Constr. Approx. (1996) 12: 67–94

CONSTRUCTIVE APPROXIMATION

Simultaneous Approximation by Algebraic Polynomials

K. Kopotun

Abstract. Some estimates for simultaneous polynomial approximation of a function and its derivatives are obtained. These estimates are exact in a certain sense. In particular, the following result is derived as a corollary:

For $f \in C^r[-1,1]$, $m \in \mathbf{N}$, and any $n \geq \max\{m+r-1, 2r+1\}$, an algebraic polynomial P_n of degree $\leq n$ exists that satisfies

$$\left|f^{(k)}(x) - P_n^{(k)}(f,x)\right| \leq C(r,m)\Gamma_{nrmk}(x)^{r-k}\omega^m\left(f^{(r)}, \Gamma_{nrmk}(x)\right),$$

for $0 \leq k \leq r$ and $x \in [-1,1]$, where $\omega^\nu(f^{(k)}, \delta)$ denotes the usual νth modulus of smoothness of $f^{(k)}$, and

$$\Gamma_{nrmk}(x) := \begin{cases} n^{-1}\sqrt{1-x^2}, & \text{if } x \in [-1+n^{-2}, 1-n^{-2}] \\ (1-x^2)^{(r-k+1)/(r-k+m)}\left(\dfrac{1}{n^2}\right)^{(m-1)/(r-k+m)}, & \text{if } x \in [-1,-1+n^{-2}] \\ & \quad \cup [1-n^{-2}, 1]. \end{cases}$$

Moreover, for no $0 \leq k \leq r$ can $(1-x^2)^{(r-k+1)/(r-k+m)}(1/n^2)^{(m-1)/(r-k+m)}$ be replaced by $(1-x^2)^{\alpha_k}n^{2\alpha_k-2}$, with $\alpha_k > (r-k+1)/(r-k+m)$.

1. Introduction

We begin by recalling some standard notations. The symmetric mth difference of a function f is given by

$$\Delta_\eta^m(f,x,[a,b]) := \begin{cases} \displaystyle\sum_{i=0}^m \binom{m}{i}(-1)^{m-i} f\left(x - \frac{m}{2}\eta + i\eta\right), & \text{if } \ x \pm \dfrac{m}{2}\eta \in [a,b], \\ 0, & \text{otherwise.} \end{cases}$$

The Ditzian-Totik modulus of smoothness is (see [5] and [6])

$$\omega_{\varphi^\lambda}^m(f,\delta,[a,b]) := \sup_{0<h\leq\delta} \|\Delta_{h\varphi(x)^\lambda}^m(f,x,[a,b])\|_{C[a,b]},$$

Date received: January 11, 1994. Date revised: November 8, 1994. Communicated by Vladimir N. Temlyakov.
AMS classification: 41A10, 41A25, 41A28.
Key words and phrases: Simultaneous approximation, Timan-Gopengauz type estimates, Ditzian-Totik moduli of smoothness.

where $\varphi(x) := \sqrt{1-x^2}$ and $[a, b] \subset [-1, 1]$. Note that if $\lambda = 0$, then

$$\omega_1^m(f, \delta, [a, b]) =: \omega^m(f, \delta, [a, b])$$

is the usual modulus. Also, we denote the set of all algebraic polynomials of degree $\le n$ by Π_n, and we let $\Delta_n(x) := n^{-1}\sqrt{1-x^2} + n^{-2}$ and $\omega_{\varphi^\lambda}^m(f, \delta) := \omega_{\varphi^\lambda}^m(f, \delta, [-1, 1])$.

The following results on simultaneous approximation of a function and its derivatives in terms of the usual moduli of smoothness are known.

Theorem A. *Let $f \in C^r[-1, 1]$ and $m \in \mathbf{N}$. Then there exists an integer $n_0 = n_0(r, m)$ such that for any $n \ge n_0$ there is a polynomial $P_n \in \Pi_n$ satisfying, for $0 \le k \le r$ and $x \in [-1, 1]$,*

$$|f^{(k)}(x) - P_n^{(k)}(x)| \le C(r, m)\Delta_n(x)^{r-k}\omega^m(f^{(r)}, \Delta_n(x)). \tag{1}$$

Theorem B. *Let $f \in C^r[-1, 1]$ and $m \in \mathbf{N}$. Then there exists an integer $n_0 = n_0(r, m)$ such that for any $n \ge n_0$, there is a polynomial $P_n \in \Pi_n$ satisfying*

$$|f^{(k)}(x) - P_n^{(k)}(x)| \le C(r, m)\left(\frac{\sqrt{1-x^2}}{n}\right)^{r-k}\omega^m\left(f^{(r)}, \frac{\sqrt{1-x^2}}{n}\right), \tag{2}$$

for $0 \le k \le \min\{r - m + 2, r\}$ and $x \in [-1, 1]$.

Moreover, the condition $k \le r - m + 2$ cannot be removed.

Remark. In Theorem A, the exact lower bound on n is $n_0 = m + r - 1$. Indeed, it is easy to see that for $n = m + r - 1$, Theorem A is valid (choose P_{m+r-1} to be a Lagrange interpolation polynomial of degree $\le m + r - 1$), and that for $n < m + r - 1$, (1) is no longer true (consider $f \in \Pi_{m+r-1}$). At the same time, the exact lower bound on n in Theorem B, as far as we know, is not found (at least not for all r and m). It follows from Corollary 2-3.1 in Section 2 (see also [22]) that $n_0 \le 2r + 1$. Also, it is not difficult to see that (2) implies $f^{(k)}(\pm 1) = P_n^{(k)}(\pm 1)$ for $k = 0, 1, \dots, \tilde{k} + [(r - \tilde{k})/2]$, where $\tilde{k} := \min\{r - m + 2, r\}$, and, therefore, $n_0 \ge \max\{m + r - 1, 2\tilde{k} + 2[(r - \tilde{k})/2] + 1\}$. Thus, in the case $3 \le m \le r + 1$ the question about the exact value of n_0 in Theorem B remains unanswered.

The following is a brief history of proofs of Theorems A and B. Estimate (1) with $k = 0$ and $m = 1$ was obtained by A. F. Timan [28] in 1951. In 1955, A. O. Gelfond [10] proved Theorem A with n^{-1} instead of $\Delta_n(x)$ and $m = 1$. In 1962, R. M. Trigub [29] showed the validity of Theorem A in the case $m = 1$ and remarked that the same proof works for $m = 2$ (see also V. N. Malosemov [23]). In 1963, Yu. A. Brudnyi [1] (see also [2]) extended Timan's result showing that (1) is valid for $k = 0$ and arbitrary $m \in \mathbf{N}$. In 1966-67, S. A. Telyakovskii [27] and I. E. Gopengauz [12] independently proved (2) in the cases $m = 1, k = 0$ and $m = 1, 0 \le k \le r$, respectively. In 1967, I. E. Gopengauz [13] proved Theorem A in general. In 1975, R. A. DeVore [4] being the first to prove estimates involving $\omega^m(\sqrt{1-x^2}/n)$ with $m > 1$, obtained (2) for $m = 2$ and $r = 0$. Eight years later, in 1983, E. Hinnemann and H. H. Gonska [16] proved the case $m = 2$, $r \ge 0$ and $k = 0$ in Theorem B. In 1985, they [11] also showed the validity of (2) for the cases $k = 0, m \le r + 2$ and $0 \le k \le r - m, m \le r$. In 1985, X. M. Yu [30] showed that (2) is not true if $k = 0$ and $m \geq r + 3$. Finally, W. Li [22] in 1986 and R. Dahlhaus [3] in 1989 independently settled Theorem B as stated.

W. Li [22] also proved the following result showing that the obtained estimate is the best possible in some sense (we refer the reader to Theorem 11 of [22] for details).

Theorem C [22]. *Let $f \in C^r[-1,1]$ and $m \geq r+2$. Then, for any $n \geq m+r-1$, there exists a linear operator $Q_n\colon C^r[-1,1] \mapsto \Pi_n$ such that*

$$(3)\quad |f^{(k)}(x) - Q_n^{(k)}(f,x)| \leq C(r,m)\left(\frac{\sqrt{1-x^2}}{n}\right)^{r-k} \omega^m\left(f^{(r)}, \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}(n\sqrt{1-x^2})^{(r+2-k)/m}\right),$$

for $0 \leq k \leq r$ and $x \in [-1,1]$.

The following result is an immediate corollary of Theorem C.

Corollary D. *Let $f \in C^r[-1,1]$ and $m \in \mathbf{N}$. Then there exists a sequence of linear operators $Q_n\colon C^r[-1,1] \mapsto \Pi_n$ such that for every $0 \leq k \leq r$ and $x \in [-1,1]$,*

$$(4)\qquad |f^{(k)}(x) - Q_n^{(k)}(f,x)| \leq C(r,m)\left(\frac{\sqrt{1-x^2}}{n}\right)^{r-k} \omega^m(f^{(r)}, \Delta_n(x)).$$

Indeed, if $m \geq r+2$, then $\sqrt{1-x^2}/n + 1/n^2(n\sqrt{1-x^2})^{(r+2-k)/m} \leq C\Delta_n(x)$ and, therefore, (4) is valid. For $1 \leq m < r+2$, (4) follows from the case $m = r+2$ and the inequality $\omega^{r+2}(f^{(r)},\delta) \leq C(r)\omega^m(f^{(r)},\delta)$. Estimate (4) is not as strong as (2), but, on the other hand, it is valid for all $0 \leq k \leq r$ while (2) is not true in general for $k > r - m + 2$.

Recently, the following analog of Theorem B and Corollary D in terms of $\omega^m_{\varphi^\lambda}$ moduli was obtained by Z. Ditzian, D. Jiang and D. Leviatan [7].

Theorem E [7]. *For a function $f \in C^r[-1,1]$, $m \in \mathbf{N}$ and $0 \leq \lambda \leq 1$ there exists a sequence of polynomials $P_n \in \Pi_n$ for which*

$$(5)\quad |f^{(k)}(x) - P_n^{(k)}(x)| \leq C\left(\frac{\sqrt{1-x^2}}{n}\right)^{r-k} \omega^m_{\varphi^\lambda}(f^{(r)}, n^{-\lambda}\Delta_n(x)^{1-\lambda}), \quad 0 \leq k \leq r$$

and

$$(6)\qquad |P_n^{(k)}(x)| \leq C\Delta_n(x)^{r-k}\omega^m_{\varphi^\lambda}(f^{(r)}, n^{-\lambda}\Delta_n(x)^{1-\lambda}), \quad k \geq m+r,$$

where $x \in [-1,1]$.

Clearly, (5) coincides with (4) when $\lambda = 0$. For $m = 1, 2$, better estimates than those in Theorem E were proved in [7].

Theorem F [7]. *For $f \in C^r[-1,1]$ and $0 \le \lambda \le 1$, there exists a sequence of polynomials $P_n \in \Pi_n$ such that for all $x \in [-1,1]$,*

$$(7)\qquad |f^{(k)}(x) - P_n^{(k)}(x)| \le C\left(\frac{\sqrt{1-x^2}}{n}\right)^{r-k} \omega_{\varphi^\lambda}^l\left(f^{(r)}, n^{-\lambda}\left(\frac{\sqrt{1-x^2}}{n}\right)^{1-\lambda}\right),$$

for $l = 1, 2$ and $0 \le k \le r$, and

$$(8)\qquad |P_n^{(k)}(x)| \le C\Delta_n(x)^{r-k}\omega_{\varphi^\lambda}^l\left(f^{(r)}, n^{-\lambda}\left(\frac{\sqrt{1-x^2}}{n}\right)^{1-\lambda}\right),$$

for $l + r \le k \le k_0$, with some $k_0 \in \mathbf{N}$.

(In Theorem 1.2 of [7] it was stated that (8) holds for all $k \ge l + r$. This is a misprint since the only polynomials such that $P^{(k)}(\pm 1) = 0$ for all $k \ge l + r$ are those of degree $\le r + l - 1$. One should add the restriction $k \le k_0$.)

Other results on simultaneous approximation of a function and its derivatives can be found in [14], [15], [17], [21], [26], and [31].

2. Main Results

In this paper we obtain some results on simultaneous polynomial approximation of a function together with its derivatives which improve the estimates quoted in Section 1. Our proof is different from those employed in the above-mentioned papers (it seems that it is closest to the one used by Yu. A. Brudnyi in [2]) and, hence, can be viewed as an alternative proof of Theorems A-F. Moreover, not only do we improve the estimates (1)–(8), but also our construction (see Theorem 2) yields a polynomial $P_n(f, x)$ (more precisely, a linear operator $P_n(f, x)$: $C^r[-1, 1] \mapsto \Pi_n$), which fits all the above cases simultaneously. Furthermore, unlike the polynomials in Theorems E and F, which are different for different λ's, $P_n(f, x)$ is constructed independently of λ (for further discussions see Theorem 2 and the comment after it).

We start with the following result which is of the same type as Theorem A, and which improves (1)–(8) inside the interval $[-1, 1]$, i.e., for $x \in [-1 + n^{-2}, 1 - n^{-2}]$.

Theorem 1. *Let $r \in \mathbf{N}_0$, $m \in \mathbf{N}$ and $f \in C^r[-1,1]$. Then for any $n \ge m + r - 1$, there exists a linear operator $P_n(f, \cdot)$: $C^r[-1,1] \mapsto \Pi_n$ such that for every $0 \le \lambda \le 1$ and $x \in [-1,1]$,*

$$(9)\qquad |f^{(k)}(x) - P_n^{(k)}(f,x)| \le C(r,m)\Delta_n(x)^{r_k-k}\omega_{\varphi^\lambda}^{m+r-r_k}(f^{(r_k)}, n^{-\lambda}\Delta_n(x)^{1-\lambda}),$$

for $0 \le k \le r$ and any $r_k \in \mathbf{N}_0$ satisfying $k \le r_k \le r$.

Also, the following estimates hold for every $0 \le \lambda \le 1$ *and* $x \in [-1,1]$:

$$(10) \qquad |P_n^{(k)}(f,x)| \le C(k)\Delta_n(x)^{\tilde{r}-k}\omega_{\varphi^\lambda}^{m+r-\tilde{r}}(f^{(\tilde{r})}, n^{-\lambda}\Delta_n(x)^{1-\lambda}),$$

for $k \ge m+r$ *and any* $\tilde{r} \in \mathbf{N}_0$, $0 \le \tilde{r} \le r$.

Note that the properties of the $\omega_{\varphi^\lambda}^s$ moduli (see Chapter 6 of [5]), unfortunately, make it impossible to simplify the assertion of Theorem 1 without losing its generality (we encounter the same problem for Theorem 2 whose assertion is even more forbidding). When $\lambda > 0$, the cases for different r_k's and $\tilde{r}$'s are independent. This is determined by the fact that the inequality

$$(11) \qquad \omega_{\varphi^\lambda}^s(g, n^{-\lambda}\Delta_n(x)^{1-\lambda}) \le C\Delta_n(x)\omega_{\varphi^\lambda}^{s-1}(g', n^{-\lambda}\Delta_n(x)^{1-\lambda})$$

is not, in general, true if $\lambda > 0$. Indeed, for any $0 \le \lambda \le 1$ and $s \in \mathbf{N}$,

$$\omega_{\varphi^\lambda}^s(x^s, \delta) = s!\min\{\delta^s, (2/s)^s\}.$$

Therefore, if $g(x) = x^s$ and $n > s/2$, then

$$\omega_{\varphi^\lambda}^s(g, n^{-\lambda}\Delta_n(x)^{1-\lambda}) = C(s)(n^{-\lambda}\Delta_n(x)^{1-\lambda})^s$$

and

$$\omega_{\varphi^\lambda}^{s-1}(g', n^{-\lambda}\Delta_n(x)^{1-\lambda}) = C(s)(n^{-\lambda}\Delta_n(x)^{1-\lambda})^{s-1}.$$

Hence, if (11) were correct for $\lambda > 0$, we would have $\Delta_n(x) \ge Cn^{-1}$ for all $x \in [-1,1]$, which, of course, is not true. On the other hand, for $\lambda = 0$ (11) is correct and, hence, the assertion of Theorem 1 for $\lambda = 0$ is much simpler (estimates (9) and (10) for $r_k > k$ and $\tilde{r} > 0$ follow from the case $r_k = k$ and $\tilde{r} = 0$).

Corollary 1.1. *For* $f \in C^r[-1,1]$, $m \in \mathbf{N}$, *and any* $n \ge m+r-1$, *a linear operator* $P_n(f,\cdot)\colon C^r[-1,1] \mapsto \Pi_n$ *exists such that for* $x \in [-1,1]$,

$$(12) \qquad |f^{(k)}(x) - P_n^{(k)}(f,x)| \le C(r,m)\omega^{m+r-k}(f^{(k)}, \Delta_n(x)),$$

for $0 \le k \le r$, *and*

$$(13) \qquad |P_n^{(k)}(f,x)| \le C(k)\Delta_n(x)^{-k}\omega^{m+r}(f, \Delta_n(x)),$$

for $k \ge m+r$.

Corollary 1.1 not only implies Theorem A, but also (12) is better than (1) since the inequality, $\omega^m(f,\delta) \le C\delta\omega^{m-1}(f',\delta)$, generally cannot be reversed.

Our next result is an analog of Theorems B–F (i.e., the rate of approximation is estimated by the quantity which is zero at the endpoints of the interval $[-1,1]$) in terms of $\omega_{\varphi^\lambda}^m$ moduli, which improves (1)–(8) near the endpoints of $[-1,1]$ (i.e., for $x \in [-1,-1+n^{-2}] \cup [1-n^{-2},1]$).

Theorem 2. *Let* $r \in \mathbf{N}_0$, $m \in \mathbf{N}$, $k_0 \ge m+r$, *and* $f \in C^r[-1,1]$. *Then for any* $n \ge \max\{m+r-1, 2r+1\}$ *there exists a linear operator* $P_n(f,\cdot)\colon C^r[-1,1] \mapsto \Pi_n$ *such that for every sequence* $\{\alpha_k\}_{k=0}^r \subset [1/m, 1]$, *and for* $0 \le \lambda \le 1$ *and* $0 \le k \le r$, *the following inequalities hold:*

$$(14) \qquad |f^{(k)}(x) - P_n^{(k)}(f,x)| \le C(k_0)\Delta_n(x)^{r-k}\omega_{\varphi^\lambda}^m(f^{(r)}, n^{-\lambda}\Delta_n(x)^{1-\lambda}),$$

for $x \in [-1+n^{-2}, 1-n^{-2}]$, *and*

$$(15)\qquad |f^{(k)}(x) - P_n^{(k)}(f,x)| \le C(k_0)n^{2-2\alpha_k m}(1-x^2)^{r-k+1-\alpha_k m} \times \omega_{\varphi^\lambda}^m(f^{(r)}, n^{-\lambda}((1-x^2)^{\alpha_k} n^{2\alpha_k-2})^{1-\lambda}),$$

for $x \in [-1, -1+n^{-2}] \cup [1-n^{-2}, 1]$.

Also, there exists a constant $n_0 = n_0(k_0)$ *such that if* $n \ge n_0$, *then for every* $\{\alpha_k\}_{k=m+r}^{k_0} \subset [1/m, 1]$, $\{r_k\}_{k=m+r}^{k_0} \subset [0, k_0]$, *and for* $0 \le \lambda \le 1$ *and* $m+r \le k < k_0$, *operator* $P_n(f,x)$ *satisfies*

$$(16)\qquad |P_n^{(k)}(f,x)| \le C(k_0)\Delta_n(x)^{r-k}\omega_{\varphi^\lambda}^m(f^{(r)}, n^{-\lambda}\Delta_n(x)^{1-\lambda}),$$

for $x \in [-1+n^{-2}, 1-n^{-2}]$, *and*

$$(17)\qquad |P_n^{(k)}(f,x)| \le C(k_0)n^{2(r_k-r+k+1-\alpha_k m)}(1-x^2)^{r_k+1-\alpha_k m} \times \omega_{\varphi^\lambda}^m(f^{(r)}, n^{-\lambda}((1-x^2)^{\alpha_k} n^{2\alpha_k-2})^{1-\lambda}),$$

for $x \in [-1, -1+n^{-2}] \cup [1-n^{-2}, 1]$.

Even though Theorem 2 looks somewhat formidable (even the case $\lambda = 0$ is rather involved), it has some nice applications, and, in particular, all the results from Section 1 (the direct part of Theorem B) immediately follow from it. Subsequently we present a few corollaries of Theorem 2 for the particular case $\lambda = 0$. To simplify the exposition we omit the estimates of the higher derivatives of approximating polynomial $P_n(x)$ (or operator $P_n(f,x)$) in these corollaries.

We emphasize one more time (it is already done in the statement of Theorem 2) that (15) and (17) are valid for *all* choices of α_k's and r_k's (which can be different for different k's) satisfying $1/m \le \alpha_k \le 1$ and $0 \le r_k \le k_0$. In fact, the upper bound k_0 in the restrictions on the r_k's is not important. We need it only to stress that the r_k's cannot be boundlessly large. Any positive integer (say, k_1) would do. In that case the construction of $P_n(f,x)$ and all the constants in (14)–(17) would depend on k_1. However, rather than introduce one more parameter we employ what is already in use, the number k_0. In that way we do not over-complicate the statement and, at the same time, it is very easy to see that by employing k_0 doing so we lose no generality of Theorem 2 (k_0 can always be chosen larger than k_1).

We also remark that the lower bound on n in Theorem 2, $n \ge \max\{m+r-1, 2r+1\}$, is exact.

In addition, the natural question, How sharp are the estimates of Theorem 2, is partially resolved in this paper. First, since Theorem 2 ($\lambda = 0$, $m \ge r+2$, and $\alpha_k = (r+2-k)/(2m)$ for $0 \le k \le r$) implies (3) which cannot be improved in some sense (see Theorem 11 of W. Li [22]), then (15) cannot be improved uniformly for all λ in the same sense. More details for the case $\lambda = 0$ are given in Theorem 3. In particular, the negative part of Theorem B follows as its corollary. In fact, the proof of Theorem 3 is based on a slight modification of the ideas of X. M. Yu [30] which, in turn, were used in the proof of Theorem B in [3].

Theorem 3. *Let $r \in \mathbf{N}_0$, $m \in \mathbf{N}$, and let $\alpha \geq 0$, $\beta, \gamma \in \mathbf{R}$ be such that $\alpha + m\beta > r + 1$. Then for every constant $K \in \mathbf{R}$, a function $f \in C^{m+r-1}[-1, 1]$ exists such that*

$$\inf_{P_n \in \Pi_n} \left\{ \max_{x \in [-1+n^{-2}, 1-n^{-2}]} \frac{|f(x) - P_n(x)|}{\Delta_n(x)^r \omega^m(f^{(r)}, \Delta_n(x))} \right.$$
$$\left. + \sup_{x \in [-1,-1+n^{-2}] \cup [1-n^{-2}, 1]} \frac{|f(x) - P_n(x)|}{(1-x^2)^\alpha n^{2\alpha - 2r} \omega^m(f^{(r)}, (1-x^2)^\beta n^\gamma)} \right\} > K.$$

We have the following corollaries of Theorems 2 and 3.

Corollary 2-3.1. *Let $f \in C^r[-1, 1]$ and $m \in \mathbf{N}$. Then for any $n \geq \max\{m + r - 1, 2r + 1\}$ there exists a linear operator $P_n(f, \cdot)\colon C^r[-1, 1] \mapsto \Pi_n$ such that for every $0 \leq k \leq r$, the following inequalities hold:*

$$|f^{(k)}(x) - P_n^{(k)}(f, x)| \leq C(r, m)\Delta_n(x)^{r-k}\omega^m(f^{(r)}, \Delta_n(x)) \tag{18}$$

for $x \in [-1 + n^{-2}, 1 - n^{-2}]$, and

$$|f^{(k)}(x) - P_n^{(k)}(f, x)| \leq C(r, m)\Gamma_{nrmk}(x)^{r-k}\omega^m(f^{(r)}, \Gamma_{nrmk}(x)), \tag{19}$$

for $x \in [-1, -1 + n^{-2}] \cup [1 - n^{-2}, 1]$, where

$$\Gamma_{nrmk}(x) := (1 - x^2)^{(r-k+1)/(r-k+m)}(1/n^2)^{(m-1)/(r-k+m)}.$$

Moreover, these estimates are exact in the sense that for no $0 \leq k \leq r$ can $\Gamma_{nrmk}(x)$ be replaced by $(1 - x^2)^{\alpha_k} n^{2\alpha_k - 2}$ with $\alpha_k > (r - k + 1)/(r - k + m)$.

Corollary 2-3.1 improves the estimates of Theorem B. First, $\Gamma_{nrmk}(x) \leq \sqrt{1 - x^2}/n$ for any $0 \leq k \leq r + 2 - m$ and for all $x \in [-1, -1 + n^{-2}] \cup [1 - n^{-2}, 1]$. Second, (18) and (19) hold for all $0 \leq k \leq r$ while (2) may not be true if $k > r + 2 - m$. It is also of interest to consider the special case $m = 1$ in Corollary 2-3.1.

Corollary 2-3.2 ($m = 1$). *For a function $f \in C^r[-1, 1]$ and any $n \geq 2r + 1$ there exists a linear operator $P_n(f, \cdot)\colon C^r[-1, 1] \mapsto \Pi_n$ such that for every $0 \leq k \leq r$ and $x \in [-1, 1]$, the following inequality holds:*

$$|f^{(k)}(x) - P_n^{(k)}(f, x)| \leq C(r)\tilde{\Gamma}_{nrk}(x)^{r-k}\omega(f^{(r)}, \tilde{\Gamma}_{nrk}(x)),$$

where $\tilde{\Gamma}_{nrk}(x) := \min\{1 - x^2, \sqrt{1 - x^2}/n\}$.

Moreover, $\tilde{\Gamma}_{nrk}(x)$ cannot be replaced by $\min\{(1 - x^2)^\alpha, \sqrt{1 - x^2}/n\}$ with $\alpha > 1$.

The following result follows from Corollary 2-3.2 by the argument used by D. Leviatan in the proof of Theorem 2 of [21].

Corollary 2-3.3. *Let $f \in C^r[-1, 1]$. Then for any $n \geq 2r + 1$ there is a polynomial $P_n \in \Pi_n$ such that for every $0 \leq k \leq r$ and $x \in [-1, 1]$,*

$$|f^{(k)}(x) - P_n^{(k)}(x)| \leq C(r)\left(\min\left\{1 - x^2, \frac{\sqrt{1 - x^2}}{n}\right\}\right)^{r-k} E_{n-r}(f^{(r)}),$$

where $E_s(g) := \inf_{P_s \in \Pi_s} \|g - P_s\|_{C[-1,1]}$.

Corollary 2-3.3 improves the estimate

$$|f^{(k)}(x) - P_n^{(k)}(x)| \leq C(r) \left(\frac{\sqrt{1-x^2}}{n} \right)^{r-k} E_{n-r}(f^{(r)}),$$

which was obtained by T. Kilgore [17].

The most general case in Theorem 2 for $\lambda = 0$ is when $\alpha_k = 1/m$ for all k. Therefore, for $\lambda = 0$, Theorem 2 (without (16) and (17)) can be restated as follows:

Corollary 2-3.4. *Let $r \in \mathbf{N}_0$, $m \in \mathbf{N}$, and $f \in C^r[-1, 1]$. Then for any $n \geq \max\{m + r - 1, 2r + 1\}$ a linear operator $P_n(f, \cdot)$: $C^r[-1, 1] \mapsto \Pi_n$ exists such that for every $0 \leq k \leq r$, the following inequalities hold*:

$$|f^{(k)}(x) - P_n^{(k)}(f, x)| \leq C(r, m)\Delta_n(x)^{r-k}\omega^m(f^{(r)}, \Delta_n(x)), \tag{20}$$

for $x \in [-1 + n^{-2}, 1 - n^{-2}]$, and

$$|f^{(k)}(x) - P_n^{(k)}(f, x)| \leq C(r, m)(1 - x^2)^{r-k}\omega^m \left(f^{(r)}, (1-x^2)^{1/m} \left(\frac{1}{n^2} \right)^{(m-1)/m} \right), \tag{21}$$

for $x \in [-1, -1 + n^{-2}] \cup [1 - n^{-2}, 1]$.

Moreover, for any $\gamma \in \mathbf{R}$ the quantity $(1 - x^2)^{1/m}(1/n^2)^{(m-1)/m}$ in (21) cannot be replaced by $(1 - x^2)^{\alpha} n^{\gamma}$ with $\alpha > 1/m$.

It should be mentioned that (20) and (21) coincide with (3) when $k = r$ and $m \geq r+2$. Even though for $k < r$ (3) is weaker than (21), it can be shown (using the Taylor expansion) that (21) with $k < r$ follows from the case $k = r$ if $P_n^{(k)}(\pm 1) = f^{(k)}(\pm 1)$ for all $k = 0, \ldots, r$. Therefore, for $m \geq r + 2$ and large n, the direct part of Corollary 2-3.4 follows from Theorem C. (Similarly, it can be shown that Corollary 2-3.3 follows from Theorem B with $m = 1$.)

In the next section we recall some definitions and introduce notations which are used throughout the paper. Then following section contains auxiliary results for the proofs of Theorems 1 and 2. In Section 5 we separately consider auxiliary results intended for the proofs of the case $\lambda > 0$ (therefore, this section can be skipped by the reader only interested in the case $\lambda = 0$). Finally, the proofs of Theorems 1 and 2, and Theorem 3 are given in Sections 6 and 7, respectively.

3. Definitions and Notations

Throughout this paper we use the following notations (cf. [8], [9], [24]–[26]):

$$x_j := \cos(j\pi/n), \qquad 0 \leq j \leq n,$$

$$I_j := [x_j, x_{j-1}], \qquad h_j := x_{j-1} - x_j, \qquad 1 \leq j \leq n.$$

(Note that $h_{j\pm1} < 3h_j$ and $\Delta_n(x) \le h_j \le 5\Delta_n(x)$ for $x \in I_j$.)

$$\tilde{I}_j := \begin{cases} [x_j, x_{j-m-r+1}], & \text{if } \ m+r-1 \le j \le n, \\ [x_{m+r-1}, 1], & \text{if } \ 0 \le j < m+r-1, \end{cases}$$

when $m+r-1 > 0$, and

$$\tilde{I}_j := \begin{cases} I_j, & \text{if } \ 1 \le j \le n, \\ I_1, & \text{if } \ j = 0, \end{cases}$$

when $m+r-1 = 0$ (i.e., when $m = 1$ and $r = 0$). Also,

$$t_j(x) := \left(\frac{\cos 2n \arccos x}{x - x_j^0}\right)^2 + \left(\frac{\sin 2n \arccos x}{x - \bar{x}_j}\right)^2$$

is the algebraic polynomial of degree $4n-2$, where $\bar{x}_j := \cos(j\pi/n - \pi/2n)$, $1 \le j \le n$, $x_j^0 := \cos(j\pi/n - \pi/4n)$, $1 \le j < n/2$ and $x_j^0 := \cos(j\pi/n - 3\pi/4n)$, $n/2 \le j \le n$. This polynomial was introduced by V. K. Dzyadyk (see also I. A. Shevchuk [26]) and extensively used in [9], [24]–[26], and [18].

Let

$$\Pi_j(n, \mu, \xi) := \int_{-1}^{1} (1 - y^2)^{\xi} t_j(y)^{\mu}\, dy,$$

$$T_j(n, \mu, \xi)(x) := \Pi_j(n, \mu, \xi)^{-1} \int_{-1}^{x} (1 - y^2)^{\xi} t_j(y)^{\mu} dy,$$

and

$$R_{j,m}(n, \mu, \xi)(x) := (x - x_j)^m T_j(n, \mu, \xi)(x).$$

We also denote

$$\chi_j(x) := \begin{cases} 1, & \text{if } \ x \ge x_j, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\psi_j := \frac{h_j}{|x - x_j| + h_j},$$

and note that $\psi_j \le 1$ for all $x \in [-1, 1]$ and $1 \le j \le n$.

$L(f, t; t_1, \dots, t_{\nu+1})$ is the Lagrange polynomial of degree $\le \nu$ which interpolates the function f at the points $t_1, \dots, t_{\nu+1}$.

Finally, all C are positive constants which are not necessarily the same, even when they occur in the same line. In order to emphasize that C depends only on the parameters $\nu_1, \dots, \nu_k$ the notation $C(\nu_1, \dots, \nu_k)$ is used.

4. Auxiliary Statements and Results

The following proposition contains simple but important inequalities which are used in almost all proofs later on.

Proposition 4. *The following inequalities hold for all $x, y \in [-1, 1]$ and $1 \le j \le n$:*

$$
\begin{aligned}
&(22) \qquad (|x - x_j| + h_j)^{-2} \le t_j(x) \le 4 \cdot 10^3 (|x - x_j| + h_j)^{-2}, \\
&(23) \qquad \Delta_n(y)^2 \le 4\Delta_n(x)\,(|x - y| + \Delta_n(x)), \\
&(24) \qquad (|x - y| + \Delta_n(x))/2 \le |x - y| + \Delta_n(y) \le 2\,(|x - y| + \Delta_n(x)), \\
&(25) \qquad \frac{1 + x}{1 + x_{j-1}} \le C\psi_j^{-1} \quad \textit{and} \quad \frac{1 - x}{1 - x_j} \le C\psi_j^{-1}, \\
&(26) \qquad C\psi_j^2 \Delta_n(x) \le \Delta_n(x_j) \le C\psi_j^{-1}\Delta_n(x),
\end{aligned}
$$

and

$$
(27) \qquad \sum_{i=1}^{n} \psi_i^{\alpha} \le C, \quad \textit{for any} \quad \alpha \ge 2.
$$

Proof. Estimates (22)–(24) and (27) are verified by straightforward computations, and can be found in [24]–[26], for example. To show the validity of (25) we write

$$
\frac{1 + x}{1 + x_{j-1}} = 1 + \frac{x - x_{j-1}}{1 + x_{j-1}} \le 1 + \frac{|x - x_{j-1}|}{h_j} \le C\psi_j^{-1},
$$

and

$$
\frac{1 - x}{1 - x_j} = 1 + \frac{x_j - x}{1 - x_j} \le 1 + \frac{|x - x_j|}{h_j} = \psi_j^{-1}.
$$

The right-hand side inequality in (26) follows from (23) and (24) since

$$
\begin{aligned}
\Delta_n(x_j) &\le 4\Delta_n(x)\frac{|x - x_j| + \Delta_n(x)}{\Delta_n(x_j)} \\
&\le 8\Delta_n(x)\frac{|x - x_j| + \Delta_n(x_j)}{\Delta_n(x_j)} \le C\psi_j^{-1}\Delta_n(x).
\end{aligned}
$$

Finally, using the last inequality, we have

$$
\begin{aligned}
\Delta_n(x_j) &\ge \frac{\Delta_n(x)^2}{4(|x - x_j| + \Delta_n(x_j))} \\
&= \frac{\Delta_n(x)}{4}\frac{\Delta_n(x)}{\Delta_n(x_j)}\frac{\Delta_n(x_j)}{|x - x_j| + \Delta_n(x_j)} \ge C\psi_j^2\Delta_n(x),
\end{aligned}
$$

which is the left-hand side inequality in (26). ■

The following proposition is needed to describe the behavior of the polynomials T_j, $R_{j,m}$, and their derivatives (see Lemmas 6 and 7 below). To some degree it is a generalization of the inequalities (17.8) and (17.10) of [26].

Proposition 5. *Let $\mu \in \mathbf{N}$ and $\xi \in \mathbf{N}_0$ be such that $\mu \geq \xi + 1$. Then the following inequalities are valid for any $1 \leq j \leq n$:*

$$C(\mu) \leq \Pi_j(n, \mu, \xi)(1 + x_{j-1})^{-\xi}(1 - x_j)^{-\xi} h_j^{2\mu-1} \leq C(\mu). \tag{28}$$

Proof. The estimates (28) can be proved using the method from [24] and [26] (i.e., by estimating separately each of the following three parts of the original integral: $\int_{-1}^{x_j}$, $\int_{x_j}^{x_{j-1}}$, and $\int_{x_{j-1}}^{1}$). However, there is a simpler proof. Since the function $(1 - y^2)^\xi t_j(y)^\mu$ does not change sign in $[-1, 1]$, then (22) implies

$$\frac{\Pi_j(n, \mu, \xi)}{(1 + x_{j-1})^\xi (1 - x_j)^\xi} \sim \int_{-1}^{1} \left(\frac{1 - y^2}{(1 + x_{j-1})(1 - x_j)} \right)^\xi \frac{1}{(|y - x_j| + h_j)^{2\mu}} dy =: \tilde{\Pi}_j.$$

Now,

$$\begin{aligned}
\tilde{\Pi}_j &\geq \int_{x_j + h_j/3}^{x_{j-1} - h_j/3} \left(\frac{1 - y^2}{(1 + x_{j-1})(1 - x_j)} \right)^\xi \frac{1}{(|y - x_j| + h_j)^{2\mu}} dy \\
&\geq \int_{x_j + h_j/3}^{x_{j-1} - h_j/3} \left(\frac{(1 + x_j + h_j/3)(1 - x_{j-1} + h_j/3)}{(1 + x_{j-1})(1 - x_j)} \right)^\xi \frac{1}{(2h_j)^{2\mu}} dy \\
&\geq C h_j^{-2\mu+1},
\end{aligned}$$

and, hence, the left-hand side inequality in (28) is proved. For the proof of the right-hand side inequality (using (25)) we write

$$\begin{aligned}
\tilde{\Pi}_j &\leq C \int_{-1}^{1} \left(\frac{h_j}{|y - x_j| + h_j} \right)^{-2\xi} \frac{1}{(|y - x_j| + h_j)^{2\mu}} dy \\
&= C \int_{-1}^{1} \left(\frac{h_j}{|y - x_j| + h_j} \right)^{2\mu - 2\xi} h_j^{-2\mu} dy \\
&\leq C h_j^{-2\mu} \int_0^\infty \left(\frac{h_j}{t + h_j} \right)^{2\mu - 2\xi} dt \leq C h_j^{-2\mu+1}.
\end{aligned}$$

■

Lemma 6. *Let $\mu, \xi \in \mathbf{N}$ be such that $\mu \geq \xi + 1$, and let $1 \leq j \leq n$ be a fixed index. Then for the polynomial*

$$T_j(x) := T_j(n, \mu, \xi)(x) = \Pi_j(n, \mu, \xi)^{-1} \int_{-1}^{x} (1 - y^2)^\xi t_j(y)^\mu \, dy,$$

of degree $\leq 4n\mu$ the following inequalities hold for $x \in [-1, 1]$:

$$|T_j^{(k)}(x)| \leq C(\mu) \left(\frac{1 - x^2}{(1 + x_{j-1})(1 - x_j)} \right)^{\xi - k + 1} \psi_j^{2\mu - 7k + 7} h_j^{-k}, \quad 1 \leq k \leq \xi \tag{29}$$

and

$$|T_j(x) - \chi_j(x)| \leq C(\mu) \left(\frac{1 - x^2}{(1 + x_{j-1})(1 - x_j)} \right)^\xi \psi_j^{2\mu - \xi - 1}. \tag{30}$$

Proof. For any $x \in [-1, 1]$ (using Proposition 5 and (22)) we have

$$|T_j'(x)| \le C\left(\frac{1-x^2}{(1+x_{j-1})(1-x_j)}\right)^{\xi} t_j(x)^{\mu} h_j^{2\mu-1}$$
$$\le C\left(\frac{1-x^2}{(1+x_{j-1})(1-x_j)}\right)^{\xi} \psi_j^{2\mu} h_j^{-1},$$

which is inequality (29) with $k = 1$.

For the proof of (30), we consider two cases: $x < x_j$ and $x \ge x_j$. First, for $x < x_j$ (using (29) with $k = 1$ and the second inequality in (25)) we have

$$\begin{aligned}|T_j(x) - \chi_j(x)| = |T_j(x)| &= \left|\int_{-1}^{x} T_j'(y)\,dy\right| \\ &\le C\int_{-1}^{x}\left(\frac{1+y}{1+x_{j-1}}\right)^{\xi}\left(\frac{h_j}{|y-x_j|+h_j}\right)^{2\mu-\xi} h_j^{-1}\,dy \\ &\le C\left(\frac{1+x}{1+x_{j-1}}\right)^{\xi} h_j^{2\mu-\xi-1}\int_{-\infty}^{x}(x_j - y + h_j)^{-2\mu+\xi}\,dy \\ &\le C\left(\frac{1-x^2}{(1+x_{j-1})(1-x_j)}\right)^{\xi}\psi_j^{2\mu-\xi-1}.\end{aligned}$$

Similarly, for $x \ge x_j$ we write

$$\begin{aligned}|T_j(x) - \chi_j(x)| = |1 - T_j(x)| &= \left|\int_{x}^{1} T_j'(y)\,dy\right| \\ &\le C\int_{x}^{1}\left(\frac{1-y}{1-x_j}\right)^{\xi}\left(\frac{h_j}{|y-x_j|+h_j}\right)^{2\mu-\xi} h_j^{-1}\,dy \\ &\le C\left(\frac{1-x}{1-x_j}\right)^{\xi} h_j^{2\mu-\xi-1}\int_{x}^{\infty}(y - x_j + h_j)^{-2\mu+\xi}\,dy \\ &\le C\left(\frac{1-x^2}{(1+x_{j-1})(1-x_j)}\right)^{\xi}\psi_j^{2\mu-\xi-1}.\end{aligned}$$

This verifies inequality (30).

Thus, it remains to prove (29) for $1 < k \le \xi$. We need the following inequality of V. K. Dzyadyk [8] (see also [9] and [26]) for algebraic polynomials:

For any $P_n \in \Pi_n$ and any $s \in \mathbf{R}$ the inequality

(31) $$\|\Delta_n(x)^{s+\nu} P_n^{(\nu)}(x)\| \le C(\nu)\|\Delta_n(x)^s P_n(x)\|$$

holds for any $\nu \in \mathbf{N}$, where $\|\cdot\| := \|\cdot\|_{C[-1,1]}$.

Now the fact that $t_j(x)$ is an algebraic polynomial of degree $\le 4n-2$, together with (31) and (23), implies

$$\begin{aligned}\|\Delta_n(x)^{\nu+2}t_j^{(\nu)}(x)\| &\le C\|\Delta_n(x)^2 t_j(x)\| \le C\left\|\left(\frac{\Delta_n(x)}{|x-x_j|+h_j}\right)^2\right\| \\ &\le C\left\|\frac{4\Delta_n(x_j)(|x-x_j|+\Delta_n(x_j))}{(|x-x_j|+h_j)^2}\right\| \le C\left\|\frac{h_j}{|x-x_j|+h_j}\right\| \le C.\end{aligned}$$

Therefore, for any $\nu \ge 1$ and $x \in [-1,1]$ (using (23) and (24)) we have

$$\text{(32)}\qquad \begin{aligned}|t_j^{(\nu)}(x)| &\le C\Delta_n(x)^{-\nu-2} \le C\left(\frac{|x-x_j|+\Delta_n(x)}{\Delta_n(x_j)^2}\right)^{\nu+2} \\ &\le C\left(\frac{|x-x_j|+h_j}{h_j}\right)^{\nu+2} h_j^{-\nu-2} \le C(\psi_j h_j)^{-\nu-2}.\end{aligned}$$

Using the last inequality, we have, for $1 \le k \le \xi$,

$$\begin{aligned}\Pi_j(n,\mu,\xi)|T_j^{(k)}(x)| &= |[(1-x^2)^\xi t_j(x)^\mu]^{(k-1)}| \\ &= \left|\sum_{\nu=0}^{k-1}\binom{k-1}{\nu}[(1-x^2)^\xi]^{(k-1-\nu)}[t_j(x)^\mu]^{(\nu)}\right| \\ &\le C\sum_{\nu=0}^{k-1}\binom{k-1}{\nu}\sum_{l=0}^{k-1-\nu}\binom{k-1-\nu}{l}[(1-x)^\xi]^{(k-1-\nu-l)}[(1+x)^\xi]^{(l)} \\ &\quad\times\sum_{\substack{i_1,\dots,i_\nu\ge 0\\ i_1+\dots+i_\nu=\nu}}|t_j^{(i_1)}(x)\cdots t_j^{(i_\nu)}(x)|t_j(x)^{\mu-\nu} \\ &\le C\sum_{\nu=0}^{k-1}\sum_{l=0}^{k-1-\nu}(1-x)^{\xi-k+1+\nu+l}(1+x)^{\xi-l} \\ &\quad\times\sum_{\substack{i_1,\dots,i_\nu\ge 0\\ i_1+\dots+i_\nu=\nu}}|(\psi_j h_j)^{-i_1-2}\cdots(\psi_j h_j)^{-i_\nu-2}|(|x-x_j|+h_j)^{-2\mu+2\nu} \\ &\le C\sum_{\nu=0}^{k-1}\sum_{l=0}^{k-1-\nu}(1-x)^{\xi-k+1+\nu+l}(1+x)^{\xi-l}\psi_j^{2\mu-5\nu}h_j^{-2\mu-\nu}.\end{aligned}$$

Therefore, Proposition 5 yields

$$\begin{aligned}|T_j^{(k)}(x)| &\le C\sum_{\nu=0}^{k-1}\sum_{l=0}^{k-1-\nu}\left(\frac{1-x^2}{(1+x_{j-1})(1-x_j)}\right)^{\xi-k+1}\left(\frac{1-x}{1-x_j}\right)^{\nu+l} \\ &\quad\times\left(\frac{1+x}{1+x_{j-1}}\right)^{k-1-l}\left(\frac{h_j}{1-x_j}\right)^{k-1-\nu-l}\left(\frac{h_j}{1+x_{j-1}}\right)^{l}\psi_j^{2\mu-5\nu}h_j^{-k}.\end{aligned}$$

Now, using (25), inequalities $1 - x_j \geq h_j$, and $1 + x_{j-1} \geq h_j$, and the fact that $\psi_j \leq 1$, we have

$$\begin{aligned}\left|T_j^{(k)}(x)\right| &\leq C\left(\frac{1-x^2}{(1+x_{j-1})(1-x_j)}\right)^{\xi-k+1}\sum_{\nu=0}^{k-1}\psi_j^{2\mu-6\nu-k+1}h_j^{-k}\\ &\leq C\left(\frac{1-x^2}{(1+x_{j-1})(1-x_j)}\right)^{\xi-k+1}\psi_j^{2\mu-7k+7}h_j^{-k}.\end{aligned}$$

The proof of (29) is complete. ■

Lemma 7. *Let $1 \leq j \leq n$ be a fixed index, and let $m_0, k_0, \mu, \xi \in \mathbf{N}$ be such that $\xi \geq 2k_0$ and $\mu \geq 10\xi + m_0$. Then for any $0 \leq m \leq m_0$ the polynomial $R_{j,m}(x) := R_{j,m}(n, \mu, \xi)(x) = (x - x_j)^m T_j(n, \mu, \xi)(x)$ satisfies the following inequalities for $x \in [-1, 1]$ and $0 \leq k \leq k_0$:*

$$|R_{j,m}^{(k)}(x) - [(x-x_j)^m]^{(k)}\chi_j(x)| \leq C(\mu)\left(\frac{1-x^2}{(1+x_{j-1})(1-x_j)}\right)^{\xi/2}\psi_j^{\mu}h_j^{m-k}. \tag{33}$$

Note that $[(x - x_j)^m]^{(k)} \equiv 0$ for $k \geq m + 1$. Therefore for these k, (33) becomes an estimate of the kth derivative of the polynomial $R_{j,m}$. This observation is used in the proofs of (10), (16), and (17).

Proof. We have

$$\begin{aligned}&|R_{j,m}^{(k)}(x) - [(x-x_j)^m]^{(k)}\chi_j(x)|\\ &\quad\leq |[(x-x_j)^m]^{(k)}(T_j(x)-\chi_j(x))| + \sum_{\nu=1}^{k}\binom{k}{\nu}|T_j^{(\nu)}(x)|\,|[(x-x_j)^m]^{(k-\nu)}|\\ &\quad= \mathcal{S}_1(x) + \mathcal{S}_2(x).\end{aligned}$$

Note that if $k \geq m + 1$, then $\mathcal{S}_1(x) \equiv 0$. For $0 \leq k \leq m$ (using (30)) we have the following:

$$\begin{aligned}\mathcal{S}_1(x) &\leq C|x-x_j|^{m-k}\left(\frac{1-x^2}{(1+x_{j-1})(1-x_j)}\right)^{\xi}\psi_j^{2\mu-\xi-1}\\ &\leq C\left(\frac{1-x^2}{(1+x_{j-1})(1-x_j)}\right)^{\xi/2}\psi_j^{2\mu-2\xi+k-m-1}h_j^{m-k}.\end{aligned}$$

In order to estimate $\mathcal{S}_2(x)$ (using Lemma 6) we write

$$\mathcal{S}_2(x) \leq C\sum_{\nu=\max\{1,k-m\}}^{k}\left(\frac{1-x^2}{(1+x_{j-1})(1-x_j)}\right)^{\xi-\nu+1}\psi_j^{2\mu-7\nu+7}h_j^{-\nu}|x-x_j|^{m-k+\nu}$$

$$\leq C\left(\frac{1-x^2}{(1+x_{j-1})(1-x_j)}\right)^{\xi/2}\sum_{\nu=\max\{1,k-m\}}^{k}\psi_j^{2\mu-\xi-2}\psi_j^{2\mu-7\nu+7}\psi_j^{k-m-\nu}h_j^{m-k}$$

$$\leq C\left(\frac{1-x^2}{(1+x_{j-1})(1-x_j)}\right)^{\xi/2}\psi_j^{2\mu-\xi-5k-m+5}h_j^{m-k}.$$

Combining the estimates for $\mathcal{S}_1(x)$ and $\mathcal{S}_2(x)$ together, and keeping in mind that $\psi_j \leq 1$, we obtain (33). ■

For the proof of Theorem 1 we need the following lemma.

Lemma G. *Let $r \in \mathbf{N}_0$, $m \in \mathbf{N}$, and $f \in C^r(\tilde{I}_j)$, where $j \geq m+r-1$ is a fixed index. Then the following inequalities hold for all $0 \leq k \leq r$:*

$$\begin{aligned}(34)\qquad &|f^{(k)}(x) - L^{(k)}(f, x; x_j, x_{j-1}, \dots, x_{j-m-r+1})|\\ &\qquad \leq C(r,m)\omega^{m+r-k}(f^{(k)}, h_j, \tilde{I}_j), \quad x \in \tilde{I}_j.\end{aligned}$$

Proof. For $r = 0$, the assertion of the lemma immediately follows from the well-known Whitney inequality. For $r > 0$, it was proved by I. A. Shevchuk (Lemma 1.4.2 of [26]). ■

The following consequence of Lemma G is used for the proof of Theorem 2.

Corollary H. *Let $r \in \mathbf{N}_0$, $m \in \mathbf{N}$, and $f \in C^r(\tilde{I}_j)$, where $j \geq m+r-1$. Then the following inequalities hold for all $0 \leq k \leq r$:*

$$\begin{aligned}(35)\qquad &|f^{(k)}(x) - L^{(k)}(f, x; x_j, x_{j-1}, \dots, x_{j-m-r+1})|\\ &\qquad \leq C(r,m)h_j^{r-k}\omega^{m}(f^{(r)}, h_j, \tilde{I}_j), \quad x \in \tilde{I}_j.\end{aligned}$$

Also, for the proof of Theorem 2 we consider Lagrange polynomials concurrently with Lagrange-Hermite polynomials when interpolating f and its derivatives at 1 (or -1).

Lemma 8. *Let $r \in \mathbf{N}_0$, $m \in \mathbf{N}$, $f \in C^r(\tilde{I}_1)$. Let $\hat{L}(f,x)$ be the Lagrange-Hermite interpolation polynomial of degree $\leq m+r-1$ such that $\hat{L}(f, x_j) = f(x_j)$, $1 \leq j \leq m-1$ and $\hat{L}^{(k)}(f,1) = f^{(k)}(1)$, $0 \leq k \leq r$. Then the following inequalities hold for all $0 \leq k \leq r$ and $x \in [1-n^{-2}, 1]$:*

$$\begin{aligned}(36)\qquad &|f^{(k)}(x) - \hat{L}^{(k)}(f,x)|\\ &\qquad \leq C(r,m)(1-x)^{r-k}\omega^{m}\left(f^{(r)}, (1-x)^{1/m}\left(\frac{1}{n^2}\right)^{(m-1)/m}, \tilde{I}_1\right).\end{aligned}$$

Changing variable x to $-x$ (i.e., considering the symmetric case), we immediately get the following result for $\tilde{L}(f,x) := \hat{L}(f(-x), -x)$.

Lemma 9. *Let $r \in \mathbf{N}_0$, $m \in \mathbf{N}$, $f \in C^r(\tilde{I}_n)$. Let $\tilde{L}(f, x)$ be the Lagrange-Hermite interpolation polynomial of degree $\le m+r-1$ such that $\tilde{L}(f, x_j) = f(x_j)$, $n-m+1 \le j \le n-1$ and $\tilde{L}^{(k)}(f, -1) = f^{(k)}(-1)$, $0 \le k \le r$. Then the following inequalities hold for all $0 \le k \le r$ and $x \in [-1, -1+n^{-2}]$:*

$$
\begin{aligned}
(37) \qquad & |f^{(k)}(x) - \tilde{L}^{(k)}(f, x)| \\
& \le C(r, m)(1+x)^{r-k} \omega^m \left(f^{(r)}, (1+x)^{1/m} \left(\frac{1}{n^2} \right)^{(m-1)/m}, \tilde{I}_n \right).
\end{aligned}
$$

Proof of Lemma 8. In the proof it is convenient to denote the interval $\tilde{I}_1$ by J. Then $|J| = \max\{1 - x_{m+r-1}, 1 - x_1\}$. Also, let

$$L(f, x) := L(f, x; x_{m+r-1}, \dots, x_0)$$

be the Lagrange interpolation polynomial of degree $\le m+r-1$ (it is important that $L(f, x)$ has the same degree as $\hat{L}(f, x)$). First, we prove that for $\hat{L}(f, x)$ an inequality similar to (35) is satisfied (i.e., to obtain the estimates of local approximation near ± 1 of the same type as (35) we can use either Lagrange or Lagrange-Hermite interpolation polynomials). The following identity is valid:

$$f(x) - \hat{L}(f, x) \equiv f(x) - L(f, x) - \hat{L}(f - L(f, \cdot), x).$$

Using the Taylor formula, we write the polynomial $\hat{L}(f - L(f, \cdot), x)$ in the form

$$\hat{L}(f - L(f, \cdot), x) = \sum_{i=0}^{r} (i!)^{-1} (f^{(i)}(1) - L^{(i)}(f, 1))(x-1)^i + (x-1)^{r+1} p_{m-2}(f, x),$$

where $p_{m-2}(f, x) \in \Pi_{m-2}$ if $m \ge 2$, and $p_{m-2}(f, x) \equiv 0$ if $m = 1$. Now $f(x_j) = L(f, x_j)$, $1 \le j \le m-1$, imply $\hat{L}(f - L(f, \cdot), x_j) = 0$, $1 \le j \le m-1$. Therefore,

$$\sum_{i=0}^{r} (i!)^{-1} (f^{(i)}(1) - L^{(i)}(f, 1))(x_j - 1)^i + (x_j - 1)^{r+1} p_{m-2}(f, x_j) = 0,$$

for $1 \le j \le m-1$. Using the fact that $n^{-2} \le |x_j - 1| \le |J| \le Cn^{-2}$ ($C = 3^{m+r}$ will do), we have, for $1 \le j \le m-1$,

$$|p_{m-2}(f, x_j)| \le Cn^{2r+2} \sum_{i=0}^{r} |f^{(i)}(1) - L^{(i)}(f, 1)| n^{-2i},$$

and, applying Corollary H, we have

$$
\begin{aligned}
|p_{m-2}(f, x_j)| & \le Cn^{2r+2} \sum_{i=0}^{r} n^{-2r+2i} \omega^m (f^{(r)}, n^{-2}, J) n^{-2i} \\
& \le Cn^2 \omega^m (f^{(r)}, n^{-2}, J).
\end{aligned}
$$

Now, since

$$p_{m-2}(f, x) = \sum_{j=1}^{m-1} \left(\prod_{1 \le i \le m-1, i \ne j} \frac{x - x_i}{x_j - x_i} \right) p_{m-2}(f, x_j),$$

the estimate

$$|p_{m-2}(f,x)| \le Cn^2\omega^m(f^{(r)},n^{-2},J)$$

follows for all $x \in J$. This implies

$$|\hat{L}(f - L(f,\cdot),x)| \le Cn^{-2r}\omega^m(f^{(r)},n^{-2},J), \quad x \in J.$$

Applying the Markov inequality we have

$$\|\hat{L}^{(k)}(f - L(f,\cdot),x)\| \le C|J|^{-k}\|\hat{L}(f - L(f,\cdot),x)\| \le Cn^{-2r+2k}\omega^m(f^{(r)},n^{-2},J).$$

Therefore, together with (35), we have

$$\begin{aligned}|f^{(k)}(x) - \hat{L}^{(k)}(f,x)| &\le |f^{(k)}(x) - L^{(k)}(f,x)| + |\hat{L}^{(k)}(f - L(f,\cdot),x)|\\ &\le Cn^{-2r+2k}\omega^m(f^{(r)},n^{-2},J), \quad x \in J.\end{aligned}$$

Now we improve the last inequality near 1 using the fact that $\hat{L}^{(k)}(f,1) = f^{(k)}(1)$, $0 \le k \le r$, and the techniques developed in [9] and [26]. First, we consider the case $k = r$. Denoting $f^{(r)}(x) - \hat{L}^{(r)}(f,x)$ by $g(x)$, we conclude that

$$|g(x)| \le C\omega^m(f^{(r)},n^{-2},J) = C\omega^m(g,n^{-2},J), \quad x \in J,$$

since $\hat{L}^{(r)}(f,x)$ is of degree $\le m-1$. Now, using the equality $g(1) = 0$, we have the following for any $x \in [1-n^{-2},1]$:

$$|g(x)| = |g(1) - g(x)| \le \omega(g,1-x,J).$$

Therefore, if $m = 1$, then (36) is proved for $k = r$. If $m \ge 2$, then using the Marchaud inequality, we have

$$\begin{aligned}|g(x)| &\le \omega(g,1-x,J)\\ &\le C(1-x)\left(\int_{1-x}^{|J|} u^{-2}\omega^m(g,u,J)\,du + |J|^{-1}\|g\|_{C(J)}\right)\\ &\le C(1-x)\left(\int_{1-x}^{|J|} u^{-2}\omega^m(g,u,J)\,du + |J|^{-1}\omega^m(g,|J|,J)\right)\\ &\le C(1-x)\int_{1-x}^{|J|} u^{-2}\omega^m(g,u,J)\,du.\end{aligned} \tag{38}$$

The last inequality is valid since, for all $u \le |J|$, we have $|J|^{-m}\omega^m(g,|J|,J) \le Cu^{-m}\omega^m(g,u,J)$. Therefore,

$$\begin{aligned}\int_{1-x}^{|J|} u^{-2}\omega^m(g,u,J)\,du &\ge C\int_{1-x}^{|J|} |J|^{-m}u^{m-2}\omega^m(g,|J|,J)\,du\\ &\ge C|J|^{-1}\omega^m(g,|J|,J).\end{aligned}$$

Estimate (38) implies

$$|f^{(r)}(x) - \hat{L}^{(r)}(f,x)| \le C(1-x)\int_{1-x}^{|J|} u^{-2}\omega^m(f^{(r)},u,J)\,du.$$

Now let us denote $\Omega_x := (1-x)^{1/m}(1/n^2)^{(m-1)/m}$. Clearly, $1-x \le \Omega_x \le n^{-2} < |J|$ for any $x \in [1-n^{-2}, 1]$. Hence,

$$\begin{aligned}
|f^{(r)}(x) - \hat{L}^{(r)}(f,x)| &\le C(1-x)\left(\int_{1-x}^{\Omega_x} + \int_{\Omega_x}^{|J|}\right) u^{-2}\omega^m(f^{(r)}, u, J)\,du \\
&\le C(1-x)\left(\int_{1-x}^{\infty} u^{-2}\omega^m(f^{(r)}, \Omega_x, J)\,du\right. \\
&\qquad \left. + \int_0^{|J|} \Omega_x^{-m} u^{m-2}\omega^m(f^{(r)}, \Omega_x, J)\,du\right) \\
&\le C\omega^m(f^{(r)}, \Omega_x, J)(1 + \Omega_x^{-m}(1-x)|J|^{m-1}) \\
&\le C\omega^m(f^{(r)}, \Omega_x, J), \quad x \in [1-n^{-2}, 1].
\end{aligned}$$

Thus, (36) is proved for any $m \in \mathbf{N}$ for the case $k = r$.

For $0 \le k \le r-1$ (using the Taylor formula) we have

$$|f^{(k)}(x) - \hat{L}^{(k)}(f,x)| = \left|\frac{f^{(r)}(\zeta_x) - \hat{L}^{(r)}(f,\zeta_x)}{(r-k)!}(x-1)^{r-k}\right|$$

for some $\zeta_x \in [x, 1]$. Finally, using the last equality and (36) with $k = r$, we have, for $0 \le k \le r-1$ and $x \in [1-n^{-2}, 1]$,

$$\begin{aligned}
|f^{(k)}(x) - \hat{L}^{(k)}(f,x)| &\le (1-x)^{r-k}\omega^m\left(f^{(r)}, (1-\zeta_x)^{1/m}\left(\frac{1}{n^2}\right)^{(m-1)/m}, J\right) \\
&\le C(1-x)^{r-k}\omega^m\left(f^{(r)}, (1-x)^{1/m}\left(\frac{1}{n^2}\right)^{(m-1)/m}, J\right),
\end{aligned}$$

which completes the proof of the lemma. ∎

The following lemma shows how any spline can be presented as a linear combination of the truncated power functions $(t-t_i)_+^\nu$.

Lemma I (Proposition 2.3.1 of [20]). *The spline $S(t)$ of degree M with the knots at t_i, $1 \le i \le N-1$ ($t_0 < t_1 < \cdots < t_N$) is uniquely presented in the form*

$$\begin{aligned}
(39)\quad S(t) &= \sum_{\nu=0}^{M} \frac{1}{\nu!} S^{(\nu)}(t_0)(t-t_0)^\nu \\
&\quad + \sum_{i=1}^{N-1}\sum_{j=0}^{M} \frac{1}{(M-j)!}(S^{(M-j)}(t_i+0) - S^{(M-j)}(t_i-0))(t-t_i)_+^{M-j}
\end{aligned}$$

for $t \in [t_0, t_N]$.

Note that if $S(x)$ has defect k_i ($1 \le k_i \le m+1$) at a knot t_i, then $S^{(m-j)}(t_i+0) - S^{(m-j)}(t_i-0) = 0$ for $k_i \le j \le m$.

5. Auxiliary Results for the Case $\lambda > 0$

This section contains results which are used only in the case $\lambda > 0$. The following lemma shows that, in fact, it does not matter whether we use ω^m or $\omega^m_{\varphi^\lambda}$ moduli to estimate the degree of local approximation. Whatever is convenient to use (in most cases, the usual modulus of smoothness) will do.

Lemma J [19]. *Let $[a, b] \subset [-1, 1]$ be such that $b - a \le C\Delta_n(a)$, where C is an absolute constant. Then for any integer m, there exists a constant $C(m)$ such that for any $\lambda \in [0, 1]$ and $x \in [a, b]$,*

$$\begin{aligned} C(m)\omega^m(f, \Delta_n(x), [a, b]) &\le \omega^m_{\varphi^\lambda}(f, n^{-\lambda}\Delta_n(x)^{1-\lambda}, [a, b]) \\ &\le C(m)\omega^m(f, \Delta_n(x), [a, b]). \end{aligned} \tag{40}$$

Note that only the first inequality in (40) is used in our proofs. For the proof of Theorem 2 we also need the following refinement of the above lemma for $x \in \{x | 1 - x^2 \le n^{-2}\}$.

Lemma 10. *For any $m \in \mathbf{N}$, $\lambda \in [0, 1]$, $0 \le \alpha \le 1$, and $x \in \{x | 1 - x^2 \le n^{-2}\}$, the following inequality holds*:

$$\omega^m(f, (1 - x^2)^\alpha n^{2\alpha-2}) \le C(m)\omega^m_{\varphi^\lambda}(f, n^{-\lambda}((1 - x^2)^\alpha n^{2\alpha-2})^{1-\lambda}). \tag{41}$$

Note that (41) cannot be reversed (at least not for all λ) since, for $x = 1$ and $\alpha \ne 0$, the left-hand side of (41) is equal to zero; at the same time, for $\lambda = 1$, the right-hand side of (41) does not vanish if f is not in Π_{m-1}.

Proof of Lemma 10. Using the definition of $\omega^m_{\varphi^\lambda}$ moduli, we have the following inequalities:

$$\begin{aligned} J_\lambda &:= \omega^m_{\varphi^\lambda}(f, n^{-\lambda}((1 - x^2)^\alpha n^{2\alpha-2})^{1-\lambda}) \\ &= \sup_h \left\{ \left\| \Delta^m_{h(\sqrt{1-y^2})^\lambda}(f, y) \right\|_{C[-1,1]}, 0 < h \le n^{-\lambda}((1 - x^2)^\alpha n^{2\alpha-2})^{1-\lambda} \right\} \\ &= \sup_{\tilde h := (hn^\lambda)^{1/(1-\lambda)}} \left\{ \left\| \Delta^m_{\tilde h^{1-\lambda} n^{-\lambda}(\sqrt{1-y^2})^\lambda}(f, y) \right\|_{C[-1,1]}, 0 < \tilde h \le (1 - x^2)^\alpha n^{2\alpha-2} \right\} \\ &\ge \sup_{0<\tilde h\le(1-x^2)^\alpha n^{2\alpha-2}} \left\| \Delta^m_{\tilde h^{1-\lambda} n^{-\lambda}(\sqrt{1-y^2})^\lambda}(f, y) \right\|_{C[-1+(m/2)^{2/(2-\lambda)}\tilde h, 1-(m/2)^{2/(2-\lambda)}\tilde h]} \end{aligned}$$

Now, note that for any $y \in [-1+(m/2)^{2/(2-\lambda)}\tilde h, 1-(m/2)^{2/(2-\lambda)}\tilde h]$ the inequality $1 - y^2 \ge (m/2)^{2/(2-\lambda)}\tilde h$ is valid, and, therefore, since $\tilde h \le n^{-2}$, then $\tilde h^{1-\lambda} n^{-\lambda}(\sqrt{1 - y^2})^\lambda \ge$

$(m/2)^{\lambda/(2-\lambda)}\tilde{h}$. Hence, we have (by the same argument as in Section 5 of [18])

$$
\begin{aligned}
J_\lambda &\geq \sup_{0<\tilde{h}\leq(1-x^2)^\alpha n^{2\alpha-2}} \|\Delta^m_{(m/2)^{\lambda/(2-\lambda)}\tilde{h}}(f,y)\|_{C[-1+(m/2)^{2/(2-\lambda)}\tilde{h},1-(m/2)^{2/(2-\lambda)}\tilde{h}]} \\
&= \sup_{0<\hat{h}:=(m/2)^{\lambda/(2-\lambda)}\tilde{h}\leq(m/2)^{\lambda/(2-\lambda)}(1-x^2)^\alpha n^{2\alpha-2}} \|\Delta^m_{\hat{h}}(f,y)\|_{C[-1+m\hat{h}/2,1-m\hat{h}/2]} \\
&= \omega^m(f,(m/2)^{\lambda/(2-\lambda)}(1-x^2)^\alpha n^{2\alpha-2}) \\
&\geq 2^{-m}\omega^m(f,(1-x^2)^\alpha n^{2\alpha-2}).
\end{aligned}
$$

Thus, the proof of the lemma is complete. ∎

The following proposition is needed only to make the constants in (9), (10), and (14)–(17) independent of λ.

Proposition K [19]. *For any integer m there exists a constant $C = C(m)$ such that for every $t > 0$, $0 \leq \lambda \leq 1$, and $\mu \geq 1$, the following inequality holds:*

$$
\omega^m_{\varphi^\lambda}(f,\mu t) \leq C(m)\mu^{2m}\omega^m_{\varphi^\lambda}(f,t). \tag{42}
$$

6. Proofs of Theorems 1 and 2

The idea behind the proofs of both Theorems 1 and 2 is quite natural, and to a certain extent was used in the literature (see [2], [4], [18], and [26], for example). Namely, if the spline S is defined to be a Lagrange or Lagrange-Hermite interpolation polynomial on I_j, then Lemmas 8, 9, and G imply that $f^{(k)}$ is sufficiently approximated by $S^{(k)}$. Using Lemma I, we construct a polynomial P_n which has the same form as the analytic representation (39) of the spline, but with $R_{i,M-j}$ instead of $(\cdot - x_i)_+^{M-j}$. Finally, using Lemma 7, we show that $P_n^{(k)}$ sufficiently approximates $S^{(k)}$ and, therefore, $f^{(k)}$.

Everywhere in this section we use the following convention which simplifies notations. Let

$$
l_j(x) := \begin{cases} L(f,x;x_j,x_{j-1},\dots,x_{j-m-r+1}), & \text{if } m+r-1\leq j\leq n, \\ L(f,x;x_{m+r-1},x_{m+r-2},\dots,x_0), & \text{if } 0\leq j< m+r-1. \end{cases}
$$

Then for any $j = 0,\dots,n$, function $l_j(x)$ is the Lagrange interpolation polynomial of degree $\leq m+r-1$.

Note that $I_j \subset \tilde{I}_j$ for all $j = 1,\dots,n$ and $|\tilde{I}_j| \leq C(r,m)|I_j|$ (therefore, the condition in the assertion of Lemma J is satisfied for any $[a,b] \subset \tilde{I}_j$). Also, note that it is sufficient to prove (9), (10), and (14)–(17) almost everywhere in $[-1,1]$, since all the functions being considered in these inequalities are assumed to be continuous. Everywhere in this section it is presumed that $x \neq x_j$, $1 \leq j \leq n-1$. Thus, for example, when we consider derivatives of the spline $S(x)$ we do not emphasize (though it is implied) that $S^{(k)}(x)$ is defined for $x \in [-1,1] \setminus \{x_1,\dots,x_{n-1}\}$.

Now we are ready to prove Theorem 1.

6.1. *Proof of Theorem* 1

Let $S(x) := l_j(x)$ for $x \in I_j$, $1 \le j \le n$. Then $S(x)$ is a spline of degree $\le m + r - 1$. Using Lemmas G and J, and properties of the classical moduli of smoothness, we have the following estimates for $x \in (x_j, x_{j-1})$, $1 \le j \le n$, and for all $0 \le k \le r$ and $k \le r_k \le r$:

$$
\begin{aligned}
(43)\qquad |f^{(k)}(x) - S^{(k)}(x)| &= |f^{(k)}(x) - l_j^{(k)}(x)| \\
&\le C\omega^{m+r-k}(f^{(k)}, \Delta_n(x), \tilde{I}_j) \\
&\le C\Delta_n(x)^{r_k-k}\omega^{m+r-r_k}(f^{(r_k)}, \Delta_n(x), \tilde{I}_j) \\
&\le C\Delta_n(x)^{r_k-k}\omega_{\varphi^\lambda}^{m+r-r_k}(f^{(r_k)}, n^{-\lambda}\Delta_n(x)^{1-\lambda}, \tilde{I}_j) \\
&\le C\Delta_n(x)^{r_k-k}\omega_{\varphi^\lambda}^{m+r-r_k}(f^{(r_k)}, n^{-\lambda}\Delta_n(x)^{1-\lambda}).
\end{aligned}
$$

Taking into account that the spline $S(x)$ is of degree at most $m + r - 1$ we get the following analytic representation (see (39)), which will be used for the construction of an approximating polynomial.

$$
(44)\qquad S(x) = p_{m+r-1}(x) + \sum_{i=1}^{n-1}\sum_{j=0}^{m+r-1} A_{ij}(x - x_i)_+^{m+r-1-j}, \quad x \in [-1, 1],
$$

where $p_{m+r-1}(x) = \sum_{\nu=0}^{m+r-1}(1/\nu!)S^{(\nu)}(-1)(x+1)^\nu$ is a polynomial of degree $\le m + r - 1$, and coefficients A_{ij} are given by

$$
A_{ij} := \frac{1}{(m+r-1-j)!}(S^{(m+r-1-j)}(x_i + 0) - S^{(m+r-1-j)}(x_i - 0)).
$$

Now let

$$
(45)\qquad P_n(x) = p_{m+r-1}(x) + \sum_{i=1}^{n-1}\sum_{j=0}^{m+r-1} A_{ij}R_{i,m+r-1-j}(x).
$$

Then $P_n(x)$ is a polynomial of degree $\le 4n\mu + m + r$ and $P_{4n\mu+m+r}(f, \cdot)\colon f \mapsto P_n$ is a linear operator.

Let us estimate $P_n^{(k)}(x) - S^{(k)}(x)$, $x \in [-1, 1]$. First, we consider A_{ij}. Using the Markov inequality first and then the Whitney inequality (Lemma G with $k = 0$), we have for $1 \le i \le n - 1$, $0 \le j \le m + r - 1$, and any $0 \le \tilde{r} \le r$,

$$
\begin{aligned}
(46)\qquad |A_{ij}| &= C|S^{(m+r-1-j)}(x_i + 0) - S^{(m+r-1-j)}(x_i - 0)| \\
&= C|l_{i+1}^{(m+r-1-j)}(x_i) - l_i^{(m+r-1-j)}(x_i)| \\
&\le Ch_i^{-m-r+1+j}\,\|l_{i+1} - l_i\|_{C(I_i)} \\
&\le Ch_i^{-m-r+1+j}(\|l_{i+1} - f\|_{C(\tilde{I}_{i+1})} + \|f - l_i\|_{C(\tilde{I}_i)}) \\
&\le Ch_i^{-m-r+1+j}\omega^{m+r}(f, \Delta_n(x_i), \tilde{I}_{i+1} \cup \tilde{I}_i) \\
&\le Ch_i^{-m-r+1+j+\tilde{r}}\omega^{m+r-\tilde{r}}(f^{(\tilde{r})}, \Delta_n(x_i), \tilde{I}_{i+1} \cup \tilde{I}_i) \\
&\le Ch_i^{-m-r+1+j+\tilde{r}}\omega_{\varphi^\lambda}^{m+r-\tilde{r}}(f^{(\tilde{r})}, n^{-\lambda}\Delta_n(x_i)^{1-\lambda}, \tilde{I}_{i+1} \cup \tilde{I}_i).
\end{aligned}
$$

Now we choose μ and ξ to be large in comparison with r and m. For example, let $\xi = 4(m+r)$ and $\mu = 50(m+r)$. We continue to write "μ" and "ξ" understanding that now these variables are functions of r and m.

For any $x \in [-1,1] \setminus \{x_1, \dots, x_{n-1}\}$, $0 \le k \le 2(m+r)$ and $0 \le \tilde{r} \le r$ (using Lemma 7, Proposition K, and estimates (46), (25), (26), and (27)) we have

$$
\begin{aligned}
(47) \qquad & |P_n^{(k)}(x) - S^{(k)}(x)| \\
& \le \sum_{i=1}^{n-1} \sum_{j=0}^{m+r-1} |A_{ij}| \left| R_{i,m+r-1-j}^{(k)}(x) - \frac{\partial^k}{\partial x^k}(x - x_i)_+^{m+r-1-j} \right| \\
& \le C \sum_{i=1}^{n-1} \sum_{j=0}^{m+r-1} h_i^{-m-r+j+1+\tilde{r}} \omega_{\varphi^\lambda}^{m+r-\tilde{r}}(f^{(\tilde{r})}, n^{-\lambda}\Delta_n(x_i)^{1-\lambda}) \\
& \times \psi_i^{\mu-\xi} h_i^{m+r-j-1-k} \\
& \le C \omega_{\varphi^\lambda}^{m+r-\tilde{r}}(f^{(\tilde{r})}, n^{-\lambda}\Delta_n(x)^{1-\lambda}) \sum_{i=1}^{n-1} h_i^{\tilde{r}-k} \psi_i^{\mu-\xi-2m-2r+2\tilde{r}} \\
& \le C \Delta_n(x)^{\tilde{r}-k} \omega_{\varphi^\lambda}^{m+r-\tilde{r}}(f^{(\tilde{r})}, n^{-\lambda}\Delta_n(x)^{1-\lambda}) \sum_{i=1}^{n-1} \psi_i^{\mu-\xi-2m-2r-2k} \\
& \le C \Delta_n(x)^{\tilde{r}-k} \omega_{\varphi^\lambda}^{m+r-\tilde{r}}(f^{(\tilde{r})}, n^{-\lambda}\Delta_n(x)^{1-\lambda}).
\end{aligned}
$$

Therefore for any $x \in [-1,1]$ and for fixed $0 \le k \le r$, choosing $\tilde{r} = r_k$ together with (43), we have

$$
|f^{(k)}(x) - P_n^{(k)}(x)| \le C \Delta_n(x)^{r_k-k} \omega_{\varphi^\lambda}^{m+r-r_k}(f^{(r_k)}, n^{-\lambda}\Delta_n(x)^{1-\lambda}),
$$

which is the desired inequality (9). Estimate (10) for $m+r \le k \le 2(m+r)$ follows from (47) since $S^{(k)}(x) \equiv 0$ a.e. for $k \ge m+r$.

Finally, for $k > 2(m+r)$, inequality (10) follows from the above estimates and Theorem 4.1 of [6]. (Note that Theorem 4.1 in [6] was proved with constants which depend on λ. However, using the inequality $\omega_{\varphi^\lambda}^s(g, \mu\delta) \le C(s)(1+\mu^{2s})\omega_{\varphi^\lambda}^s(g,\delta)$ (see Proposition K), instead of $\omega_{\varphi^\lambda}^s(g,\mu\delta) \le C(s,\lambda)(1+\mu^s)\omega_{\varphi^\lambda}^s(g,\delta)$, and following its proof word for word one can show that this dependence on λ is not necessary and can be eliminated.)

The proof of Theorem 1 is now complete for sufficiently large n, say, $n \ge n_0$ (in fact we proved (9) and (10) for $n \ge 201(m+r)$). For $m+r-1 \le n < n_0$ the assertion of Theorem 1 follows from the case $n = m+r-1$ for which it is sufficient to choose $P_{m+r-1}(f,x) := L(f,x;-1,-1+2/(m+r-1),\dots,1)$.

6.2. *Proof of Theorem* 2

For the proof of Theorem 2 we change the construction of the spline $S(x)$ from the previous subsection near the endpoints of the interval $[-1, 1]$. Namely, let

$$S(x) := \begin{cases} l_j(x), & x \in I_j, 2 \le j \le n-1, \\ \hat{L}(f, x), & x \in I_1, \\ \tilde{L}(f, x), & x \in I_n. \end{cases} \tag{48}$$

where $\hat{L}(f, x)$ and $\tilde{L}(f, x)$ are the Lagrange-Hermite interpolation polynomials of degree $\le m + r - 1$ defined in Lemmas 8 and 9, respectively.

Inequality (43) with $r_k = r$, together with the estimates (36) and (37) implies for any $0 \le k \le r$:

$$|f^{(k)}(x) - S^{(k)}(x)| \le C \begin{cases} \Delta_n(x)^{r-k} \omega^m(f^{(r)}, \Delta_n(x), \tilde{I}_j), \text{ if } x \in (x_j, x_{j-1}), \\ \qquad 2 \le j \le n-1, \\ (1-x^2)^{r-k} \omega^m \left(f^{(r)}, (1-x^2)^{1/m} \left(\dfrac{1}{n^2} \right)^{(m-1)/m}, \tilde{I}_j \right), \\ \qquad x \in (x_j, x_{j-1}), j = 1 \text{ or } n. \end{cases} \tag{49}$$

Now since $S(x)$ is a spline of degree $\le m + r - 1$, it has the analytic representation (44). Let the polynomial $P_n(x)$ be defined by (45) with $S(x)$ given by (48). Since the Lagrange (Lagrange-Hermite) interpolation process is a linear mapping from $C[-1, 1]$ ($C^r[-1, 1]$) to the subspace of the algebraic polynomials of some degree (of degree $\le m + r - 1$ in our case), then the operator $P_{4n\nu+m+r}(f, \cdot)$: $f \mapsto P_n$ is also a linear operator.

Inequality (46) with $\tilde{r} = r$ implies that for any $2 \le i \le n-2$ and $0 \le j \le m+r-1$,

$$|A_{ij}| \le C h_i^{-m+1+j} \omega_{\varphi^\lambda}^m (f^{(r)}, n^{-\lambda} \Delta_n(x_i)^{1-\lambda}, \tilde{I}_{i+1} \cup \tilde{I}_i). \tag{50}$$

Inequality (50) also holds for $i = 1$ and $i = n - 1$, since for $i = 1$ (the case $i = n - 1$ is treated similarly), we have (using Lemma 8 and Corollary H)

$$\begin{aligned} (m + r - 1 - j)! |A_{1j}| &= |S^{(m+r-1-j)}(x_1 + 0) - S^{(m+r-1-j)}(x_1 - 0)| \\ &= |\hat{L}^{(m+r-1-j)}(f, x_1) - l_2^{(m+r-1-j)}(x_1)| \\ &\le C h_1^{-m-r+1+j} \|\hat{L}(f, \cdot) - l_2\|_{C(I_1)} \\ &\le C h_1^{-m-r+1+j} (\|\hat{L}(f, \cdot) - f\|_{C(\tilde{I}_1)} + \|f - l_2\|_{C(\tilde{I}_2)}) \\ &\le C h_1^{-m-r+1+j} h_1^r \omega^m (f^{(r)}, \Delta_n(x_1), \tilde{I}_1 \cup \tilde{I}_2) \\ &\le C h_1^{-m+1+j} \omega_{\varphi^\lambda}^m (f^{(r)}, n^{-\lambda} \Delta_n(x_1)^{1-\lambda}, \tilde{I}_1 \cup \tilde{I}_2). \end{aligned}$$

Therefore, for any $x \in [-1, 1]$ and $0 \le k \le k_0$, (47) with $\tilde{r} = r$ holds for the above-defined $P_n(x)$ and $S(x)$. Now, considerations similar to those from the previous subsection imply (14) and (16). It remains to prove (15) and (17) for $x \in [-1, -1+n^{-2}] \cup [1-n^{-2}, 1] =: \mathcal{E}_n$. The inequality

$$(51) \qquad t_1^{-m}\omega^m(g, t_1) \le 2^m t_2^{-m}\omega^m(g, t_2), \quad t_1 \ge t_2,$$

yields, for any $x \in \mathcal{E}_n$ and $1 \le i \le n-1$,

$$(52) \qquad \omega^m(f^{(r)}, h_i) \le \frac{h_i^m n^{2m-2}}{1-x^2}\omega^m\left(f^{(r)}, (1-x^2)^{1/m}\left(\frac{1}{n^2}\right)^{(m-1)/m}\right).$$

Therefore, for any $0 \le k \le k_0$, $0 \le r_k \le k_0$, and $x \in \mathcal{E}_n$ (choosing $\xi = 4k_0$ and $\mu = 50k_0$, for example, and using Lemma 7, (50) with $\lambda = 0$, (52), (26), and (27)) we have

$$\begin{aligned}
(53) \quad & \left|P_n^{(k)}(x) - S^{(k)}(x)\right| \\
&\le \sum_{i=1}^{n-1}\sum_{j=0}^{m+r-1}|A_{ij}|\left|R_{i,m+r-1-j}^{(k)}(x) - \frac{\partial^k}{\partial x^k}(x-x_i)_+^{m+r-1-j}\right| \\
&\le C\sum_{i=1}^{n-1}\sum_{j=0}^{m+r-1} h_i^{-m+j+1}\omega^m(f^{(r)}, h_i) \\
&\quad \times \left(\frac{1-x^2}{(1+x_{i-1})(1-x_i)}\right)^{\xi/2}\psi_i^\mu h_i^{m+r-j-1-k} \\
&\le C\omega^m\left(f^{(r)}, (1-x^2)^{1/m}\left(\frac{1}{n^2}\right)^{(m-1)/m}\right) \\
&\quad \times \sum_{i=1}^{n-1}\frac{h_i^m n^{2m-2}}{1-x^2}\left(\frac{1-x^2}{h_i}\right)^{\xi/2}\psi_i^\mu h_i^{r-k} \\
&= C(1-x^2)^{r_k}\omega^m\left(f^{(r)}, (1-x^2)^{1/m}\left(\frac{1}{n^2}\right)^{(m-1)/m}\right) \\
&\quad \times \sum_{i=1}^{n-1}\left(\frac{1-x^2}{h_i}\right)^{\xi/2-r_k-1}(n^2h_i)^{m-1}h_i^{r-k-r_k}\psi_i^\mu \\
&\le C(1-x^2)^{r_k}\Delta_n(x)^{r-k-r_k}\omega^m\left(f^{(r)}, (1-x^2)^{1/m}\left(\frac{1}{n^2}\right)^{(m-1)/m}\right) \\
&\quad \times \sum_{i=1}^{n-1}\left(\frac{1}{n^2h_i}\right)^{\xi/2-r_k-m}\psi_i^{\mu-2r-2k-2r_k} \\
&\le C(1-x^2)^{r_k}\Delta_n(x)^{r-k-r_k}\omega^m\left(f^{(r)}, (1-x^2)^{1/m}\left(\frac{1}{n^2}\right)^{(m-1)/m}\right).
\end{aligned}$$

Now, choosing r_k in (53) to be $r-k$ (in the case $0 \le k \le r$) together with (49), we have, for any $x \in \mathcal{E}_n$ and $0 \le k \le r$,

$$|f^{(k)}(x) - P_n^{(k)}(x)| \le C(1-x^2)^{r-k}\omega^m\left(f^{(r)}, (1-x^2)^{1/m}\left(\frac{1}{n^2}\right)^{(m-1)/m}\right), \tag{54}$$

which is (15) for $\lambda = 0$ and $\alpha_k = 1/m$. Also, since $S^{(k)}(x) \equiv 0$ a.e. for $k \ge m+r$, then (53) implies, for $m+r \le k \le k_0$ and any $0 \le r_k \le k_0$,

$$|P_n^{(k)}(x)| \le C(1-x^2)^{r_k}\Delta_n(x)^{r-k-r_k}\omega^m\left(f^{(r)}, (1-x^2)^{1/m}\left(\frac{1}{n^2}\right)^{(m-1)/m}\right). \tag{55}$$

Now (51) implies, for $x \in \mathcal{E}_n$ and all $\alpha \in [1/m, 1]$,

$$\begin{aligned} &\omega^m\left(f^{(r)}, (1-x^2)^{1/m}\left(\frac{1}{n^2}\right)^{(m-1)/m}\right) \\ &\qquad \le C(n^2(1-x^2))^{1-\alpha m}\omega^m(f^{(r)}, (1-x^2)^{\alpha}n^{2\alpha-2}). \end{aligned} \tag{56}$$

Finally, (54) and (56), together with Lemma 10, imply

$$\begin{aligned} &|f^{(k)}(x) - P_n^{(k)}(f,x)| \\ &\qquad \le Cn^{2-2\alpha_k m}(1-x^2)^{r-k+1-\alpha_k m}\omega^m_{\varphi^\lambda}(f^{(r)}, n^{-\lambda}((1-x^2)^{\alpha_k}n^{2\alpha_k-2})^{1-\lambda}), \end{aligned}$$

for $x \in \mathcal{E}_n$, $0 \le k \le r$, and any $\alpha_k \in [1/m, 1]$.

Similarly, using Lemma 10 and (55) and (56), we obtain (17). The proof of Theorem 2 is now complete for sufficiently large n ($n \ge n_0 := 201k_0$).

For $\max\{m+r-1, 2r+1\} \le n < n_0$, the assertion of Theorem 2 follows from the case $\tilde{n} = \max\{m+r-1, 2r+1\}$, for which it is sufficient to choose $P_{\tilde{n}}(f,x) := P_{\tilde{n}}(x)$, where $P_{\tilde{n}}(x)$ is the polynomial such that $P_{\tilde{n}}^{(k)}(\pm 1) = f^{(k)}(\pm 1)$ for all $k = 0, 1, \dots, r$; and, if $m > r+2$, $P_{\tilde{n}}(-1+2i/(m-r-1)) = f(-1+2i/(m-r-1))$ for all $i = 1, 2, \dots, m-r-2$. Using considerations similar to those employed for the proof of Lemma 8, one can show that $P_{\tilde{n}}(f,x)$ satisfies (14) and (15). ■

7. Proof of Theorem 3

As we have already mentioned, the method of the construction of a counterexample and the proof is a minor modification of the method introduced by X. M. Yu [30]. For completeness of exposition and, since the proof is not long, we adduce it here.

Suppose that the assertion of the theorem is not correct. Then there is a constant $K \in \mathbf{R}$ such that for every $f \in C^{m+r-1}[-1,1]$, a polynomial $P_n \in \Pi_n$ exists satisfying

$$|f(x) - P_n(x)| \le K\Delta_n(x)^r\omega^m(f^{(r)}, \Delta_n(x)), \tag{57}$$

for $x \in [-1+n^{-2}, 1-n^{-2}]$, and

$$|f(x) - P_n(x)| \le K(1-x^2)^{\alpha}n^{2\alpha-2r}\omega^m(f^{(r)}, (1-x^2)^{\beta}n^{\gamma}), \tag{58}$$

for $x \in [-1, -1+n^{-2}] \cup [1-n^{-2}, 1]$.

Let

$$f(x) := \begin{cases} (x - 1 + An^{-2})^{m+r}, & x \in [1 - An^{-2}, 1], \\ 0, & x \in [-1, 1 - An^{-2}], \end{cases}$$

where $A \leq 1$ is a constant which will be chosen later. Then $f \in C^{m+r-1}[-1, 1]$, and the following inequalities are satisfied:

$$|f(x)| \leq (An^{-2})^{m+r}, \quad -1 \leq x \leq 1, \tag{59}$$

and

$$\omega^m(f^{(r)}, t) \leq (m + r)! \min\{t^m, (An^{-2})^m\}, \quad t > 0. \tag{60}$$

The last inequality implies

$$|f(x) - P_n(x)| \leq K(m + r)!(1 - x^2)^{\alpha + m\beta} n^{2\alpha - 2r + m\gamma},$$

for $x \in [1 - n^{-2}, 1]$, and, since $\alpha + m\beta > r + 1$, we have $f^{(k)}(1) = P_n^{(k)}(1)$ for all $k = 0, 1, \ldots, r + 1$. Also, (57), (58), and (60) yield

$$|f(x) - P_n(x)| \leq K(m + r)! \Delta_n(x)^r (An^{-2})^m,$$

for $x \in [-1 + n^{-2}, 1 - n^{-2}]$, and

$$|f(x) - P_n(x)| \leq K(m + r)! 2^\alpha n^{-2r} (An^{-2})^m,$$

for $x \in [-1, -1 + n^{-2}] \cup [1 - n^{-2}, 1]$. Therefore,

$$|f(x) - P_n(x)| \leq K(m + r)! 2^\alpha \Delta_n(x)^r (An^{-2})^m,$$

for all $x \in [-1, 1]$. Hence, for every $x \in [-1, 1]$ (applying (59)) we have

$$|P_n(x)| \leq |f(x)| + |f(x) - P_n(x)| \leq (1 + K(m + r)!)\, 2^\alpha \Delta_n(x)^r (An^{-2})^m.$$

Now applying the Dzyadyk inequality (see (31)), we conclude that

$$\|\Delta_n(x) P_n^{(r+1)}(x)\| \leq C_r \|\Delta_n(x)^{-r} P_n(x)\| \leq C_r\, (1 + K(m + r)!)\, 2^\alpha (An^{-2})^m,$$

and thus,

$$|P_n^{(r+1)}(1)| \leq C_r\, (1 + K(m + r)!)\, 2^\alpha A^m n^{-2m+2}. \tag{61}$$

On the other hand,

$$|P_n^{(r+1)}(1)| = |f^{(r+1)}(1)| = \frac{(m + r)!}{(m - 1)!} A^{m-1} n^{-2m+2}. \tag{62}$$

Now, choosing

$$A = \min\left\{1, \frac{(m + r)!}{2C_r(m - 1)!\,(1 + K(m + r)!)\, 2^\alpha}\right\},$$

we conclude that (61) and (62) cannot hold simultaneously, thus obtaining a contradiction. The proof is complete. ∎

References

1. YU. A. BRUDNYI (1963): *Generalization of a theorem of A. F. Timan.* Soviet Math. Dokl., **4**:244–247.
2. YU. A. BRUDNYI (1968): *Approximation of functions by algebraic polynomials.* Izv. Akad. Nauk SSSR, Ser. Mat., **32**:780–787 (in Russian).
3. R. DAHLHAUS (1989): *Pointwise approximation by algebraic polynomials.* J. Approx. Theory, **57**:274–277.
4. R. A. DEVORE (1977): *Pointwise approximation by polynomials and splines.* In: Theory of Approximation of Functions (S. B. Stechkin and S. A. Telyakovskii, eds.). Proc. Intern. Conf., Kaluga, 1975, Nauka, Moskva, pp. 132–141.
5. Z. DITZIAN, V. TOTIK (1987): Moduli of Smoothness. Berlin: Springer-Verlag.
6. Z. DITZIAN, D. JIANG (1992): *Approximation of functions by polynomials in* $C[-1, 1]$. Canad. J. Math., **44**(5):924–940.
7. Z. DITZIAN, D. JIANG, D. LEVIATAN (1993): *Simultaneous polynomial approximation.* SIAM J. Math. Anal., **24**(6):1652–1661.
8. V. K. DZYADYK (1956): *Constructive characterization of functions satisfying the condition* $Lip\alpha(0 < \alpha < 1)$ *on a finite segment of the real axis.* Izv. Akad. Nauk SSSR, Ser. Mat., **20**:623–642 (in Russian).
9. V. K. DZYADYK (1977): Introduction to the Theory of Uniform Approximation of Functions by Polynomials. Moscow: Nauka (in Russian).
10. A. O. GELFOND (1955): *On uniform approximation by polynomials with integer coefficients.* Uspekhi Mat. Nauk, **1**(63):41–65 (in Russian).
11. H. GONSKA, E. HINNEMANN (1985): *Pointwise estimations of approximations by algebraic polynomials.* Acta Math. Hungar., **46**(3–4):243–254 (in German).
12. I. E. GOPENGAUZ (1967): *A theorem of A. F. Timan on the approximation of functions by polynomials on a finite segment.* Mat. Zametki, **1**(2):163–172 (in Russian) (English translation (1967)) Math. Notes, **1**:110–116.
13. I. E. GOPENGAUZ (1967): *A question concerning the approximation of functions on a segment and a region with corners.* Theor. Funkcii Funkcional. Anal. Pril., **4**:204–210 (in Russian).
14. M. HASSON (1980): *Derivatives of the algebraic polynomials of best approximation.* J. Approx. Theory, **29**:91–102.
15. M. HEILMANN (1989): *On simultaneous approximation by optimal algebraic polynomials.* Results in Mathematics, **16**:77–81.
16. E. HINNEMANN, H. H. GONSKA (1983): *Generalization of a theorem of DeVore.* In: Approximation Theory IV (C. K. Chui, L. L. Schumaker, J. W. Ward, eds.). New York: Academic Press, pp. 527–532.
17. T. KILGORE (1993): *An elementary simultaneous approximation theorem.* Proc. Amer. Math. Soc., **118**(2):529–536.
18. K. A. KOPOTUN (1994): *Pointwise and uniform estimates for convex approximation of functions by algebraic polynomials.* Constr. Approx., **10**:153–178.
19. K. A. KOPOTUN (1995): *Unconstrained and convex polynomial approximation in* $C[-1, 1]$. Approx. Theory Appl., **11**(2):41–58.
20. N. P. KORNEJCHUK (1987): Exact Constants in Approximation Theory. Moscow: Nauka (English translation (1991) Cambridge: University Press.).
21. D. LEVIATAN (1982): *The behavior of the derivatives of the algebraic polynomials of best approximation.* J. Approx. Theory, **35**:169–176.
22. W. LI (1986): *On Timan type theorems in algebraic polynomial approximation.* Acta Math. Sinica, **29**(4):544–549 (in Chinese).
23. V. N. MALOSEMOV (1966): *Joint approximation of a function and its derivatives by algebraic polynomials.* Soviet Math. Dokl., **7**(5):1274–1276.
24. I. A. SHEVCHUK (1992): *Approximation of monotone functions by monotone polynomials.* Mat. Sb., **183**(5):63–78 (in Russian) (English translation (1993): Russian Acad. Sci. Sb. Math., **76**(1):51–64).
25. I. A. SHEVCHUK (1989): *On coapproximation of monotone functions.* Dokl. Akad. Nauk USSR, **308**(3):537–541.
26. I. A. SHEVCHUK (1992): Approximation by Polynomials and Traces of the Functions Continuous on an Interval. Kiev: Naukova dumka (in Russian).
27. S. A. TELYAKOVSKII (1966): *Two theorems on the approximation of functions by algebraic poly-*

nomials. Mat. Sb., **70**:252–265 (in Russian) (English translation (1966): American Mathematical Society Translations, Series 2, **77**:163–178).

28. A. F. TIMAN (1951): *A strengthening of Jackson's theorem on the best approximation of continuous functions on a finite segment of the real axis*. Dokl. Akad. Nauk USSR, **78**:17–20 (in Russian).
29. R. M. TRIGUB (1962): *Approximation of functions by polynomials with integral coefficients*. Izv. Akad. Nauk SSSR, Ser. Mat., **26**:261–280 (in Russian).
30. X. M. YU (1985): *Pointwise estimate for algebraic polynomial approximation*. Approx. Theory Appl., **1**(3):109–114.
31. S. P. ZHOU (1988): *On simultaneous approximation of a function and its derivatives*. Dokl. Akad. Nauk BSSR, **32**(6):493–495 (in Russian).

K. Kopotun
Department of Mathematical Sciences
University of Alberta
Edmonton
Alberta
Canada T6G 2G1