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CONSTRUCTIVE APPROXIMATION

Whitney Theorem of Interpolatory Type for k-Monotone Functions

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Abstract. One of the main results of this paper is the following Whitney theorem of interpolatory type for k-monotone functions (i.e., functions f such that divided differences $f[x_0, \ldots, x_k]$ are nonnegative for all choices of $(k+1)$ distinct points $x_0, \ldots, x_k$).

Theorem. *Let $0 < p \le \infty$, $k \ge 1$, $m \le k$, and let $f \in L_p[a,b]$ be a k-monotone function on $[a,b]$. Then, if a polynomial p_{m-1} of degree $\le m-1$ interpolates f at m arbitrary points in $J_A = [a + A(b-a), b - A(b-a)]$, where $A < \frac{1}{2}$ is a strictly positive constant, then*

$$\|f - p_{m-1}\|_{L_p[a,b]} \le C \inf_{q:\deg q \le m-1} \|f - q\|_{L_p[a,b]},$$

where the constant C is independent of the location of interpolation points. Except for the case $m = 1$ and $p = \infty$, the above statement is no longer true if $A = 0$.

We also show that the above theorem is not valid if $m \ge k+1$. It is well known that results of this (interpolatory) type are not true in general for non-k-monotone functions.

1. Introduction

The following theorem is well known, and is now called the *Whitney* (or *Whitney-type*) *theorem*. It was proved by Burkill [3] ($k = 2$, $p = \infty$), Whitney [9], [10] ($p = \infty$), Brudnyi [1] ($1 \le p \le \infty$), and Storozhenko [8] ($0 < p < 1$). It has applications in many areas, and has been further generalized to various classes of functions and other approximating spaces. However, this paper deals with its "classical version," which we now state.

Theorem A. *Let $f \in L_p[a,b]$, $0 < p \le \infty$. Then there exists $q_{k-1} \in \Pi_{k-1}$, a polynomial of degree $\le k-1$, such that*

$$\|f - q_{k-1}\|_{L_p[a,b]} \le C\omega_k(f, b-a, [a,b])_p. \tag{1}$$

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Moreover, in the case $p = \infty$, *we can choose* $q_{k-1}(x) := L_{k-1}(f, x; a, a+(b-a)/(k-1), \dots, b)$, *the Lagrange polynomial interpolating* f *at* k *equidistant points in* $[a, b]$. *Here,* ω_k *is the* k*th modulus of smoothness defined by*

$$\omega_k(f, t, [a, b])_p := \sup_{0<h\le t} \|\Delta_h^k(f, \cdot, [a, b])\|_{L_p[a,b]}, \qquad 0 < p \le \infty,$$

where $\Delta_h^k(f, x, [a, b])$ *is the* k*th finite difference.*

Of course, in the case $p < \infty$, the polynomial $L_{k-1}(f)$ does not have to satisfy (1) even if we assume that $f \in \mathbf{C}^\infty[a, b]$. To see that, one can, for example, consider a function $f \in \mathbf{C}^\infty$ such that $0 \le f(x) \le 1$, $x \in [a, b]$, $f(a + i(b-a)/(k-1)) = 1$ for $0 \le i \le k-1$, and $f(x) = 0$ if $|x - (a + i(b-a)/(k-1)| > \varepsilon$ for all $i = 0, \dots, k-1$. Then $L_{k-1}(f, x) \equiv 1$, $\|f - L_{k-1}(f)\|_{L_p[a,b]} \sim$ const., and, at the same time, $\omega_k(f, b-a, [a, b])_p \le C\|f\|_p \le C(k\int_{-\varepsilon}^{\varepsilon} 1\, dx)^{1/p} = C\varepsilon^{1/p}$, and, hence, (1) does not hold in general for $q_{k-1} = L_{k-1}(f)$. In the case $p = \infty$, the interpolation points do not really have to be equidistant in order for (1) to be true, but, at the same time, it is well known that they cannot be too close to each other (see [5], for example).

One of the often-used consequences of Theorem 1 is the fact that the kth modulus of smoothness of f is equivalent to the error of best approximation of f by polynomials of degree $\le k-1$:

$$\omega_k(f, b-a, [a, b])_p \sim E_{k-1}(f)_p := \inf_{q\in\Pi_{k-1}} \|f - q\|_{L_p[a,b]}. \tag{2}$$

Thus, any polynomial $q \in \Pi_{k-1}$ that satisfied (1) is a "near-best" approximant to f in the L_p, $0 < p \le \infty$, metric in the sense that

$$\|f - q\|_{L_p[a,b]} \le C E_{k-1}(f)_p.$$

In particular, the Lagrange polynomial interpolating f at k equidistant points in $[a, b]$ is a near-best approximant to f in the uniform metric. It is not generally a near-best approximant in other L_p, $p < \infty$, metrics.

In many applications, it is desirable that the mathematical model preserves certain geometric properties of the data such as monotonicity, 2-monotonicity (convexity), and, in general, k-monotonicity. This is the subject that the so-called "shape-preserving approximation" deals with.

In this paper, we show, in particular, that when a function f is k-monotone, the inequality (1) is true for all $0 < p \le \infty$, and any polynomial $q_{k-1} \in \Pi_{k-1}$ interpolating f at k *arbitrary* points which are "not too close" to the endpoints of the interval $[a, b]$. This result will have applications in the area of shape-preserving approximation, since it implies that any k-monotone function (polynomial, spline, etc.) interpolating a k-monotone function at certain rather arbitrary points has good local approximation properties (in L_p for all $0 < p \le \infty$). We remark that some partial results in this direction were obtained in [5].

We now recall the definition of k-monotone functions, and discuss some of their properties.

A function $f : [a, b] \mapsto \mathbf{R}$ is said to be k-monotone, $k \ge 1$, on $[a, b]$ iff for all choices of $(k+1)$ distinct points $x_0, \dots, x_k$ in $[a, b]$ the inequality $f[x_0, \dots, x_k] \ge 0$

holds, where $f[x_0, \ldots, x_k] = \sum_{j=0}^{k} f(x_j)/w'(x_j)$ denotes the kth divided difference of f at $x_0, \ldots, x_k$, and $w(x) = \prod_{j=0}^{k}(x - x_j)$. Note that 0-monotone, 1-monotone, and 2-monotone functions are just nonnegative, nondecreasing, and convex functions, respectively. We denote the class of all k-monotone functions on $[a, b]$ by $\Delta^k[a, b]$. If $f \in \mathbf{C}^k[a, b]$, then $f \in \Delta^k[a, b]$ iff $f^{(k)}(x) \geq 0$, $x \in [a, b]$.

Also, we denote by $\Delta^k(a, b)$ the class of all k-monotone functions that are not required to be defined at the endpoints of $[a, b]$ (and, thus, do not have to be bounded on $[a, b]$). For example, $(-1)^k x^{-1} \in \Delta^k(0, 1)$ for all $k \in \mathbf{N}$. Clearly, $\Delta^k[a, b] \subset \Delta^k(a, b)$.

The following theorem about the properties of k-monotone functions will be used throughout this paper. In particular, we often use the fact that for all $k \geq 2$, if $f \in \Delta^k(a, b)$, then f is $(k-2)$-times continuously differentiable on (a, b). Also, in the case $k = 1$, if $f \in \Delta^1(a, b)$, then $f \in L_p[\xi, \zeta]$ for any $0 < p \leq \infty$, and any closed interval $[\xi, \zeta]$ contained in (a, b).

Theorem B ([7], [6]). *Suppose for some $k \geq 2$ that $f : [a, b] \mapsto \mathbf{R}$ is k-monotone. Then $f^{(j)}(x)$, the derivative of order j, exists on (a, b) for $j \leq k - 2$ and is $(k - j)$-monotone. In particular, $f^{(k-2)}(x)$ exists, is convex, and therefore satisfies a Lipschitz condition on any closed interval $[\xi, \zeta]$ contained in (a, b), is absolutely continuous on $[\xi, \zeta]$, is continuous on (a, b), and has left and right (nondecreasing) derivatives, $f_-^{(k-1)}(x)$ and $f_+^{(k-1)}(x)$ on (a, b). Moreover, the set E where $f^{(k-1)}(x)$ fails to exist is countable, and $f^{(k-1)}$ is continuous on $(a, b) \backslash E$.*

2. Main Results

One of our main results is the following interpolatory version of the Whitney theorem for k-monotone functions (see also Theorem 6 below).

Theorem 1. *Let $f \in \Delta^k(a, b) \cap L_p[a, b]$, $k \geq 1$, $0 < p \leq \infty$, and let $p_{k-1}(f) \in \Pi_{k-1}$ interpolate f at k points in $J_A = [a + A(b - a), b - A(b - a)]$, where $A < \frac{1}{2}$ is a strictly positive constant. Then,*

$$\|f - p_{k-1}(f)\|_{L_p[a,b]} \leq C\omega_k(f, b - a, [a, b])_p, \tag{3}$$

where the constant C depends only on A, k, p (if $p < 1$), and does not depend on the (location of) interpolation points. Except for the case $k = 1$, $p = \infty$, the statement of the theorem is no longer true if $A = 0$.

The proof of Theorem 1 turns out to be rather technical and is postponed until Section 3. We remark that (2) implies that $p_{k-1}(f)$ is a near-best approximant to $f \in \Delta^k$ in the L_p metric for all $0 < p \leq \infty$, and that the same remark is also valid for Theorems 3 and 6.

The following natural question now arises: since condition $f \in \Delta^k$ is strong enough to yield the Whitney estimate for Lagrange interpolation polynomials of degree $\leq k - 1$, what can be said about Lagrange polynomials of degree $\leq m - 1$ if $m \neq k$? Do we still have the Whitney inequality? Also, do interpolation points have to be "far" from the endpoints of $[a, b]$ in the case $m \neq k$ as well?

The main purpose of this paper is to provide answers to the above questions. First of all, it turns out that the case when $p = \infty$ and $m = 1$ (or $k = 1$) is very much different from the others. The following lemma is trivial, and is only stated here for completeness.

Lemma 2. *Let $f : [a, b] \mapsto \mathbf{R}$ be bounded on $[a, b]$. Then, for any $\xi \in [a, b]$:*

$$\| f(x) - f(\xi)\|_{L_\infty[a,b]} \le \omega(f, b-a, [a, b])_\infty.$$

It turns out that we cannot have the Whitney estimate for Lagrange interpolation polynomials of degree ≥ 1 (if $p = \infty$) or degree ≥ 0 (if $p < \infty$) without an additional assumption that the interpolation points are "not too close" to the endpoints of $[a, b]$. Also, as we show below, we need to assume that $m \le k$ for the Whitney estimate to hold. It is not surprising that, if $m \ge k + 1$, then the condition $f \in \Delta^k$ is "not strong enough."

Theorem 3 ($m \le k$). *Let $f \in \Delta^k(a, b) \cap L_p[a, b]$, $0 < p \le \infty$, $k \ge 1$, $m \le k$, and let $p_{m-1}(f) \in \Pi_{m-1}$ interpolate f at m points in $J_A = [a + A(b-a), b - A(b-a)]$, where $A < \frac{1}{2}$ is a strictly positive constant. Then*

$$\| f - p_{m-1}(f)\|_{L_p[a,b]} \le C\omega_m(f, b-a, [a, b])_p, \tag{4}$$

where the constant C depends only on A, k, p (if $p < 1$), and does not depend on the (location of) interpolation points. Except for the case $m = 1$, $p = \infty$, the above statement is no longer true if $A = 0$.

Proof of Theorem 3. We will show how the case $m < k$ can be reduced to $m = k$, which is considered in Theorem 1.

First of all, we note that it is enough to show that, for any $f \in \Delta^k(a, b) \cap L_p[a, b]$:

$$\| p_{m-1}(f)\|_{L_p[a,b]} \le C\| f\|_{L_p[a,b]}. \tag{5}$$

Indeed, suppose that (5) is true, and let q_{m-1} be any polynomial in Π_{m-1} such that $\| f - q_{m-1}\|_{L_p[a,b]} \le C\omega_m(f, b-a, [a, b])_p$. Then, taking into account that $f - q_{m-1} \in \Delta^k(a, b) \cap L_p[a, b]$ we have

$$\begin{aligned}
\| f - p_{m-1}(f)\|_{L_p[a,b]} &= \| f - q_{m-1} - p_{m-1}(f - q_{m-1})\|_{L_p[a,b]} \\
&\le C\| f - q_{m-1}\|_{L_p[a,b]} + C\| p_{m-1}(f - q_{m-1})\|_{L_p[a,b]} \\
&\le C\| f - q_{m-1}\|_{L_p[a,b]} \le C\omega_m(f, b-a, [a, b])_p.
\end{aligned}$$

To prove (5) we use the following rather trivial statement:

> Let Q be an arbitrary polynomial, and let $p_{m-1}(Q)$ be a polynomial of degree $\le m-1$ interpolating Q at m points inside $[a, b]$. Then, for any $0 < p \le \infty$:
>
> $$\| p_{m-1}(Q)\|_{L_p[a,b]} \le C\| Q\|_{L_p[a,b]},$$
>
> where the constant C depends only on m, $\deg Q$, p (if $p < 1$), and is independent of the interpolation points.

The above statement can be easily proved using Markov's inequality and the fact that norms in finitely dimensional spaces are equivalent (which allows us to only prove this statement in the case $p = \infty$). We omit the details.

To prove (5) we suppose that Q is a polynomial of degree $\le k-1$ which interpolates f at a set of k points in J_A which is a superset of the interpolation set for $p_{m-1}(f)$ (consisting of $m < k$ points). This immediately implies that $p_{m-1}(f) = p_{m-1}(Q)$. Now, using the fact that $\|f - Q\|_{L_p[a,b]} \le C\omega_k(f, b-a, [a,b])_p$ (Theorem 1), we have

$$\begin{aligned}
\|p_{m-1}(f)\|_{L_p[a,b]} &= \|p_{m-1}(Q)\|_{L_p[a,b]} \le C\|Q\|_{L_p[a,b]} \\
&\le C\|f - Q\|_{L_p[a,b]} + C\|f\|_{L_p[a,b]} \\
&\le C\omega_k(f, b-a, [a,b])_p + C\|f\|_{L_p[a,b]} \\
&\le C\|f\|_{L_p[a,b]}.
\end{aligned}$$

This proves (5). The proof of Theorem 3 is now complete. ∎

We would like to emphasize that the condition that J_A is "in the center" of $[a,b]$ is essential, and cannot be removed. We have the following proposition to illustrate that.

Proposition 4 ($m \in \mathbf{N}$, J near endpoints of $[a,b]$). *Let $0 < p \le \infty$, $k \in \mathbf{N}$, $m \in \mathbf{N}$ be such that*

$$m \ge \begin{cases} 2 & \text{if } p = \infty, \\ 1 & \text{if } p < \infty, \end{cases}$$

and let the interval $J = [\widetilde{a}, \widetilde{b}] \subset [a,b]$ be such that $\operatorname{dist}(J, \{a,b\}) = 0$ (*i.e., either $\widetilde{a} = a$ or $\widetilde{b} = b$*). *Then for any constant B, there exists a function $f \in \Delta^k[a,b]$ and a set of m points in J such that, if p_{m-1} is a polynomial of degree $\le m-1$ interpolating f at these points, then*

$$\|p_{m-1}\|_{L_p[a,b]} \ge B\|f\|_{L_p[a,b]}. \tag{6}$$

Hence, the inequality

$$\|f - p_{m-1}\|_{L_p[a,b]} \le C\omega_m(f, b-a, [a,b])_p$$

is not true in general for Lagrange polynomial p_{m-1} interpolating $f \in \Delta^k[a,b]$ at m points which are close to the endpoints of $[a,b]$.

Lemma 2 shows that m in Proposition 4 cannot be 1 if $p = \infty$.

Proof of Proposition 4. Let $J = [\widetilde{a}, \widetilde{b}] \subset [a,b]$ be any interval such that $\operatorname{dist}(J, b) = 0$ (i.e., $\widetilde{b} = b$), and define $f_\varepsilon(x) = \varepsilon^{1-k}(x - b + \varepsilon)_+^{k-1}$. Note that $f_\varepsilon \in \Delta^k[a,b]$.

First of all, we prove the proposition in the case $p < \infty$ and $m \ge 1$. Let p_{m-1} interpolate f_ε at $\{\widetilde{a} + i[(b - \widetilde{a})/(m-1)]\}_{i=0}^{m-1}$ ($\{b\}$ if $m = 1$). Then, for all sufficiently small ε ($\varepsilon < (b - \widetilde{a})/(m-1)$), $\|p_{m-1}\|_{L_p[a,b]} = \text{const.}$ (since $f(b) = 1$, and f is 0 at all other interpolation points), and, at the same time, $\|f_\varepsilon\|_{L_p[a,b]} \le C\varepsilon^{1/p} \to 0$ if $\varepsilon \to 0$.

Now, suppose that $p = \infty$ and $m \geq 2$. Let p_{m-1} interpolate f_ε at the points $\{t_j\}_{j=0}^{m-1}$, where $t_j = b - 2\varepsilon + [\varepsilon/(m-1)]j$, $j = 0, \ldots, m-2$, and $t_{m-1} = b$. Using the Lagrange interpolation formula, we conclude that

$$\begin{aligned}\|p_{m-1}\|_{\mathbf{C}[a,b]} &\geq |p_{m-1}(a)| = \prod_{j=0}^{m-2} \left| \frac{a - t_j}{b - t_j} \right| \\ &\geq \prod_{j=0}^{m-2} \left| \frac{a - b + 2\varepsilon}{2\varepsilon} \right| \geq C\varepsilon^{1-m} \to \infty \qquad \text{as} \quad \varepsilon \to 0,\end{aligned}$$

and, at the same time, $\|f_\varepsilon\|_{\mathbf{C}[a,b]} = f_\varepsilon(b) = 1$. This completes the proof of the proposition. ∎

We now show that, if $m \geq k+1$, then the Whitney estimate does not necessarily hold for a Lagrange polynomial of degree $\leq m-1$ interpolating $f \in \Delta^k$ at m points even if we require that these points are "inside" $[a, b]$ (i.e., far from the endpoints of $[a, b]$).

Proposition 5 ($m \geq k+1$, arbitrary $J \subset [a, b]$). *Let* $0 < p \leq \infty$, $k \in \mathbf{N}$, $m \geq k+1$, *and* $J = [\widetilde{a}, \widetilde{b}] \subset [a, b]$. *Then, for any constant* B, *there exists a function* $f \in \Delta^k[a, b]$ *and a set of* m *points in* J *such that, if* p_{m-1} *is a polynomial of degree* $\leq m-1$ *interpolating* f *at these points, then*

$$\|p_{m-1}\|_{L_p[a,b]} \geq B\|f\|_{L_p[a,b]}. \tag{7}$$

Hence, the inequality

$$\|f - p_{m-1}\|_{L_p[a,b]} \leq C\omega_m(f, b-a, [a, b])_p$$

is not true in general for Lagrange polynomial p_{m-1} *interpolating* $f \in \Delta^k[a, b]$ *at* $m \geq k+1$ *arbitrary points in* J.

Proof of Proposition 5. Let $\xi = (\widetilde{a} + \widetilde{b})/2$. Without loss of generality we may assume that $\xi \geq (a+b)/2$. Now, let $f(x) := (x - \xi)_+^{k-1}$, and note that $f \in \Delta^k[a, b]$ and that $\|f\|_{L_p[a,b]} = C$, where the constant C depends only on ξ and the interval $[a, b]$. Now let $t_j = \xi - (m-2-j)\varepsilon$, $j = 0, \ldots, m-1$, where $(m-2)\varepsilon < \min\{|J|/2, (b-a)/8\}$, and suppose that p_{m-1} interpolates f at $\{t_j\}_{j=0}^{m-1}$. Taking into account that $f(t_j) = 0$ for $j = 0, \ldots, m-2$, and $f(t_{m-1}) = \varepsilon^{k-1}$ we have $p_{m-1}(x) = \varepsilon^{k-1} \prod_{j=0}^{m-2}(x - t_j)/(t_{m-1} - t_j)$ and, hence, for every $x \in [a, a + (b-a)/4]$:

$$|p_{m-1}(x)| \geq \varepsilon^{k-1} \prod_{j=0}^{m-2} \frac{(b-a)/8}{(m-1)\varepsilon} = C\varepsilon^{k-m}.$$

Therefore,

$$\|p_{m-1}\|_{L_p[a,b]} \geq \|p_{m-1}\|_{L_p[a,a+(b-a)/4]} \geq C\varepsilon^{k-m} \to \infty \qquad \text{as} \quad \varepsilon \to 0,$$

and (7) follows if we choose ε sufficiently small. ∎

Finally, as an immediate consequence of Theorems 1 and 3, we obtain the following theorem on simultaneous approximation of $f \in \Delta^k$ and its derivatives $f^{(i)}$ by p_{m-1} and $p_{m-1}^{(i)}$.

Theorem 6. *Let $f \in \Delta^k(a,b)$, $k \ge 1$, $m \le k$, and let $p_{m-1} \in \Pi_{m-1}$ interpolate f at m points in $J_A = [a + A(b-a), b - A(b-a)]$, where $A < \frac{1}{2}$ is a strictly positive constant. Then, for $i = 0, \dots, m-1$:*

$$\| f^{(i)} - p_{m-1}^{(i)} \|_{L_p[a,b]} \le C\omega_{m-i}(f^{(i)}, b-a, [a,b])_p, \tag{8}$$

whenever $f^{(i)} \in L_p[a,b]$, $0 < p \le \infty$, where the constants C depend only on A, k, p (if $p < 1$), and do not depend on the (location of) interpolation points. The statement of the theorem is no longer true in general if $A = 0$ (unless $m = 1$ and $p = \infty$).

Proof of Theorem 6. If $i = 0$, then (8) immediately follows from Theorem 3. In the case $1 \le i \le \min\{m-1, k-2\}$, Theorem 6 also follows from Theorem 3 taking into account that $f^{(i)} \in \Delta^{k-i}(a,b) \cap \mathbf{C}(a,b)$ and, by Rolle's theorem, $p_{m-1}^{(i)}$ interpolates $f^{(i)}$ at $m-i$ points inside J_A. Finally, in the case $m = k$, $i = k-1 \ge 1$, even though $f^{(k-1)}$ may fail to exist at countably many points, it is in Δ^1, and hence, (8) follows. ∎

3. Proof of Theorem 1

In order to prove Theorem 1 we first need to consider a couple of auxiliary results. First, recall the following lemma which can be found in Bullen [2].

Lemma C ([2]). *Let $f \in \Delta^k$, $k \ge 1$, and let $l_{k-1}(x)$ interpolate f at $z_1, \dots, z_k$, then $f - l_{k-1}$ changes sign at $z_1, \dots, z_k$ and, in particular, $f(x) - l_{k-1}(x) \ge 0$ for $x \ge \max\{z_1, \dots, z_k\}$, and $(-1)^k(f(x) - l_{k-1}(x)) \ge 0$ for $x \le \min\{z_1, \dots, z_k\}$.*

We also remark that Lemma C immediately follows from the definition of Δ^k and the identity

$$f(x) - l_{k-1}(x) = f[x, z_1, \dots, z_k](x - z_1)\dots(x - z_k), \qquad x \ne z_i.$$

Lemma 7. *Let $f \in \Delta^k(a,b)$, $k \ge 1$, be such that $f(t_1) = \dots = f(t_k) = 0$, where $a < t_1 < \dots < t_k < b$. Then $f \in \Delta^j(t_k, b)$ and $(-1)^{k-j} f \in \Delta^j(a, t_1)$ for all $j = 0, \dots, k-1$.*

Proof of Lemma 7. If $j = 0$, then the lemma is an immediate consequence of Lemma C. If $1 \le j \le k-2$ then, by Theorem B, the function $f^{(j)} \in \Delta^{k-j}(a,b) \cap \mathbf{C}(a,b)$ and, by Rolle's theorem, $f^{(j)}$ has $k-j$ zeros in $[t_1, t_k]$. Therefore, Lemma C implies that $f^{(j)}(x) \ge 0$ for $x \ge t_k$, and $(-1)^{k-j} f^{(j)}(x) \ge 0$ for $x \le t_1$. Hence, $f \in \Delta^j(t_k, b)$ and $(-1)^{k-j} f \in \Delta^j(a, t_1)$. This completes the proof in the case $j \le k-2$. In the case $j = k-1$, we note that the function $f^{(k-2)}$ is convex on (a,b) and has two zeros in $[t_1, t_k]$. Therefore, $f^{(k-2)}$ is nondecreasing for $x \ge t_k$, nonincreasing for $x \le t_1$, and, hence, $f \in \Delta^{k-1}(t_k, b)$ and $-f \in \Delta^{k-1}(a, t_1)$. The lemma is now proved. ∎

Lemma 8. *Let $k \geq 1$, $0 < p \leq \infty$, $J := [c, d] \subset D$, and $f \in L_p(D)$. Define $\widetilde{J} = [c, d + \delta]$ (or $\widetilde{J} = [c - \delta, d]$), where $0 < \delta \leq (d - c)/k$. Then, if $\widetilde{J} \subset D$, the following estimate is valid*:

$$\|f\|_{L_p(\widetilde{J})} \leq C\|f\|_{L_p(J)} + C\omega_k(f, \delta, \widetilde{J})_p. \tag{9}$$

Proof of Lemma 8. The lemma is well known. However, since its proof is simple and short, we include it here for completeness. We only consider the case $\widetilde{J} = [c, d+\delta]$, since the case $\widetilde{J} = [c-\delta, d]$ is completely analogous. Let $x \in [d-k\delta, d-(k-1)\delta] \subset [c, d]$. Then, since

$$\Delta_\delta^k(f, x) = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f(x + i\delta),$$

we have

$$|f(x + k\delta)| \leq |\Delta_\delta^k(f, x)| + \sum_{i=0}^{k-1} \binom{k}{i} |f(x + i\delta)|. \tag{10}$$

Taking supremum of both sides of this inequality immediately yields (9) for $p = \infty$. If $0 < p < \infty$, raising (10) to power p, taking into account that $(\mu + \nu)^p \leq C(\mu^p + \nu^p)$ for $0 < p < \infty$, and integrating with respect to x over $[d - k\delta, d - (k-1)\delta]$, we have

$$\begin{aligned}\int_{d-k\delta}^{d-(k-1)\delta} |f(x + k\delta)|^p\,dx \ &\leq\ C\|\Delta_\delta^k(f, x, \widetilde{J})\|_{L_p(\widetilde{J})}^p \\ &\qquad + C2^{pk}\sum_{i=0}^{k-1}\int_{d-k\delta}^{d-(k-1)\delta} |f(x + i\delta)|^p\,dx \\ &\leq\ C\|\Delta_\delta^k(f, x, \widetilde{J})\|_{L_p(\widetilde{J})}^p + C\sum_{i=0}^{k-1}\int_{d-(k-i)\delta}^{d-(k-1-i)\delta} |f(x)|^p\,dx,\end{aligned}$$

and, hence,

$$\|f\|_{L_p[d,d+\delta]}^p \leq C\|f\|_{L_p(J)}^p + C\omega_k(f, \delta, \widetilde{J})_p^p,$$

where C depends only on k and p (if $p < 1$). This immediately implies (9). ∎

Corollary 9. *Let $k \geq 1$, $0 < p \leq \infty$, $J = [c, d] \subset D$, and $f \in L_p(D)$. Let $\widetilde{J} \subset D$ be such that $J \subset \widetilde{J}$. Then*

$$\|f\|_{L_p(\widetilde{J})} \leq C\|f\|_{L_p(J)} + C\omega_k(f, |\widetilde{J}|, \widetilde{J})_p, \tag{11}$$

where C depends on the ratio $|\widetilde{J}|/|J|$.

Corollary 9 is used to prove the following Lemma 10 which is a simpler (but not as general) version of Theorem 1. In turn, the proof of Theorem 1 is based on this lemma.

Lemma 10. *Let $f \in \Delta^k(a,b)$, $k \ge 1$, and let J be an interval of length $\le (b-a)/(4k+1)$ in the center of $[a,b]$ (i.e., $\operatorname{dist}(J,a) = \operatorname{dist}(J,b)$). Let $p_{k-1} \in \Pi_{k-1}$ interpolate f at k points in J. If $f \in L_p[a,b]$, $0 < p \le \infty$, then*

$$\|f - p_{k-1}\|_{L_p[a,b]} \le C\omega_k(f, b-a, [a,b])_p, \tag{12}$$

where the constant C depends only on k, p (if $p < 1$), and does not depend on the (location of) interpolation points.

Proof of Lemma 10. For an interval $J = [\widetilde{a}, \widetilde{b}]$ we denote $|J| = \widetilde{b} - \widetilde{a}$, and

$$[(2\nu+1)J] := [\widetilde{a} - \nu|J|, \widetilde{b} + \nu|J|].$$

First of all, we note that it is sufficient to prove Lemma 10 for the interval J such that $|J| = (b-a)/(4k+1)$. This can be easily seen if we recall that the only condition that we have on interpolation points is that they are inside J. Thus, enlarging J if necessary, we can assume that $[(4k+1)J] = [a,b]$.

Now, let $f \in \Delta^k(a,b)$, $k \ge 1$, and let p_{k-1} interpolate f at k points in J:

$$\|f - p_{k-1}\|_{L_p[3J]} \le C\omega_k(f, b-a, [a,b])_p.$$

Now, taking into account that $|J| \sim b-a$, and using Corollary 9 we have

$$\begin{aligned} \|f - p_{k-1}\|_{L_p[a,b]} &\le C\|f - p_{k-1}\|_{L_p[3J]} + C\omega_k(f, b-a, [a,b])_p \\ &\le C\omega_k(f, b-a, [a,b])_p. \end{aligned}$$

This completes the proof of Lemma 10. ■

We are going to use extensively the fact that, for $0 < p \le \infty$, and for any $\widetilde{J}$ such that $J \subset \widetilde{J}$ and $|\widetilde{J}| \sim |J|$, if p is a polynomial of degree $\le k-1$, then (see, e.g., [4]):

$$\|p\|_{L_p(J)} \sim \|p\|_{L_p(\widetilde{J})}.$$

The following lemma and its corollary are used to derive Theorem 1 from Lemma 10.

Lemma 11. *Let $J = [\widetilde{a}, \widetilde{b}] \subset [a,b]$, and let $f \in L_p[a,b]$, $0 < p \le \infty$, be such that $f \in \Delta^k(a,b)$, $k \ge 1$, and f has k zeros to the left of the interval J. Let $p \in \Pi_{k-1}$ interpolate f at k points inside J. Then, for any constant $\mu > 0$, such that $\widetilde{b} + \mu|J| \le b$:*

$$\|p\|_{L_p[\widetilde{a}, \widetilde{b}+\mu|J|]} \le C\|f\|_{L_p[\widetilde{a}, \widetilde{b}+\mu|J|]}, \tag{13}$$

where $C = C(k, \mu, p)$. In the case $\mu = 0$ this statement is no longer true.

Proof. Suppose that p interpolates f at the points $y_0 < \cdots < y_{k-1}$ inside J. Then $p(x) = \sum_{i=0}^{k-1}\{f[y_0,\ldots,y_i]\prod_{j=0}^{i-1}(x - y_j)\}$ and since, by Lemma 7, $f[y_0,\ldots,y_i] \ge 0$ for all $i = 0,\ldots,k-1$ (because $f \in \Delta^i(\widetilde{a}, b)$), we know that p is nondecreasing for

$x \geq y_{k-1}$, and, hence, $p(x) \geq 0$ for $x \geq y_{k-1}$ (since $f(y_{k-1}) \geq 0$ by Lemma 7). Thus, it follows from Lemma C that $0 \leq p(x) \leq f(x)$ for $\widetilde{b} \leq x \leq b$. Therefore,

$$\|p\|_{L_p[\widetilde{b},\widetilde{b}+\mu|J|]} \leq \|f\|_{L_p[\widetilde{b},\widetilde{b}+\mu|J|]}.$$

Now, since $|[\widetilde{b}, \widetilde{b}+\mu|J|]| \sim |J| = |\widetilde{b}-\widetilde{a}|$ we conclude that

$$\|p\|_{L_p[\widetilde{a},\widetilde{b}+\mu|J|]} \leq C\|p\|_{L_p[\widetilde{b},\widetilde{b}+\mu|J|]} \leq C\|f\|_{L_p[\widetilde{b},\widetilde{b}+\mu|J|]},$$

which completes the proof of the lemma. ■

Again, we would like to emphasize that "the endpoints cause difficulties" (except for the case $k = 1$ and $p = \infty$). Namely, it is essential that the constant μ in Lemma 11 is assumed to be *strictly positive*. The inequality (13) is no longer valid if $\mu = 0$. We can apply ideas similar to those used in the proof of Proposition 4 to show that. (In fact, we would not have Proposition 4 if Lemma 11 were true with $\mu = 0$.) Suppose that the interval $J = [\widetilde{a}, \widetilde{b}] \subset [a, b]$ is fixed, and let $f(x) = (x - \widetilde{b} + \varepsilon)_+^{k-1}$, $\varepsilon < |J|/2$. Let p interpolate f at k points in $[\widetilde{b} - \varepsilon, \widetilde{b}] \subset J$. Then $p(x) = (x - \widetilde{b} + \varepsilon)^{k-1}$ and, hence,

$$\|p\|_{L_p(J)} \geq \|p\|_{L_p[\widetilde{a},\widetilde{b}-\varepsilon]} = C(|J| - \varepsilon)^{k-1+1/p} \geq C|J|^{k-1+1/p}.$$

At the same time,

$$\|f\|_{L_p(J)} = \|f\|_{L_p[\widetilde{b}-\varepsilon,\widetilde{b}]} \leq C\varepsilon^{k-1+1/p}.$$

Choosing ε sufficiently small shows that for no constant C is the inequality $\|p\|_{L_p(J)} \leq C\|f\|_{L_p(J)}$ true in general.

The following corollary immediately follows from Lemma 11 using the fact that $f(x) \in \Delta^k(a, b)$ iff $(-1)^k f(a + b - x) \in \Delta^k(a, b)$.

Corollary 12. *Let $J = [\widetilde{a}, \widetilde{b}] \subset [a, b]$, and let $f \in L_p[a, b]$, $0 < p \leq \infty$, be such that $f \in \Delta^k(a, b)$, $k \geq 1$, and f has k zeros to the right of the interval J. Let $p \in \Pi_{k-1}$ interpolate f at k points inside J. Then, for any constant $\mu > 0$, such that $\widetilde{a} - \mu|J| \geq a$:*

$$\|p\|_{L_p[\widetilde{a}-\mu|J|,\widetilde{b}]} \leq C\|f\|_{L_p[\widetilde{a}-\mu|J|,\widetilde{b}]}, \tag{14}$$

where $C = C(k, \mu, p)$. In the case $\mu = 0$ this statement is no longer true.

We are now ready to prove Theorem 1.

Proof of Theorem 1. Let $A > 0$ be fixed, and let $\widetilde{J}$ be the interval of length $(b - a)/(4k+1)$ in the center of $[a, b]$ (i.e., $\mathrm{dist}(\widetilde{J}, a) = \mathrm{dist}(\widetilde{J}, b)$). Let $q_{k-1} \in \Pi_{k-1}$ interpolate f at k points inside $\widetilde{J} \cap J_A$. Then, Lemma 10 implies that

$$\|f - q_{k-1}\|_{L_p[a,b]} \leq C\omega_k(f, b - a, [a, b])_p.$$

Now, let $r_{k-1} = r_{k-1}(f) \in \Pi_{k-1}$ interpolate f at k points in $[b - A(b-a), b - \frac{1}{2}A(b-a)]$. Then,

$$\begin{aligned}
\|f - r_{k-1}(f)\|_{L_p[a,b]} &= \|f - q_{k-1} - r_{k-1}(f - q_{k-1})\|_{L_p[a,b]} \\
&\leq C\|f - q_{k-1}\|_{L_p[a,b]} + C\|r_{k-1}(f - q_{k-1})\|_{L_p[a,b]} \\
&\leq C\|f - q_{k-1}\|_{L_p[a,b]} + C\|r_{k-1}(f - q_{k-1})\|_{L_p[b-A(b-a),b]} \\
&\leq C\|f - q_{k-1}\|_{L_p[a,b]} + C\|f - q_{k-1}\|_{L_p[b-A(b-a),b]},
\end{aligned}$$

where, in the last inequality, we used Lemma 11 with $\widetilde{a} = b - A(b-a)$, $\widetilde{b} = b - \frac{1}{2}A(b-a)$, and $\mu = 1$. Hence,

$$\| f - r_{k-1}(f)\|_{L_p[a,b]} \le C\| f - q_{k-1}\|_{L_p[a,b]} \le C\omega_k(f, b-a, [a,b])_p.$$

Now, we are ready to estimate $\| f - p_{k-1}(f)\|_{L_p[a,b]}$:

$$\begin{aligned}
\| f - p_{k-1}(f)\|_{L_p[a,b]} &\le \| f - r_{k-1} - p_{k-1}(f - r_{k-1})\|_{L_p[a,b]} \\
&\le C\| f - r_{k-1}\|_{L_p[a,b]} + C\| p_{k-1}(f - r_{k-1})\|_{L_p[a,b]} \\
&\le C\| f - r_{k-1}\|_{L_p[a,b]} + C\| p_{k-1}(f - r_{k-1})\|_{L_p[a,b-A(b-a)]} \\
&\le C\| f - r_{k-1}\|_{L_p[a,b]} + C\| f - r_{k-1}\|_{L_p[a,b-A(b-a)]},
\end{aligned}$$

where Corollary 12 was applied with $\widetilde{a} = a + A(b-a)$, $\widetilde{b} = b - A(b-a)$, $\mu = A/(1-2A)$. Thus,

$$\| f - p_{k-1}(f)\|_{L_p[a,b]} \le C\| f - r_{k-1}\|_{L_p[a,b]} \le C\omega_k(f, b-a, [a,b])_p,$$

which completes the proof of the theorem. ■

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