{P_n\in\Pi_n\cap\Delta^k} \|f-P_n\|_p$.

The following Theorem A was proved by A. S. Shvedov [15] in the case $i=0$. Recently, S. P. Manya noticed that the same proof works for all $i \le k-1$ (see [13], for example).

Theorem A. *Let $A>0$, $k\ge 1$, $0<p\le\infty$, $n\ge k+1$ and $0\le i\le k-1$ be given. Then there exists a function $f\in C^k[0,1]\cap\Delta^k$, such that*

$$E_n^{(k)}(f)_p > A\omega_{k+2-i}(f^{(i)},1)_p\,. \tag{1}$$

In the case for monotone approximation, Theorem A implies that the estimate $E_n^{(1)}(f)_p \le C\omega_3(f,n^{-1})_p$ is not true in general. At the same time the following (direct) result is known.

Theorem B. *If $f\in L_p[0,1]\cap\Delta^1$, $0<p\le\infty$, then for any $n\ge 1$*

$$E_n^{(1)}(f)_p \le C\omega_2(f,n^{-1})_p\,. \tag{2}$$

Therefore, in the case $1\le p\le\infty$, if $f\in L_p^1[0,1]\cap\Delta^1$, then

$$E_n^{(1)}(f)_p \le Cn^{-1}\omega(f',n^{-1})_p\,. \tag{3}$$

Theorem B was proved by R. A. DeVore [2] for $p=\infty$, by A. S. Shvedov [15] in the case $1\le p<\infty$, and by R. A. DeVore, D. Leviatan and X. M. Yu [4] for $0<p<1$.

In the case $p=\infty$, it was proved by I. A. Shevchuk [11] and, independently, by Y. P. Ma and X. M. Yu [16] that, if f is continuously differentiable, then a much better estimate than (3) is true. Namely, the following result was established.

Theorem C. *Let $f\in C^1[0,1]\cap\Delta^1$, then*

$$E_n^{(1)}(f)_\infty \le Cn^{-1}\omega_m(f',n^{-1})_\infty \text{ for any } n\ge m\,. \tag{4}$$

Thus, the natural question on whether one can improve the estimate (3) in the case $0<p<\infty$ arises. We will show that it is impossible, that is, ω in (3) can not be replaced by ω_m with $m\ge 2$. In fact, we will prove the following negative result in the general k-monotone case. (This result is a generalization of Theorem 1 of [7], and its proof is based on a modification of the counterexample from [8].)

Theorem 1. *Let $k \in N$ and $0 < p < \infty$ be fixed, and let $\nu \in \mathcal{N}$ and $m \in \mathcal{N} \cup \{0\}$ be such that $\max\{k+2-m, k\} \le \nu < k + p^{-1}$. Then for any $n \in \mathcal{N}$, $0 < \varepsilon \le 1$ and $A > 0$ there exists a function $f \in C^\infty[0,1]$, $f^{(k)}(x) \ge 0$, $x \in [0,1]$ such that for every $P_n \in \Pi_n$, $P_n^{(k)}(0) \ge 0$ the following inequality holds*

$$\|f - P_n\|_{L_p[0,\varepsilon]} > A\omega_m(f^{(\nu)}, 1)_p. \tag{5}$$

Corollary 2. *The estimate*

$$E_n^{(k)}(f)_p \le C\omega_2(f^{(k)}, 1)_p \tag{6}$$

is not true in general for $0 < p < \infty$ and $f \in C^\infty[0,\, 1] \cap \Delta^k$, $k \in \mathcal{N}$.

In the case $1 \le p < \infty$ Theorem A is an immediate consequence of Corollary 2. However, we can not say the same if $0 < p < 1$, since the estimate $\omega_k(f, \delta)_p \le C\delta\omega_{k-1}(f', \delta)_p$ is no longer valid if $p < 1$ (though it is easy to modify the proof of Theorem 1 to yield (1) for all $i \le k-1$ and $p < \infty$). It is well known that L_p, $0 < p < 1$ spaces are "pathological in nature". For example, they are not Banach spaces, there are no linear continuous functionals in L_p (except the zero functional), etc. It was recently shown by Z. Ditzian, V. H. Hristov and K. Ivanov [6] that the Peetre K-functional between L_p and W_p^r is identically zero.

Theorem D. *For $0 < p < 1$, $r \in \mathcal{N}$, $t > 0$ and any $f \in L_p[0,1]$ we have $K_r(f, t^r; L_p, W_p^r) := \inf_{g \in C^r[0,1]} \left(\|f - g\|_p + t^r\|g^{(r)}\|_p\right) = 0$.*

Z. Ditzian [5] proved that the rate of simultaneous approximation of a function and its derivatives is very bad if $0 < p < 1$.

Theorem E. *For $0 < p < 1$ and $f \in AC[0,1]$ we can not have $P_n \in \Pi_n$ such that $\|f - P_n\|_p \le C\,\omega_2(f, n^{-1})_p$ and $\|f' - P_n'\|_p \le C\,\omega(f', n^{-1})_p$ simultaneously with constants independent of f and n.*

Thus, the following result is not surprising.

Theorem 3. *For every $A > 0$, $B \in \mathcal{R}$, $0 < p < 1$ and $n \in \mathcal{N}$ there exists an absolutely continuous function f ($f \in AC[0,1]$), such that*

$$E_n(f)_p > An^B\|f'\|_p\,. \tag{7}$$

Corollary 4. *The estimate $E_n(f)_p \le Cn^{-1}\omega_m(f', n^{-1})_p$ is not true in general for $0 < p < 1$ and any $m \in \mathcal{N} \cup \{0\}$.*

At the same time, the condition $f \in \Delta^k$ is rather strong (at least for $k \ge 2$) in the sense that it eliminates those functions f which "bring anomalous properties" into L_p for $p < 1$. To illustrate this we only mention that employing the method of the proof used by R. A. DeVore and D. Leviatan [3] one can show the validity of the following (direct) result on convex polynomial approximation in L_p for $0 < p < 1$.

Theorem F. *Let $f \in L^1_p[0,1]$ $(0 < p < 1)$ be convex, then for every $n \geq 1$*

$$E_n^{(2)}(f)_p \leq Cn^{-1}\omega(f', n^{-1})_p\,. \tag{8}$$

It is well known that for unconstrained polynomial approximation one has the following estimate

$$E_n(f)_p \leq Cn^{-1}E_{n-1}(f')_p, \qquad 1 \leq p \leq \infty\,. \tag{9}$$

It follows from Theorem 3 that (9) is not true if $0 < p < 1$.

Corollary 5. *For every $A > 0$, $B \in \mathcal{R}$ and $0 < p < 1$, there exists $n \in \mathcal{N}$ and $f \in AC[0,1]$ such that*

$$E_n(f)_p > An^B E_{n-1}(f')_p\,. \tag{10}$$

Now, the natural question is whether the estimate

$$E_n^{(1)}(f)_p \leq Cn^{-1}E_{n-1}(f')_p \tag{11}$$

is true for $f \in L^1_p[0,1] \cap \Delta^1$. In view of the estimate (4) it seems that it would be reasonable to expect that (11) is true at least in the case $p = \infty$. Moreover, it is not difficult to see that, if $p = \infty$, then the following inequality, which is weaker then (11), holds:

$$E_n^{(1)}(f)_\infty \leq CE_{n-1}(f')_\infty\,. \tag{12}$$

Indeed, the following proof, based on a slight modification of one of the proofs in O. Shisha's paper [14], was proposed by D. Leviatan. Let $f \in C^1[0,1] \cap \Delta^1$, and let $p_n(x) := E_{n-1}(f')_\infty x + q_n(x) + f(0) - q_n(0)$, where q_n' is the best approximation to f' from Π_{n-1}. Then $p_n \in \Pi_n \cap \Delta^1$, and

$$\begin{aligned} E_n^{(1)}(f)_\infty &\leq \|f - p_n\|_\infty = \left\| \int_0^x (f'(y) - p_n'(y))\,dy \right\|_\infty \\ &\leq \|f' - p_n'\|_\infty = \|f' - q_n' - E_{n-1}(f')_\infty\|_\infty \leq CE_{n-1}(f')_\infty\,. \end{aligned} \tag{13}$$

Despite all the above, it was recently proved by I. A. Shevchuk [12] that (11) is not true for $p = \infty$. Namely, there exists an absolute constant C_0 $(C_0 = \frac{1}{200})$ and a function $f \in C^1[0,1] \cap \Delta^1$ such that

$$E_n^{(1)}(f)_\infty \geq C_0 E_{n-1}(f')_\infty\,. \tag{14}$$

Thus, in a sense, (12) is the best possible estimate of this type. If $p < \infty$, then the rate of approximation deteriorates even further, and even the estimate (12) is no longer valid. The following is a consequence of Theorem 1.

Corollary 6. *For any $n \in \mathcal{N}$, $k \in \mathcal{N}$, $0 < p < \infty$ and $A > 0$ there exists $f \in C^\infty[0,1] \cap \Delta^k$ such that*

$$E_n^{(k)}(f)_p > A E_{n-k}(f^{(k)})_p \,. \tag{15}$$

Another corollary of Theorem 1 is the fact that one can not have the estimate

$$\mathcal{E}_{r,n}^{(k)}(f)_p \le C\omega_2(f^{(k)}, 1)_p, \qquad 0 < p < \infty \tag{16}$$

for k-monotone spline approximation, where

$$\mathcal{E}_{r,n}^{(k)}(f)_p := \inf_{s \in \mathcal{S}(r,n) \cap \Delta^k} \|f - s\|_p \,.$$

Thus, the following result on monotone spline approximation in L_p, $1 \le p \le \infty$ is the best possible in the sense of the orders of moduli of smoothness. (In the case $k = 1$ and $1 < p < \infty$ it was also recently proved by X. M. Yu and S. P. Zhou [17].)

Theorem G (monotone spline approximation). *The following estimates are valid:*

$$\mathcal{E}_{r,n}^{(1)}(f)_\infty \le Cn^{-1}\omega_{r-1}(f', n^{-1})_\infty,$$

if $f \in C^1[0,1] \cap \Delta^1$ and $r \ge 2$ (D. Leviatan and H. N. Mhaskar [9]),

$$\mathcal{E}_{r,n}^{(1)}(f)_p \le Cn^{-2}\omega_{r-2}(f'', n^{-1})_p,$$

if $f \in L_p^2[0,1] \cap \Delta^1$ $(1 \le p < \infty)$ and $r \ge 3$ ([9]),

$$\mathcal{E}_{r,n}^{(1)}(f)_p \le Cn^{-1}\omega(f', n^{-1})_p,$$

if $f \in L_p^1[0,1] \cap \Delta^1$ $(1 \le p < \infty)$ and $r \ge 2$ (C. K. Chui, P. W. Smith and J. D. Ward [1]).

§2 Proofs

Proof of Theorem 1. We now construct the counterexample described in Theorem 1. This counterexample is a modification of the one used in the proof of Theorem 2 of [8] (see also Theorem 1 of [7]).

Let $n \in \mathcal{N}$, $0 < \varepsilon \le 1$, $A > 0$ and $0 < p < \infty$ be fixed, and define

$$f_\xi(x) := \int_0^x \int_0^{x_1} \dots \int_0^{x_{k-1}} \left(\xi x_k - \ln(x_k + e^{-\xi}) - \ln \xi\right) \, dx_k \dots dx_1$$

$$= \frac{1}{(k-1)!} \int_0^x (x-y)^{k-1} \left(\xi y - \ln(y+e^{-\xi}) - \ln \xi\right) \, dy \,,$$

where $\xi \geq 1$ will be chosen later. Clearly, $f_\xi \in C^\infty[0,1]$, and it is easy to check that $f_\xi^{(k)}(x) = \xi x - \ln(x+e^{-\xi}) - \ln \xi \geq 0$, $x \in [0,1]$.

Suppose that the assertion of the theorem is not true, *i.e.*, that for every $\xi \geq 1$ there exists a polynomial $P_{\xi,n}(x) = a_0 + a_1 x + \ldots + a_n x^n \in \Pi_n$ such that $P_{\xi,n}^{(k)}(0) = k! a_k \geq 0$, and

$$\|f_\xi - P_{\xi,n}\|_{L_p[0,\varepsilon]} \leq A \omega_m(f_\xi^{(\nu)}, 1)_p \,. \tag{17}$$

Now, note that there exists a constant C_0 which depends only on p, ν and k, such that

$$\omega_m(f_\xi^{(\nu)}, 1)_p \leq C_0 \,. \tag{18}$$

Indeed, if $\nu = k$ (which is possible only in the case $m \geq 2$), then

$$\omega_m(f_\xi^{(\nu)}, 1)_p^p = \omega_m\left(\left(\ln(x+e^{-\xi})\right), 1\right)_p^p \leq \left\|\ln(x+e^{-\xi})\right\|_p^p$$

$$= \int_0^1 \left|\ln(x+e^{-\xi})\right|^p \, dx \leq \int_0^2 |\ln x|^p \, dx \leq 1 + \Gamma(p+1) \,.$$

If $\nu > k$, then

$$\omega_m(f_\xi^{(\nu)}, 1)_p^p \;=\; \omega_m\left(\left(\ln(x+e^{-\xi})\right)^{(\nu-k)}, 1\right)_p^p \leq \left\|\left(\ln(x+e^{-\xi})\right)^{(\nu-k)}\right\|_p^p$$

$$= \;(\nu-k-1)!\; \left\|\left(x+e^{-\xi}\right)^{k-\nu}\right\|_p^p \leq (\nu-k-1)! \int_0^1 \frac{dx}{x^{(\nu-k)p}} = \frac{(\nu-k-1)!}{1-(\nu-k)p} \,,$$

since $(\nu-k)p - 1 < 0$. Now, using the estimate

$$\left\|\int_0^x (x-y)^{k-1} \ln(y+e^{-\xi}) \, dy\right\|_p \leq \int_0^1 (1-y)^{k-1} \left|\ln(y+e^{-\xi})\right| \, dy$$

$$\leq \int_0^1 \left|\ln(y+e^{-\xi})\right| \, dy \leq \int_0^2 |\ln y| \, dy = 2\ln 2 \,,$$

together with (17) and (18), we have

$$\left\|\frac{1}{(k-1)!} \int_0^x (x-y)^{k-1} \left(\xi y - \ln \xi\right) \, dy - P_{\xi,n}(x)\right\|_{L_p[0,\varepsilon]}$$

$$\leq 2^{\max\{1,1/p\}} (A C_0 + 2\ln 2)$$

and, therefore (see Lemma 7.3 of [10]),

$$\left\| \frac{1}{(k-1)!} \int_0^x (x-y)^{k-1} \left(\xi y - \ln \xi\right) \, dy - P_{\xi,n}(x) \right\|_{L_\infty[0,\varepsilon]}$$
$$\le C \varepsilon^{-1/p} 2^{\max\{1,1/p\}} (AC_0 + 2\ln 2) =: C_1 \,.$$

Now, applying Markov's inequality, we get

$$\left\| \xi x - \ln \xi - P_{\xi,n}^{(k)}(x) \right\|_{L_\infty[0,\varepsilon]} \le \varepsilon^{-k} n^{2k} C_1$$

and, in particular, $\left| \ln \xi + P_{\xi,n}^{(k)}(0) \right| \le \varepsilon^{-k} n^{2k} C_1$. Therefore, choosing $\xi = \exp\{\varepsilon^{-k} n^{2k} C_1\} + 1$ we get $P_{\xi,n}^{(k)}(0) < \varepsilon^{-k} n^{2k} C_1 - \ln \xi < 0$, thus, obtaining a contradiction. ■

Proof of Theorem 3. Let $\xi \le n^{-2}/4$ be a parameter which will be chosen later. For $x \in [0, n^{-2}]$ we define

$$f_\xi(x) := \begin{cases} x\xi^{-1}, & x \in [0, \xi], \\ 1, & x \in [\xi, n^{-2}/2], \\ n^{-2}\xi^{-1}/2 + 1 - x\xi^{-1}, & x \in [n^{-2}/2, n^{-2}/2 + \xi], \\ 0, & x \in [n^{-2}/2 + \xi, n^{-2}]. \end{cases}$$

Now, let $f_\xi(x) := f_\xi\left(x - [xn^{-2}]n^{-2}\right)$ for $x \in [n^{-2}, 1]$. Clearly, $f_\xi \in AC[0,1]$. Let us denote $\mathcal{Y}_1 := \{x \in [0,1] : x - [xn^{-2}]n^{-2} \in [\xi, \frac{1}{2n^2}]\} = \cup_{i=0}^{n^2-1} [\frac{i}{n^2} + \xi, \frac{i}{n^2} + \frac{1}{2n^2}]$, $\mathcal{Y}_2 := \{x \in [0,1] : x - [xn^{-2}]n^{-2} \in [\frac{1}{2n^2} + \xi, \frac{1}{n^2}]\} = \cup_{i=0}^{n^2-1} [\frac{i}{n^2} + \frac{1}{2n^2} + \xi, \frac{i+1}{n^2}]$ and $\mathcal{Y}_3 := [0,1] \setminus \{\mathcal{Y}_1 \cup \mathcal{Y}_2\}$. Then $f_\xi(x) = 1$ if $x \in \mathcal{Y}_1$, $f_\xi(x) = 0$ if $x \in \mathcal{Y}_2$, and $|f_\xi'(x)| = \xi^{-1}$ if $x \in \mathcal{Y}_3$. Hence, $\|f_\xi'\|_p^p = \xi^{-p} meas\{\mathcal{Y}_3\} = 2n^2\xi^{1-p}$. At the same time, since every polynomial P_n of degree $\le n$ has not more than $n-1$ points of monotonicity change, there exists an interval $[a,b] \subset [0,1]$ of the length at least $1/n$ such that P_n is monotone on $[a,b]$. In turn, this implies the existence of an interval $[\alpha,\beta] \subset [a,b]$ of the length $1/(2n)$ (in fact, $[\alpha,\beta]$ is $[a,(a+b)/2]$ or $[(a+b)/2, b]$) such that either $P_n(x) \ge 1/2$ or $P_n(x) \le 1/2$ for $x \in [\alpha,\beta]$. Suppose that $P_n(x) \ge 1/2$, $x \in [\alpha,\beta]$ (the other case is treated similarly). Then

$$\|f_\xi - P_n\|_p^p \ge \int_{\mathcal{Y}_2 \cap [\alpha,\beta]} |f_\xi(x) - P_n(x)|^p \, dx = \int_{\mathcal{Y}_2 \cap [\alpha,\beta]} |P_n(x)|^p \, dx$$
$$\ge 2^{-p} meas\{\mathcal{Y}_2 \cap [\alpha,\beta]\} \ge 2^{-p-1} n (n^{-2}/2 - \xi) \ge 2^{-p-3} n^{-1} \,.$$

(In fact, using more precise calculations one can show that $\|f_\xi - P_n\|_p^p \ge C_p$.) Finally, it is sufficient to choose

$$\xi \le \min\left\{ n^{-2}/4, \left(2^{-p-4} A^{-p} n^{-Bp-3}\right)^{1/(1-p)} \right\}$$

in order to complete the proof of the theorem. ■

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