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CONSTRUCTIVE APPROXIMATION

Pointwise and Uniform Estimates for Convex Approximation of Functions by Algebraic Polynomials

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Abstract. Let Δ^q be the set of functions f for which the qth difference is nonnegative on the interval $[-1, 1]$, P_n is the set of algebraic polynomials of degree not exceeding n, $\tau_k(f, \delta)_p$ is the averaged Sendov–Popov modulus of smoothness in the $L_p[-1, 1]$ metric for $1 \le p \le \infty$, $\omega_k(f, \delta)$ and $\omega_\phi^k(f, \delta)$, $\phi(x) := \sqrt{1 - x^2}$, are the usual modulus and the Ditzian–Totik modulus of smoothness in the uniform metric, respectively. For a function $f \in C[-1, 1] \cap \Delta^2$ we construct a polynomial $p_n \in P_n \cap \Delta^2$ such that

$$|f(x) - p_n(x)| \le C\omega_3(f, n^{-1}\sqrt{1 - x^2} + n^{-2}), \qquad x \in [-1, 1];$$

$$\|f - p_n\|_\infty \le C\omega_\phi^3(f, n^{-1});$$

$$\|f - p_n\|_p \le C\tau_3(f, n^{-1})_p.$$

As a consequence, for a function $f \in C^2[-1, 1] \cap \Delta^3$ a polynomial $p_n^* \in P_n \cap \Delta^3$ exists such that

$$\|f - p_n^*\|_\infty \le Cn^{-1}\omega_2(f', n^{-1}),$$

where $n \ge 2$ and C is an absolute constant.

1. Introduction and Main Results

Let us recall that coapproximation is the approximation of functions f for which the qth forward difference, given by

$$\bar{\Delta}_h^q(f, x, [a, b]) := \begin{cases} \sum_{i=0}^{q} (-1)^{q-i}\binom{q}{i} f(x + ih) & \text{if } [x, x + qh] \subset [a, b], \\ 0, & \text{otherwise,} \end{cases}$$

is nonnegative for given $q \in N$, for all $0 \le h \le (b - a)/q$ and $x \in [a, b]$, by poly-

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nomials with nonnegative qth derivatives. Let $\Delta^q[a, b]$ be the set of such functions f (note that if $f \in C^q[a, b]$, then $f \in \Delta^q[a, b]$ if and only if $f^{(q)}(x) \geq 0$, $x \in [a, b]$).

Denote $I := [-1, 1]$, $\Delta^q := \Delta^q(I)$, $\bar{\Delta}_h^q(f, x) := \bar{\Delta}_h^q(f, x, I)$. For $k \in N$ we denote H_k^φ as the class of functions $f \in C(I)$ whose kth modulus of smoothness does not exceed the k-majorant $\varphi = \varphi(t)$ (i.e., $\varphi = \varphi(t)$, $t \geq 0$, is a continuous and nondecreasing function satisfying the conditions $\varphi(0) = 0$ and $t^{-k}\varphi(t)$ nonincreasing), that is,

$$\omega_k(f, t) := \omega_k(f, t; I)_\infty \leq \varphi(t).$$

Here we denote

$$\begin{aligned} \omega_k(f, t; [a, b]) &:= \omega_k(f, t; [a, b])_\infty \\ &:= \sup_{0<h\leq t} \|\bar{\Delta}_h^k(f, x, [a, b])\|_{C[a, b]}. \end{aligned}$$

$$W^r H_k^\varphi := \{f : f^{(r)} \in H_k^\varphi\},$$

$$\Delta_n(x) := n^{-1}\sqrt{1 - x^2} + n^{-2},$$

P_n is the set of algebraic polynomials of degree at most n, and C is an absolute constant.

In the monotone case the following analog of the direct theorems for unconstrained polynomial approximation is known.

Theorem A. *Let $k \in N$ if $r \in N$, and $k = 1$ or 2 if $r = 0$. Then, for $f \in W^r H_k^\varphi \cap \Delta^1$ and an arbitrary $n \in N$, $n \geq k + r - 1$, a polynomial $p_n \in P_n \cap \Delta^1$ satisfying*

$$|f(x) - p_n(x)| \leq C(\Delta_n(x))^r \varphi(\Delta_n(x)), \qquad C = C(k), \quad x \in I, \tag{1}$$

exists.

An immediate consequence of A. S. Shvedov [12] is the fact that Theorem A is not correct for $r = 0$, $k \geq 3$. For $r = 0$, $k = 1$ or 2 Theorem A is a consequence of the work of R. A. DeVore and X. M. Yu [1] who constructed the sequence of polynomials $p_n \in P_n \cap \Delta^1$ which approximate a function $f \in C(I) \cap \Delta^1$ so that

$$|f(x) - p_n(x)| \leq C\omega_2(f, n^{-1}\sqrt{1 - x^2}), \qquad x \in I. \tag{2}$$

For $r \in N$, $k \in N$ Theorem A was proved by I. A. Shevchuk [9], [10].

For convex approximation the following result is known.

Theorem B. *Let $k \in N$ if $r \geq 2$, $k = 1$ if $r = 1$, and $k = 1$ or 2 if $r = 0$. Then, for $f \in W^r H_k^\varphi \cap \Delta^2$ and an arbitrary $n \in N$, $n \geq k + r - 1$, a polynomial $p_n \in P_n \cap \Delta^2$ satisfying* (1) *exists.*

A. S. Shvedov showed in [12] (see also [11]) that Theorem B is not correct for $r = 0$, $k \geq 4$ and $r = 1$, $k \geq 3$. For $r = 0$, $k = 1$ or 2, and $r = k = 1$ Theorem B is a consequence of D. Leviatan [6], where the estimate (2) was obtained for convex approximation. For $r \geq 2$, $k \in N$ Theorem B was proved by S. P. Manya and

I. A. Shevchuk (see [11], for example). There is a gap in Theorem B as nothing is known for $r = 0$, $k = 3$, and $r = 1$, $k = 2$.

Let us write the number $\alpha > 0$ as $\alpha = r + \beta$ where r is a nonnegative integer and $0 < \beta \leq 1$. Denote by $Lip^* \alpha$ the class of all functions $f(x)$ on I such that

$$\omega_2(f^{(r)}, t) = O(t^\beta).$$

A consequence of Theorem B and also classical converse theorems (see, for example, p. 263 of [3]) is

Theorem C. *For $\alpha > 0$, $\alpha \neq 2$, a function $f = f(x)$ is convex on I and belongs to $Lip^* \alpha$ if and only if, for each $n \geq r + 1$, a convex polynomial on I, $p_n = p_n(x)$ of at most degree n, exists such that*

$$|f(x) - p_n(x)| \leq C(\Delta_n(x))^\alpha, \qquad x \in I. \tag{3}$$

For $\alpha = 2$ the result of Theorem C is not complete as this case corresponds to $r = 1$, $k = 2$ in Theorem B.

In this paper it is shown that Theorem B is correct for $r = 0$, $k = 3$ (and therefore for $r = 1$, $k = 2$), and hence Theorem C is correct for $\alpha = 2$. Namely, they are consequences of the following theorem.

Theorem 1. *For a convex function $f \in C(I)$ and every $n \geq 2$ a convex polynomial $p_n = p_n(x)$ of degree not exceeding n exists such that*

$$|f(x) - p_n(x)| \leq C\omega_3(f, \Delta_n(x)), \qquad x \in I. \tag{4}$$

If $f \in C^1(I)$, then the following estimate also holds:

$$|f'(x) - p_n'(x)| \leq C\omega_2(f', \Delta_n(x)), \qquad x \in I. \tag{5}$$

Moreover, for $f \in C^2(I)$ there is also the following estimate:

$$|f''(x) - p_n''(x)| \leq C\omega(f'', \Delta_n(x)), \qquad x \in I. \tag{6}$$

Corollary 1. *If $f \in C^1(I) \cap \Delta^2$, then, for every $n \geq 2$, $p_n \in P_n \cap \Delta^2$ exists such that*

$$|f(x) - p_n(x)| \leq C\Delta_n(x)\omega_2(f', \Delta_n(x)), \qquad x \in I. \tag{7}$$

Remark 1. Estimate (4) can be improved to some degree (see the method in [11], for example). Namely,

$$|f(x) - p_n(x)| \leq C\begin{cases} \omega_3\left(f, \dfrac{\sqrt{1-x^2}}{n}\right), & x \in [-1 + n^{-2}, 1 - n^{-2}], \\ \omega_3\left(f, \sqrt[3]{\dfrac{1-x^2}{n}\dfrac{1}{n}}\right), & x \in [-1, -1 + n^{-2}) \cup (1 - n^{-2}, 1]. \end{cases} \tag{8}$$

All the estimates above are pointwise. The uniform estimates in terms of the usual moduli of continuity are rather imperfect because as can be seen from inequalities (1)–(8) the degree of approximation improves as the endpoints of the interval I are approached. The modulus of smoothness ω_ϕ^k introduced and used extensively by Z. Ditzian and V. Totik [2] is given by

$$\omega_\phi^k(f, t)_p = \sup_{0<h\le t} \|\Delta_{h\phi(x)}^k f(x)\|_p,$$

where

$$\Delta_h^k f(x) := \begin{cases} \sum_{i=0}^{k} (-1)^{k-i}\binom{k}{i} f\left(x + \frac{kh}{2} - ih\right) & \text{if } x \pm \frac{kh}{2} \in I, \\ 0, & \text{elsewhere} \end{cases}$$

($\Delta_h^k f(x)$ is the kth symmetric difference). It is obvious that under the proper conditions on the function $\phi = \phi(x)$ (for example, for $\phi(x) = \sqrt{1 - x^2}$) the step of the difference is decreasing near the endpoints of $[-1, 1]$. So uniform estimates in terms of the Ditzian–Totik modulus of smoothness are more exact than the usual ones (see [2]).

For $f \in C(I)$, $\phi(x) = \sqrt{1 - x^2}$, $x \in I$ in uniform metric the following definition of a "nonuniform" modulus of smoothness $\bar{\omega}_\phi^k(f, t)$ will be used (see [11], for example):

$$\bar{\omega}_\phi^k(f, t) := \sup_{0<h\le t} \|\bar{\Delta}_\rho^k(f, x)\|_{C[-1,1]}, \qquad t \ge 0,$$

where $\rho := \rho(x, h) := \sqrt{1 - x^2}h + h^2$. It is easy to see that $\bar{\omega}_\phi^k(f, t) \sim \omega_\phi^k(f, t)_\infty$ with $\phi(x) = \sqrt{1 - x^2}$ (see also [2]).

Theorem 2. *For a function $f \in C(I) \cap \Delta^2$ and each $n \ge 2$, $p_n \in P_n \cap \Delta^2$ exists such that*

$$\|f - p_n\|_\infty \le C\bar{\omega}_\phi^3(f, n^{-1}). \tag{9}$$

If $f \in C^1(I)$, then the following estimate also holds:

$$\|f' - p_n'\|_\infty \le C\bar{\omega}_\phi^2(f', n^{-1}). \tag{10}$$

Moreover, for $f \in C^2(I)$ there is also the following estimate:

$$\|f'' - p_n''\|_\infty \le C\bar{\omega}_\phi(f'', n^{-1}). \tag{11}$$

Theorem 2 improves the estimate of convex approximation

$$\|f - p_n\|_\infty \le C\bar{\omega}_\phi^2(f, n^{-1}) \tag{12}$$

which was obtained by D. Leviatan [6].

Corollary 2. *If $f \in C^1(I) \cap \Delta^2$, then, for every $n \geq 2$, $p_n \in P_n \cap \Delta^2$ exists such that*

$$\|f - p_n\|_\infty \leq Cn^{-1}\bar{\omega}_\phi^2(f', n^{-1}). \tag{13}$$

Let us recall that the integral modulus of the kth order of function $f \in L_p[-1, 1]$, $1 \leq p \leq \infty$, is the function

$$\omega_k(f, \delta)_p := \sup_{0<h\leq\delta} \left\{\frac{1}{2}\int_{-1}^{1-kh} |\bar{\Delta}_h^k(f, x)|^p\,dx\right\}^{1/p}, \qquad \delta \in [0, 2k^{-1}].$$

For a function f bounded on $[-1, 1]$ the local modulus of smoothness of order k at the point $x \in [-1, 1]$ is the function (see Definition 1.4 of [8])

$$\omega_k(f, x; \delta) := \sup\left\{|\bar{\Delta}_h^k(f, t)| : t, t + kh \in \left[x - \frac{k\delta}{2}, x + \frac{k\delta}{2}\right]\right\}.$$

The kth-order averaged Sendov–Popov modulus of smoothness of a function f bounded and measurable on $[-1, 1]$ is (see Definition 1.5 of [8])

$$\tau_k(f, \delta)_p := \|\omega_k(f, \cdot, \delta)\|_p = \left\{\frac{1}{2}\int_{-1}^{1} (\omega_k(f, x, \delta))^p\,dx\right\}^{1/p}, \qquad \delta \in [0, 2k^{-1}].$$

The following properties of τ_k are used (see Theorems 1.4 and 1.5 of [8]):

$$\omega_k(f, \delta)_p \leq \tau_k(f, \delta)_p \leq \omega_k(f, \delta). \tag{14}$$

(15) A constant $C(k)$ depending only on $k \geq 2$ exists such that, for each function f absolutely continuous on $[a, b]$, the following inequality holds:

$$\tau_k(f, \delta)_p \leq C(k)\delta\omega_{k-1}(f', \delta)_p.$$

In this paper the following theorem is proved.

Theorem 3. *Let $1 \leq p \leq \infty$. For a function $f \in C(I) \cap \Delta^2$ and each $n \geq 2$, $p_n \in P_n \cap \Delta^2$ exists such that*

$$\|f - p_n\|_p \leq C\tau_3(f, n^{-1})_p. \tag{16}$$

If $f \in C^1(I)$, then the following estimate also holds:

$$\|f' - p_n'\|_p \leq C\tau_2(f', n^{-1})_p. \tag{17}$$

Moreover, for $f \in C^2(I)$ there is also the following estimate:

$$\|f'' - p_n''\|_p \leq C\tau(f'', n^{-1})_p. \tag{18}$$

Corollary 3. *By* (15) *and* (16), *for $f \in C^1(I) \cap \Delta^2$ and every $n \geq 2$, $p_n \in P_n \cap \Delta^2$ exists such that*

$$\|f - p_n\|_p \leq Cn^{-1}\omega_2(f', n^{-1})_p, \qquad 1 \leq p < \infty. \tag{19}$$

By the methods of [12], using (15) and also estimate (16), it is easy to prove:

Theorem 4. *For a function* $f \in C^2(I) \cap \Delta^3$ *and each* $n \geq 2$, $p_n \in P_n \cap \Delta^3$ *exists such that*

$$\|f - p_n\|_\infty \leq Cn^{-1}\omega_2(f', n^{-1})_\infty. \tag{20}$$

2. Definitions and Notation

Throughout this paper we use the following notation (see [4] and [9]–[11]):

$$\Delta_n(x) := \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}, \qquad x \in I;$$

$$x_j := \cos\frac{j\pi}{n}, \quad j = \overline{0, n}; \qquad \bar{x}_j := \cos\left(\frac{j\pi}{n} - \frac{\pi}{2n}\right), \quad j = \overline{1, n};$$

$$x_j^\circ := \cos\left(\frac{j\pi}{n} - \frac{\pi}{4n}\right) \quad \text{if} \quad j < \frac{n}{2}, \qquad x_j^\circ := \cos\left(\frac{j\pi}{n} - \frac{3\pi}{4n}\right) \quad \text{if} \quad j \geq \frac{n}{2};$$

$$I_j := [x_j, x_{j-1}], \qquad h_j := x_{j-1} - x_j, \qquad j = \overline{1, n};$$

$$t_j(x) := (x - x_j^\circ)^{-2}\cos^2 2n \arccos x + (x - \bar{x}_j)^{-2}\sin^2 2n \arccos x$$

is the algebraic polynomial of degree not exceeding $4n - 2$ (see [3] and [9]);

$$\Pi_j(\xi, \zeta, \mu) := \int_{-1}^{1} (y - x_j)^\xi (x_{j-1} - y)^\zeta t_j^\mu(y)\, dy,$$

where ξ, ζ, and μ are integers. It is shown in Proposition 2 that if $\xi \geq 0$, $\zeta \geq 0$, and μ is big enough in comparison with ξ and ζ, then $\Pi_j(\xi, \zeta, \mu) \sim h_j^{-2\mu+\xi+\zeta+1}$. This permits us to define

$$Q_j(x) := \frac{\displaystyle\int_{-1}^{x} (y - x_j)t_j^{10}(y)\, dy}{\displaystyle\int_{-1}^{1} (y - x_j)t_j^{10}(y)\, dy}, \qquad \bar{Q}_j(x) := \frac{\displaystyle\int_{-1}^{x} (x_{j-1} - y)t_j^{10}(y)\, dy}{\displaystyle\int_{-1}^{1} (x_{j-1} - y)t_j^{10}(y)\, dy},$$

$$T_j(x) := \frac{\displaystyle\int_{-1}^{x} t_j^9(y)\, dy}{\displaystyle\int_{-1}^{1} t_j^9(y)\, dy}$$

are algebraic polynomials of degree not exceeding $40n$. (The polynomials Q_j, $\bar{Q}_j$, and T_j are well defined because the denominators in the expressions of their definitions are never zeros.)

For brevity, denote

$$\psi_j := \frac{h_j}{|x - x_j| + h_j}.$$

To emphasize the dependence on n we use a double index. For example, $x_{j,n}$, $h_{j,n}$, $I_{j,n}$, etc.

$L(x, f; a_0, a_1, \ldots, a_k)$ denotes the Lagrangean polynomial, of degree not exceeding k, which interpolates a function $f(x)$ at the points $a_0, a_1, \ldots, a_k$. We also denote

$$L(x, f; x_i) := L(x, f; x_i, x_{i-1}, x_{i-2})$$

and

$$\tilde{L}(x, f; [a, b]) := L\left(x, f; a, \frac{a+b}{2}, b\right).$$

$$\chi[a, b](x) := \begin{cases} 1, & x \in [a, b], \\ 0, & \text{otherwise.} \end{cases}$$

C are positive absolute constants which are not necessarily the same even when they occur on the same line. $C_0(\mu)$ denote constants which depend on μ only and remain fixed (for certain values of μ) throughout the paper. (Thus $C_0(9)$, for example, denotes the absolute constant which corresponds to $\mu = 9$.) Without further mentioning the inequalities $h_{j\pm 1} < 3h_j$; $\Delta_n(x) < h_j < 5\Delta_n(x)$ for $x \in I_j$ are used.

3. Auxiliary Results

Proposition 1 (see [9] and [10], for example). *The following inequalities hold*:

(21)
$$\min\{(x - x_j^\circ)^{-2}, (x - \bar{x}_j)^{-2}\} \le t_j(x) \le \max\{(x - x_j^\circ)^{-2}, (x - \bar{x}_j)^{-2}\}, \qquad x \in I,$$

(22)
$$t_j(x) \le 10^3 h_j^{-2}, \qquad x \in I,$$

(23)
$$x_j^\circ - x_j > \frac{\bar{x}_j - x_j}{2} > \tfrac{1}{4}h_j, \qquad x_{j-1} - \bar{x}_j > \tfrac{1}{4}h_j, \qquad \bar{x}_j - x_j^\circ \le \tfrac{3}{8}h_j \qquad \text{if } j \le \frac{n}{2},$$

(24)
$$x_{j-1} - x_j^\circ > \frac{x_{j-1} - \bar{x}_j}{2} > \tfrac{1}{4}h_j, \qquad \bar{x}_j - x_j > \tfrac{1}{4}h_j, \qquad x_j^\circ - \bar{x}_j \le \tfrac{3}{8}h_j \qquad \text{if } j > \frac{n}{2},$$

(25)
$$\max\{(x - x_j^\circ)^{-2}, (x - \bar{x}_j)^{-2}\} \le 64(|x - x_j| + h_j)^{-2}, \qquad x \notin I_j,$$

and

(26)
$$(|x - x_j| + h_j)^{-2} \le t_j(x) \le 4 \cdot 10^3(|x - x_j| + h_j)^{-2}, \qquad x \in I.$$

Proposition 1 can be verified by simple calculations using the definitions of the points x_j, x_j°, $\bar{x}_j$ and properties of the trigonometric functions sin and cos.

The following proposition permits us to define the polynomials $Q_j(x)$, $\bar{Q}_j(x)$, and $T_j(x)$ (note that in these definitions we need only $0 \le \xi, \zeta \le 1$).

Proposition 2. *The following inequalities hold:*

$$C_0(\mu)^{-1} h_j^{-2\mu+\xi+\zeta+1} \le \Pi_j(\xi, \zeta, \mu)$$

$$:= \int_{-1}^{1} (y - x_j)^\xi (x_{j-1} - y)^\zeta t_j^\mu(y)\, dy \le C_0(\mu) h_j^{-2\mu+\xi+\zeta+1},$$

where ξ, ζ, and μ are integers satisfying $\xi \ge 0$, $\zeta \ge 0$, and

$$\mu \ge \max\{3 \max\{\xi, \zeta\} + 4, 9\}.$$

Proof (see [10]). These estimates are proved for $j \le n/2$. For $j > n/2$ the proof is analogous with the only difference that instead of (23) inequalities (24) should be used. We write

$$\Pi_j(\xi, \zeta, \mu) = \left\{ \int_{-1}^{x_j} + \int_{x_j}^{x_{j-1}} + \int_{x_{j-1}}^{1} \right\} (y - x_j)^\xi (x_{j-1} - y)^\zeta t_j^\mu(y)\, dy =: \vartheta_1 + \vartheta_2 + \vartheta_3.$$

Now using the estimates (22) and (21) we get

$$\vartheta_2 \le (x_{j-1} - x_j)^{\xi+\zeta+1} 10^{3\mu} h_j^{-2\mu} = 10^{3\mu} h_j^{-2\mu+\xi+\zeta+1}$$

and

$$\vartheta_2 \ge \int_{x_j^\circ}^{\bar{x}_j} (y - x_j)^\xi (x_{j-1} - y)^\zeta \min\{(y - x_j^\circ)^{-2\mu}, (y - \bar{x}_j)^{-2\mu}\}\, dy$$

$$\ge 2(x_j^\circ - x_j)^\xi (x_{j-1} - \bar{x}_j)^\zeta \int_{x_j^\circ}^{(\bar{x}_j + x_j^\circ)/2} (y - \bar{x}_j)^{-2\mu}\, dy$$

$$= \frac{2}{2\mu - 1} (2^{2\mu-1} - 1)(x_j^\circ - x_j)^\xi (x_{j-1} - \bar{x}_j)^\zeta (\bar{x}_j - x_j^\circ)^{-2\mu+1},$$

respectively.

Similarly, using (21) we have

$$|\vartheta_1| \le \int_{-1}^{x_j} (x_j - y)^\xi (x_{j-1} - y)^\zeta t_j^\mu(y)\, dy$$

$$\le \int_{-\infty}^{x_j} (x_j^\circ - y)^\xi 4^\zeta (x_j^\circ - y)^\zeta (x_j^\circ - y)^{-2\mu}\, dy$$

$$= \frac{4^\zeta}{2\mu - \xi - \zeta - 1} (x_j^\circ - x_j)^{-2\mu+\xi+\zeta+1}$$

and

$$\begin{aligned}|\vartheta_3| &\le \int_{x_{j-1}}^{1} (y - x_j)^{\xi}(y - x_{j-1})^{\zeta} t_j^{\mu}(y)\, dy \\ &\le \int_{x_{j-1}}^{\infty} 4^{\xi}(y - \bar{x}_j)^{\xi}(y - \bar{x}_j)^{\zeta}(y - \bar{x}_j)^{-2\mu}\, dy \\ &= \frac{4^{\xi}}{2\mu - \xi - \zeta - 1}(x_{j-1} - \bar{x}_j)^{-2\mu+\xi+\zeta+1}.\end{aligned}$$

Now using (23) we have

$$\begin{aligned}\Pi_j(\xi, \zeta, \mu) &= \vartheta_1 + \vartheta_2 + \vartheta_3 \\ &\le \left(\frac{4^{2\mu-\xi-1} + 4^{2\mu-\zeta-1}}{2\mu - \xi - \zeta - 1} + 10^{3\mu}\right) h_j^{-2\mu+\xi+\zeta+1} \\ &\le 10^{3\mu+1} h_j^{-2\mu+\xi+\zeta+1}.\end{aligned}$$

Finally the estimates in the other direction are the following:

$$\begin{aligned}\Pi_j(\xi, \zeta, \mu) &= \vartheta_1 + \vartheta_2 + \vartheta_3 \\ &\ge \left(-\frac{4^{2\mu-\xi-1} + 4^{2\mu-\zeta-1}}{2\mu - \xi - \zeta - 1} + \frac{2}{2\mu - 1}(2^{2\mu-1} - 1)4^{-\xi-\zeta}(\tfrac{8}{3})^{2\mu-1}\right) h_j^{-2\mu+\xi+\zeta+1} \\ &\ge \mu^{-1} 2^{4\mu-2\xi-2\zeta-3}((\tfrac{4}{3})^{2\mu-1} - 2^{2\xi+1} - 2^{2\zeta+1}) h_j^{-2\mu+\xi+\zeta+1} \\ &\ge h_j^{-2\mu+\xi+\zeta+1}.\end{aligned}$$

Thus the proposition is proved. ■

Lemma 1. *The following inequalities hold:*

$$0 \le T_j'(x) \le C\psi_j^{18} h_j^{-1}, \tag{27}$$

$$|\chi_j(x) - T_j(x)| \le C\psi_j^{17}, \tag{28}$$

$$1 - x_{j-1} < \int_{-1}^{1} T_j(y)\, dy < 1 - x_j, \tag{29}$$

$$|Q_j'(x)| \le C\psi_j^{19} h_j^{-1}, \tag{30}$$

$$0 \le \chi_j(x) - Q_j(x) \le C\psi_j^{18}, \tag{31}$$

$$1 - x_{j-1} < \int_{-1}^{1} Q_j(y)\, dy < 1 - x_j, \tag{32}$$

$$|\bar{Q}_j'(x)| \le C\psi_j^{19} h_j^{-1}, \tag{33}$$

$$0 \le \bar{Q}_j(x) - \chi_{j-1}(x) \le C\psi_j^{18}, \tag{34}$$

and

$$(35) \qquad 1 - x_{j-1} < \int_{-1}^{1} \bar{Q}_j(y)\,dy < 1 - x_j,$$

where $x \in I$ and $\chi_j(x) := \chi[x_j, 1](x)$.

Proof. Let us note that $Q_j'(x) < 0, x < x_j$; $Q_j'(x) > 0, x > x_j$; $Q_j(-1) = 0$; $Q_j(1) = 1$ and $\bar{Q}_j'(x) > 0$, $x < x_{j-1}$; $\bar{Q}_j'(x) < 0$, $x > x_{j-1}$; $\bar{Q}_j(-1) = 0$; $\bar{Q}_j(1) = 1$. This yields $Q_j(x) \le \chi_j(x)$ and $\chi_{j-1}(x) \le \bar{Q}_j(x)$, which are the left-hand side inequalities in (31) and (34), respectively. Now taking integrals of both parts of these inequalities we get the left-hand side inequality in (35) and the right-hand side one in (32). The other inequalities could be verified by simple calculation with the use of Proposition 1. These proofs are either given in [10] and [11] (see also [4] and [9]), or the method of proof is the same. However, because of the importance of this lemma in our considerations the complete proof is given.

First, using integration by parts we get the following identities:

$$\begin{aligned}
\int_{-1}^{1} T_j(y)\,dy < 1 - x_j \quad &\Leftrightarrow \quad \int_{-1}^{1}\int_{-1}^{x} t_j^9(y)\,dy\,dx < (1 - x_j)\int_{-1}^{1} t_j^9(y)\,dy \\
&\Leftrightarrow \quad \int_{-1}^{1} (1 - y)t_j^9(y)\,dy < (1 - x_j)\int_{-1}^{1} t_j^9(y)\,dy \\
&\Leftrightarrow \quad \Pi_j(1, 0, 9) = \int_{-1}^{1} (y - x_j)t_j^9(y)\,dy > 0,
\end{aligned}$$

and analogously

$$\int_{-1}^{1} T_j(y)\,dy > 1 - x_{j-1} \quad \Leftrightarrow \quad \Pi_j(0, 1, 9) = \int_{-1}^{1} (x_{j-1} - y)t_j^9(y)\,dy > 0.$$

Together with Proposition 2 this yields (29).

Similarly,

$$\begin{aligned}
&\int_{-1}^{1} Q_j(y)\,dy > 1 - x_{j-1} \\
&\Leftrightarrow \quad \int_{-1}^{1}\int_{-1}^{x} (y - x_j)t_j^{10}(y)\,dy\,dx > (1 - x_{j-1})\int_{-1}^{1} (y - x_j)t_j^{10}(y)\,dy \\
&\Leftrightarrow \quad \int_{-1}^{1} (1 - y)(y - x_j)t_j^{10}(y)\,dy > (1 - x_{j-1})\int_{-1}^{1} (y - x_j)t_j^{10}(y)\,dy \\
&\Leftrightarrow \quad \Pi_j(1, 1, 10) = \int_{-1}^{1} (y - x_j)(x_{j-1} - y)t_j^{10}(y)\,dy > 0
\end{aligned}$$

and

$$\int_{-1}^{1} \bar{Q}_j(y)\,dy < 1 - x_j$$

$$\Leftrightarrow \quad \int_{-1}^{1}\int_{-1}^{x} (x_{j-1} - y)t_j^{10}(y)\,dy\,dx < (1 - x_j)\int_{-1}^{1} (x_{j-1} - y)t_j^{10}(y)\,dy$$

$$\Leftrightarrow \quad \int_{-1}^{1} (1 - y)(x_{j-1} - y)t_j^{10}(y)\,dy\,dx < (1 - x_j)\int_{-1}^{1} (x_{j-1} - y)t_j^{10}(y)\,dy$$

$$\Leftrightarrow \quad \Pi_j(1, 1, 10) = \int_{-1}^{1} (y - x_j)(x_{j-1} - y)t_j^{10}(y)\,dy > 0.$$

Together with Proposition 2 this implies the left-hand side inequality in (32) and the right-hand side one in (35).

Estimates (27), (30), and (33) are the consequence of (26) and Proposition 2 as, for any $x \in I$, we have

$$\begin{aligned} 0 \le T_j'(x) &\le Ch_j^{17}t_j^9(x) \\ &\le C \cdot 4^9 \cdot 10^{27}h_j^{17}(|x - x_j| + h_j)^{-18} \\ &\le C\psi_j^{18}h_j^{-1}, \end{aligned}$$

$$\begin{aligned} |Q_j'(x)| &\le Ch_j^{18}|x - x_j|t_j^{10}(x) \\ &\le C \cdot 4^{10} \cdot 10^{30}h_j^{18}(|x - x_j| + h_j)^{-19} \\ &\le C\psi_j^{19}h_j^{-1}, \end{aligned}$$

and, similarly,

$$\begin{aligned} |\bar{Q}_j'(x)| &\le Ch_j^{18}|x_{j-1} - x|t_j^{10}(x) \\ &\le C \cdot 4^{10} \cdot 10^{30}h_j^{18}(|x - x_j| + h_j)^{-19} \\ &\le C\psi_j^{19}h_j^{-1}. \end{aligned}$$

Now let us prove the remaining inequalities which are (28) and the right-hand side inequalities in (31) and (34). First, let $x < x_j$. Then with (27), (30), and (33) we have

$$\begin{aligned} |\chi_j(x) - T_j(x)| = |T_j(x)| &= \left|\int_{-1}^{x} T_j'(y)\,dy\right| \\ &\le Ch_j^{17}\int_{-\infty}^{x} (|y - x_j| + h_j)^{-18}\,dy \le C\psi_j^{17}, \end{aligned}$$

$$\begin{aligned} \chi_j(x) - Q_j(x) = -Q_j(x) &= -\int_{-1}^{x} Q_j'(y)\,dy \\ &\le Ch_j^{18}\int_{-\infty}^{x} (|y - x_j| + h_j)^{-19}\,dy \le C\psi_j^{18} \end{aligned}$$

and

$$\bar{Q}_j(x) - \chi_{j-1}(x) = \bar{Q}_j(x) = \int_{-1}^{x} \bar{Q}_j'(y)\,dy$$
$$\leq Ch_j^{18} \int_{-\infty}^{x} (|y - x_j| + h_j)^{-19}\,dy \leq C\psi_j^{18},$$

respectively.

For $x \geq x_j$, similarly, we have

$$|\chi_j(x) - T_j(x)| = |1 - T_j(x)| = \left| \int_{x}^{1} T_j'(y)\,dy \right|$$
$$\leq Ch_j^{17} \int_{x}^{\infty} (|y - x_j| + h_j)^{-18}\,dy \leq C\psi_j^{17},$$

$$\chi_j(x) - Q_j(x) = 1 - Q_j(x) = \int_{x}^{1} Q_j'(y)\,dy$$
$$\leq Ch_j^{18} \int_{x}^{\infty} (|y - x_j| + h_j)^{-19}\,dy \leq C\psi_j^{18},$$

and

$$\bar{Q}_j(x) - \chi_{j-1}(x) = \bar{Q}_j(x) - 1 + \chi[x_j, x_{j-1}]$$
$$\leq -\int_{x}^{1} \bar{Q}_j'(y)\,dy + C\psi_j^{18}$$
$$\leq Ch_j^{18} \int_{x}^{\infty} (|y - x_j| + h_j)^{-19}\,dy + C\psi_j^{18} \leq C\psi_j^{18}.$$

This completes the proof of inequalities (28), (31), and (34). Thus the lemma is proved. ■

It follows from inequalities (29), (32), and (35) that $\alpha, \beta, \gamma \in [0, 1]$ can be chosen so that for polynomials

$$\delta_j(x) := \int_{-1}^{x} (\alpha Q_j(y) + (1 - \alpha)Q_{j+1}(y))\,dy,$$

$$\bar{\delta}_j(x) := \int_{-1}^{x} (\beta \bar{Q}_j(y) + (1 - \beta)\bar{Q}_{j+1}(y))\,dy,$$

and

$$\sigma_j(x) := \int_{-1}^{x} (\gamma T_j(y) + (1 - \gamma)T_{j+1}(y))\,dy,$$

the following equations occur:

$$\delta_j(1) = \bar{\delta}_j(1) = \sigma_j(1) = 1 - x_j.$$

Let $R_j(x) := (x - x_j)\delta_j(x)$ and $\bar{R}_j(x) := (x - x_j)\bar{\delta}_j(x)$. Polynomials $R_j(x)$, $\bar{R}_j(x)$, and $\sigma_j(x)$ and their derivatives give sufficient approximation of the truncated power functions $\chi_j(x)$, $(x - x_j)_+$, and $(x - x_j)_+^2$ (definitions of the truncated power functions are given in Lemma 2). Taking into account the fact that analytic representation of any spline of degree 2 contains only these functions, this enables us to obtain a good approximation of any spline of second degree by polynomials with controlled derivatives (see Section 4).

Lemma 2. *The following inequalities hold*:

(36) $$|\delta_j''(x)| \le C\psi_j^{19} h_j^{-1},$$

(37) $$|\bar{\delta}_j''(x)| \le C\psi_j^{19} h_j^{-1},$$

(38) $$0 \le \sigma_j''(x) \le C\psi_j^{18} h_j^{-1},$$

(39) $$|\delta_j'(x) - \chi_j(x)| \le C\psi_j^{18},$$

(40) $$|\bar{\delta}_j'(x) - \chi_j(x)| \le C\psi_j^{18},$$

(41) $$|\sigma_j'(x) - \chi_j(x)| \le C\psi_j^{17},$$

(42) $$\delta_j'(x) \le \chi_{j+1}(x),$$

$$\bar{\delta}_j'(x) \ge \chi_{j-1}(x),$$

(43) $$|(x - x_j)_+ - \delta_j(x)| \le C\psi_j^{17} h_j,$$

(44) $$|(x - x_j)_+ - \bar{\delta}_j(x)| \le C\psi_j^{17} h_j,$$

(45) $$|(x - x_j)_+ - \sigma_j(x)| \le C\psi_j^{16} h_j,$$

(46) $$|(x - x_j)_+^2 - R_j(x)| \le C\psi_j^{16} h_j^2,$$

(47) $$|(x - x_j)_+^2 - \bar{R}_j(x)| \le C\psi_j^{16} h_j^2,$$

(48) $$|2(x - x_j)_+ - R_j'(x)| \le C\psi_j^{17} h_j,$$

(49) $$|2(x - x_j)_+ - \bar{R}_j'(x)| \le C\psi_j^{17} h_j,$$

(50) $$|R_j''(x) - 2\chi_j(x)| \le C\psi_j^{18},$$

and

(51) $$|\bar{R}_j''(x) - 2\chi_j(x)| \le C\psi_j^{18},$$

where $x \in I$ and $(x - x_j)_+^k := (x - x_j)^k \chi[x_j, 1](x)$.

Proof. First, (42) is a consequence of the left-hand side inequalities in (31) and (34) as

$$\begin{aligned}\delta_j'(x) &= \alpha Q_j(x) + (1-\alpha)Q_{j+1}(x)\\ &\le \alpha\chi_j(x) + (1-\alpha)\chi_{j+1}(x)\\ &\le \chi_{j+1}(x)\end{aligned}$$

and

$$\begin{aligned}\bar\delta_j'(x) &= \beta \bar Q_j(x) + (1-\beta)\bar Q_{j+1}(x)\\ &\ge \beta\chi_{j-1}(x) + (1-\beta)\chi_j(x)\\ &\ge \chi_{j-1}(x).\end{aligned}$$

Inequalities (36)–(38) are immediate consequences of (30), (33), and (27) and the observation that $\psi_{j\pm1} < 18\psi_j$. Also, since $\psi_j \sim C$ if $x \in I_j$, then inequalities (39)–(41) are consequences of (31), (34), and (28) as

$$\begin{aligned}|\delta_j'(x) - \chi_j(x)| &\le |Q_j(x) - \chi_j(x)| + |Q_{j+1}(x) - \chi_j(x)|\\ &\le |Q_j(x) - \chi_j(x)| + |Q_{j+1}(x) - \chi_{j+1}(x)| + \chi[x_{j+1}, x_j](x)\\ &\le C\psi_j^{18}.\end{aligned}$$

The proofs of (43)–(45) are similar so we only prove (43). For $x \le x_j$ we have

$$\begin{aligned}|(x-x_j)_+ - \delta_j(x)| &\le \int_{-1}^{x} (\alpha|Q_j(y) - \chi_j(y)| + (1-\alpha)|Q_{j+1}(y) - \chi_j(y)|)\,dy\\ &\le C\int_{-1}^{x}\left(\frac{h_j}{|y-x_j| + h_j}\right)^{18} dy \le Ch_j^{18}(|x-x_j| + h_j)^{-17}.\end{aligned}$$

For $x \ge x_j$ we have the estimate

$$\begin{aligned}|(x-x_j)_+ - \delta_j(x)| &= |(\delta_j(1) - \delta_j(x)) - ((1-x_j) - (x-x_j))|\\ &= |\delta_j(1) - \delta_j(x) - (1-x)|\\ &= \left|\int_x^1 (\alpha Q_j(y) + (1-\alpha)Q_{j+1}(y) - 1)\,dy\right|\\ &\le \int_x^1 (\alpha|Q_j(y) - \chi_j(y)| + (1-\alpha)|Q_{j+1}(y) - \chi_{j+1}(y)|)\,dy\\ &\le C\int_x^{\infty}\left(\frac{h_j}{|y-x_j| + h_j}\right)^{18} dy \le Ch_j\psi_j^{17}.\end{aligned}$$

Thus inequality (43) is proved.

Inequalities (46) and (47) follow immediately from (43) and (44), respectively, and inequalities (48) and (49) are consequences of (43), (39), (44), and (40). Finally (50) and (51) follow from (36), (39), (37), and (40). The proof of the lemma is now complete. ■

4. Proof of Theorem 1

Following the ideas of [1] we construct a convex spline $S(x)$ of degree ≤ 2 which sufficiently approximates the convex function $f = f(x)$, $f \in C(I)$, that is,

$$|f(x) - S(x)| \leq C\omega_3(f, \Delta_n(x)), \qquad x \in I.$$

Then we approximate $S(x)$ by a convex algebraic polynomial so that

$$|S(x) - p_n(x)| \leq C\omega_3(f, \Delta_n(x)), \qquad x \in I.$$

This proves the estimate (4).

Construction of the Convex Spline

Let

$$S(x) := \max\{L(x, f; x_j), L(x, f; x_{j+1})\}, \qquad x \in I_j, \quad j = \overline{2, n-1},$$

$$S(x) := L(x, f; x_2), \qquad x \in I_1,$$

and

$$S(x) := L(x, f; x_n), \qquad x \in I_n.$$

It is easy to see that $S(x)$ is a convex spline of degree ≤ 2 with knots x_j, $j = \overline{0, n}$.

Now we consider the index j to be fixed and denote

$$a_v := L'(x_j, f; x_v), \qquad v = j, j+1, j+2.$$

Let us call a knot x_j, $j = \overline{2, n-2}$, "*a knot of type* **I**" if

$$a_{j+1} \leq a_{j+2}, \qquad a_{j+1} < a_j. \tag{52}$$

That is,

$$S(x) = L(x, f; x_v), \qquad x \in [x_v, x_{v-1}], \quad v = j+1, j.$$

Note that inequalities (52) are equivalent to the following ones:

$$[x_j, x_{j-1}, x_{j-2}; f] < [x_{j+1}, x_j, x_{j-1}; f] \leq [x_{j+2}, x_{j+1}, x_j; f],$$

where square brackets denote the divided difference of f.

A knot x_j is "*a knot of type* **II**" if

$$a_{j+2} < a_{j+1}, \qquad a_j \leq a_{j+1}, \tag{53}$$

which is equivalent to

$$[x_{j+2}, x_{j+1}, x_j; f] < [x_{j+1}, x_j, x_{j-1}; f] \leq [x_j, x_{j-1}, x_{j-2}; f].$$

In this case

$$S(x) = L(x, f; x_{v+1}), \qquad x \in [x_v, x_{v-1}], \quad v = j+1, j.$$

Let x_j be "*a knot of type* **III**" if

$$a_{j+2} < a_{j+1} < a_j \tag{54}$$

or equivalently

$$[x_{j+2}, x_{j+1}, x_j; f] < [x_{j+1}, x_j, x_{j-1}; f]$$

and

$$[x_j, x_{j-1}, x_{j-2}; f] < [x_{j+1}, x_j, x_{j-1}; f].$$

In this case $S(x) = L(x, f; x_{j+2})$, $x \in [x_{j+1}, x_j]$, and $S(x) = L(x, f; x_j)$, $x \in [x_j, x_{j-1}]$.

Let the knots, which are not knots of type **I**, **II**, or **III**, be "*knots of type* **IV**." It is not difficult to see that if x_j is a knot of type **IV**, then

$$S(x) = L(x, f; x_{j+1}), \qquad x \in [x_{j+1}, x_{j-1}]. \tag{55}$$

Let x_1 be a knot of type **II** if $a_3 < a_2$, x_{n-1} is a knot of type **I** if $a_n < a_{n-1}$, otherwise they are knots of type **IV**.

From (52)–(55) it follows that the spline $S(x)$ has defect 2 in knots **I**, **II**, and **III** (i.e., the first derivative of the continuous spline $S(x)$ does not exist in these knots) and does not have it in knots of type **IV** (S, S', and S'' exist and are continuous in these knots). Taking this into consideration, we get the following analytic representation of the spline $S(x)$ with truncated power functions $(x - x_j)_+$ and $(x - x_j)_+^2$ (for an analytic representation of splines see, for example, Section 2.3 of [5]):

$$\begin{aligned} S(x) = f(-1) + A_0(x+1) &+ [x_n, x_{n-1}, x_{n-2}; f](x+1)^2 \\ &+ \sum_{\substack{i=\overline{2,n-1} \\ x_i \in \mathbf{I} \cup \mathbf{III}}} A_i\{(x_{i-1} - x_i)(x - x_i)_+ - (x - x_i)_+^2\} \\ &+ \sum_{\substack{i=\overline{1,n-2} \\ x_i \in \mathbf{II} \cup \mathbf{III}}} B_i\{(x_i - x_{i+1})(x - x_i)_+ + (x - x_i)_+^2\}, \end{aligned}$$

where

$$\begin{aligned} A_0 &:= [x_n, x_{n-1}; f] - [x_{n-1}, x_{n-2}; f] + [x_{n-2}, x_n; f], \\ A_i &:= [x_{i+1}, x_i, x_{i-1}; f] - [x_i, x_{i-1}, x_{i-2}; f] \qquad \text{for} \quad i = \overline{2, n-1}, \end{aligned}$$

and

$$B_i := -A_{i+1} \qquad \text{for} \quad i = \overline{1, n-2}.$$

Note that $A_i > 0$ for $x_i \in \mathbf{I} \cup \mathbf{III}$ (i.e., if knot x_i is a knot of type **I** or **III**) and $B_i > 0$ if $x_i \in \mathbf{II} \cup \mathbf{III}$.

Now let us estimate the value $|f(x) - S(x)|$, $x \in I$. For this we need the well-known Whitney inequality (see, for example, Section 2.1 of [8]):

$$|g(x) - L(x, g; a_0, a_1, \ldots, a_k)| \le 3\omega_{k+1}\left(g, \frac{a_k - a_0}{k+1}, [a_0, a_k]\right), \tag{56}$$

where $g \in C([a, b])$, $a_{i+1} - a_i = a_i - a_{i-1}$, $i = \overline{1, k-1}$, and $x \in [a_0, a_k]$.

For $x \in [x_i, x_{i-2}]$ we have

$$(57)\quad |f(x) - L(x, f; x_i)| = |f(x) - \tilde{L}(x, f; [x_i, x_{i-2}]) - L(x, f - \tilde{L}; x_i)|$$
$$\le \|f - \tilde{L}\|_{C[x_i, x_{i-2}]}\left(1 + \left|\frac{(x - x_i)(x - x_{i-2})}{(x_{i-1} - x_i)(x_{i-1} - x_{i-2})}\right|\right)$$
$$\le 3\omega_3\left(f, \frac{h_i + h_{i-1}}{3}, [x_i, x_{i-2}]\right)\left(1 + \frac{(h_i + h_{i-1})^2}{4h_i h_{i-1}}\right)$$
$$\le 40\omega_3(f, 7\Delta_n(x)) \le 10^5\omega_3(f, \Delta_n(x)).$$

This yields

$$(58)\qquad |f(x) - S(x)| \le C\omega_3(f, \Delta_n(x)), \qquad x \in I.$$

Construction of the Convex Polynomial

Let us fix n, denote $n_1 := Mn$, where an absolute constant M is an integer and will be chosen later, and choose i_1 so that $x_{i_1, n_1} = x_{i,n}$.

Using the analytic representation of $S(x)$ and also the approximation of the truncated power functions $(x - x_i)_+$ and $(x - x_i)_+^2$ given in Lemma 2, we write the following algebraic polynomial of degree $\le 50Mn$:

$$p_n(x) := f(-1) + A_0(x + 1) + [x_n, x_{n-1}, x_{n-2}; f](x + 1)^2$$
$$+ \sum_{\substack{i=\overline{2,n-1} \\ x_i \in \mathbf{I} \cup \mathbf{III}}} A_i\{(x_{i-1} - x_i)\sigma_{i_1, n_1}(x) - R_{i_1, n_1}(x)\}$$
$$+ \sum_{\substack{i=\overline{1,n-2} \\ x_i \in \mathbf{II} \cup \mathbf{III}}} B_i\{(x_i - x_{i+1})\sigma_{i_1, n_1}(x) + \bar{R}_{i_1, n_1}(x)\}.$$

(The distance between $S(x)$ and this polynomial is estimated in inequality (64) below.)

Now we show that it is possible to choose M so that this polynomial will be convex on I. For this it is enough to choose M so that the following inequalities hold:

$$(59)\qquad (x_{i-1} - x_i)\sigma''_{i_1, n_1}(x) - R''_{i_1, n_1}(x) \ge -2\chi_i(x), \qquad x \in I;$$
$$(x_i - x_{i+1})\sigma''_{i_1, n_1}(x) + \bar{R}''_{i_1, n_1}(x) \ge 2\chi_i(x), \qquad x \in I.$$

Indeed, using (59) and taking into account inequalities $A_i > 0$ for $x_i \in \mathbf{I} \cup \mathbf{III}$ and $B_i > 0$ for $x_i \in \mathbf{II} \cup \mathbf{III}$, we have, for $x \in I \setminus \{x_1, \ldots, x_{n-1}\}$,

$$p''_n(x) \ge 2[x_n, x_{n-1}, x_{n-2}; f] + \sum_{\substack{i=\overline{2,n-1} \\ x_i \in \mathbf{I} \cup \mathbf{III}}} A_i\{-2\chi_i(x)\} + \sum_{\substack{i=\overline{1,n-2} \\ x_i \in \mathbf{II} \cup \mathbf{III}}} 2B_i\chi_i(x)$$
$$= S''(x).$$

As $S(x)$ is convex on each interval I_j, $j = \overline{1, n}$, then $S''(x) \ge 0$ for $x \in (x_i, x_{i-1})$,

$i = \overline{1, n}$ and hence $p_n''(x) \geq 0$ for $x \in I \backslash \{x_1, \ldots, x_{n-1}\}$. As $p_n(x)$ is a polynomial, i.e., it has a continuous second derivative, then $p_n''(x) \geq 0$ for $x \in I$, and therefore $p_n \in \Delta^2$.

Thus it is sufficient to prove (59). Inequalities (59) are consequences of the following estimates:

$$\text{(60)} \quad \min\{T'_{i_1,n_1}(x), T'_{i_1+1,n_1}(x)\} \min\{h_i, h_{i+1}\} > 4 \qquad \text{for} \quad x \in I_{i_1+1,n_1} \cup I_{i_1,n_1}$$

and

$$\text{(61)} \quad \min\{T'_{i_1,n_1}(x), T'_{i_1+1,n_1}(x)\} \min\{h_i, h_{i+1}\} \geq 2|x - x_i| \max\{|Q'_{i_1,n_1}(x)|, |Q'_{i_1+1,n_1}(x)|, |\bar{Q}'_{i_1,n_1}(x)|, |\bar{Q}'_{i_1+1,n_1}(x)|\}$$

for every $x \in I$.

Indeed, suppose that (60) and (61) are true. Then for any $x \in I$ together with (42) we have, for $x \notin I_{i_1+1,n_1} \cup I_{i_1,n_1}$,

$$\begin{aligned}
&(x_{i-1} - x_i)\sigma''_{i_1,n_1}(x) - R''_{i_1,n_1}(x) \\
&\quad = h_i(\gamma T'_{i_1,n_1}(x) + (1-\gamma)T'_{i_1+1,n_1}(x)) \\
&\qquad - (x - x_i)(\alpha Q'_{i_1,n_1}(x) + (1-\alpha)Q'_{i_1+1,n_1}(x)) - 2\delta'_{i_1,n_1}(x) \\
&\quad \geq h_i \min\{T'_{i_1,n_1}(x), T'_{i_1+1,n_1}(x)\} \\
&\qquad - |x - x_i| \max\{|Q'_{i_1,n_1}(x)|, |Q'_{i_1+1,n_1}(x)|\} - 2\chi[x_{i_1+1,n_1}, 1](x) \\
&\quad \geq -2\chi[x_{i_1+1,n_1}, 1](x) \\
&\quad = -2\chi_i(x).
\end{aligned}$$

For $x \in I_{i_1+1,n_1} \cup I_{i_1,n_1}$ taking into account (60) and (61) we have the following estimate:

$$\begin{aligned}
(x_{i-1} - x_i)\sigma''_{i_1,n_1}(x) - R''_{i_1,n_1}(x) &\geq \frac{h_i}{2} \min\{T'_{i_1,n_1}(x), T'_{i_1+1,n_1}(x)\} - 2\chi[x_{i_1+1,n_1}, 1](x) \\
&\geq 2 - 2\chi[x_{i_1+1,n_1}, 1](x) \\
&\geq -2\chi_i(x).
\end{aligned}$$

This proves the first estimate in (59). Considerations for the proof of the second estimate in (59) are analogous.

Thus our problem is reduced to the following one: find such an integer constant M that, for $n_1 := Mn$, inequalities (60) and (61) are valid.

It follows from (21) that, for $x \in I_{i_1+1,n_1} \cup I_{i_1,n_1}$,

$$\begin{aligned}
t_{i_1,n_1}(x) &> \min\{(x - x^\circ_{i_1,n_1})^{-2}, (x - \bar{x}_{i_1,n_1})^{-2}\} \\
&> (h_{i_1+1,n_1} + h_{i_1,n_1})^{-2} > \frac{h^{-2}_{i_1,n_1}}{16}.
\end{aligned}$$

From Proposition 2 ($\xi = 0$, $\zeta = 0$, and $\mu = 9$) we have

$$\int_{-1}^{1} t^9_{i_1,n_1}(y)\, dy < C_0(9) h^{-17}_{i_1,n_1}.$$

Thus we have, for example, the following estimate:

$$T'_{i_1,n_1}(x)h_i = \frac{t^9_{i_1,n_1}(x)}{\displaystyle\int_{-1}^{1} t^9_{i_1,n_1}(y)\,dy}\,h_i > C_0(9)^{-1}16^{-9}\frac{h_i}{h_{i_1,n_1}}.$$

Now it is sufficient to choose the number $n_1 \in N$ so that $h_i \geq 4\cdot 16^9 C_0(9)h_{i_1,n_1}$. This verifies inequality (60).

Using the same idea, Proposition 2, and also inequalities (26) to prove (61) we write, for example,

$$\begin{aligned} &h_i T'_{i_1,n_1}(x) - 2|x - x_i|\,|\bar{Q}'_{i_1,n_1}(x)| \\ &\quad\geq C_0(9)^{-1}h_i h^{17}_{i_1,n_1} t^9_{i_1,n_1}(x) - 2C_0(10)|x - x_i|\,|x_{i_1-1,n_1} - x|\,h^{18}_{i_1,n_1}t^{10}_{i_1,n_1}(x) \\ &\quad\geq C_0(9)^{-1}h_i h^{17}_{i_1,n_1}(|x - x_{i_1,n_1}| + h_{i_1,n_1})^{-18} \\ &\qquad - 2\cdot 4^{10}\cdot 10^{30}C_0(10)h^{18}_{i_1,n_1}(|x - x_{i_1,n_1}| + h_{i_1,n_1})^{-18} \\ &\quad\geq \psi^{18}_{i_1,n_1}h^{-1}_{i_1,n_1}\{C_0(9)^{-1}h_i - 2\cdot 4^{10}\cdot 10^{30}C_0(10)h_{i_1,n_1}\}. \end{aligned}$$

Thus (61) is verified if the number $n_1 \in N$ is chosen so that

$$h_i \geq 2\cdot 4^{10}\cdot 10^{30}C_0(9)C_0(10)h_{i_1,n_1}.$$

Taking into account that $h_i/h_{i_1,n_1} \geq n_1/5n$ we can conclude that inequalities (60) and (61) are true for $n_1 = [10^{50}C_0(9)C_0(10)]n =: Mn$.

It remains only to estimate $|p_n(x) - S(x)|$. Similarly to (57) using

$$[x_i, x_{i-1}, x_{i-2}, x_{i-3}; f] = \frac{f(x_{i-1}) - L(x_{i-1}, f; x_i)}{(x_{i-1} - x_i)(x_{i-1} - x_{i-2})(x_{i-1} - x_{i-3})}$$

we have

$$[x_i, x_{i-1}, x_{i-2}, x_{i-3}; f] \leq Ch_i^{-3}\omega_3(f, \Delta_n(x_i))$$

and hence

$$|A_i| = |[x_{i+1}, x_i, x_{i-1}, x_{i-2}; f](x_{i-2} - x_{i+1})| \leq Ch_i^{-2}\omega_3(f, \Delta_n(x_i)).$$

It also follows from the last inequality that

$$|A_i| \leq Ch_i^{-1}\omega_2(f', \Delta_n(x_i)) \qquad \text{for} \quad f \in C^1(I)$$

and

$$|A_i| \leq C\omega(f'', \Delta_n(x_i)) \qquad \text{for} \quad f \in C^2(I).$$

Now let us note that the estimate $h_{i_1,n_1} < h_i < (M^2/5)h_{i_1,n_1}$ implies that

$$\psi_{i_1,n_1} < 5M^2\psi_i. \tag{62}$$

Using the inequalities (see [9] and [10])

$$\Delta_n^2(y) < 4\Delta_n(x)(|x - y| + \Delta_n(x))$$

and

$$2(|x-y|+\Delta_n(x)) > |x-y|+\Delta_n(y) > \tfrac{1}{2}(|x-y|+\Delta_n(x)), \qquad x \in I, \quad y \in I,$$

and also the properties of the modulus of smoothness we have

$$\begin{aligned}(63) \qquad \omega_3(f, \Delta_n(x_i)) &\le \omega_3(f, 2\sqrt{\Delta_n(x)(|x-x_i|+\Delta_n(x))}) \\ &= \omega_3\left(f, 2\Delta_n(x)\sqrt{\frac{|x-x_i|+\Delta_n(x)}{\Delta_n(x)}}\right) \\ &\le 64\left(\frac{|x-x_i|+\Delta_n(x)}{\Delta_n(x)}\right)^{3/2} \omega_3(f, \Delta_n(x)) \\ &\le 10^6\left(\frac{|x-x_i|+h_i}{h_i}\right)^3 \omega_3(f, \Delta_n(x)).\end{aligned}$$

From (45)–(47), (62), and (63) we have

(64)

$$\begin{aligned}|p_n(x)-S(x)| &\le \sum_{\substack{i=\overline{2,n-1}\\ x_i \in \mathrm{I}\cup\mathrm{III}}} |A_i|\{h_i|(x-x_i)_+ - \sigma_{i_1,n_1}(x)| + |R_{i_1,n_1}(x)-(x-x_i)_+^2|\} \\ &\quad + \sum_{\substack{i=\overline{1,n-2}\\ x_i \in \mathrm{II}\cup\mathrm{III}}} |B_i|\{h_{i+1}|(x-x_i)_+ - \sigma_{i_1,n_1}(x)| + |\bar{R}_{i_1,n_1}(x)-(x-x_i)_+^2|\} \\ &\le C\sum_{i=1}^{n-1} \omega_3(f, \Delta_n(x))\psi_i^{13} \\ &\le C\omega_3(f, \Delta_n(x)).\end{aligned}$$

Inequalities (58) and (64) complete the proof of the estimate (4) as

$$\begin{aligned}|f(x)-p_n(x)| &\le |f(x)-S(x)| + |S(x)-p_n(x)| \\ &\le C\omega_3(f, \Delta_n(x)), \qquad x \in I\end{aligned}$$

To prove (5) the following equations are used:

$$\begin{aligned}S'(x) &= A_0 + 2[x_n, x_{n-1}, x_{n-2}; f](x+1) \\ &\quad + \sum_{\substack{i=\overline{2,n-1}\\ x_i \in \mathrm{I}\cup\mathrm{III}}} A_i\{h_i\chi_i(x) - 2(x-x_i)_+\} \\ &\quad + \sum_{\substack{i=\overline{1,n-2}\\ x_i \in \mathrm{II}\cup\mathrm{III}}} B_i\{h_{i+1}\chi_i(x) + 2(x-x_i)_+\}.\end{aligned}$$

It follows from (41), (48), and (49) that

$$
\begin{aligned}
(65)\quad & |p_n'(x) - S'(x)| \\
&\le \sum_{\substack{i=2,n-1 \\ x_i \in \mathrm{I} \cup \mathrm{III}}} |A_i| \{h_i |\sigma_{i_1,n_1}'(x) - \chi_i(x)| + |R_{i_1,n_1}'(x) - 2(x - x_i)_+|\} \\
&\quad + \sum_{\substack{i=1,n-2 \\ x_i \in \mathrm{II} \cup \mathrm{III}}} |B_i| \{h_{i+1} |\sigma_{i_1,n_1}'(x) - \chi_i(x)| + |\bar{R}_{i_1,n_1}'(x) - 2(x - x_i)_+|\} \\
&\le C \sum_{i=1}^{n-1} \omega_2(f', \Delta_n(x)) \psi_i^{14} \\
&\le C\omega_2(f', \Delta_n(x)).
\end{aligned}
$$

Now (5) follows from the following estimate for $x \in \tilde{I}_j := [x_j, x_{j-2}]$:

$$
(66)\qquad |f'(x) - L'(x, f; x_j)| \le C\omega_2(f', h_j, \tilde{I}_j).
$$

In order to prove inequality (66) (see Lemma 1.4.2 of [11]) let us denote

$$
\bar{L}(x) := f(x_j) + \int_{x_j}^{x} L(u, f', x_j, x_{j-2})\, du
$$

($\bar{L}(x)$ is an algebraic polynomial of degree 2) and note that

$$
f'(x) - L'(x, f; x_j) = f'(x) - \bar{L}'(x) - L'(x, f - \bar{L}; x_j).
$$

The following estimate is a consequence of Whitney's inequality (56):

$$
|f'(x) - L(x, f'; x_j, x_{j-2})| \le C\omega_2(f', h_j, \tilde{I}_j), \qquad x \in \tilde{I}_j.
$$

This implies, for any $x \in \tilde{I}_j$,

$$
\begin{aligned}
|f(x) - \bar{L}(x)| &= \left| \int_{x_j}^{x} f'(u) - L(u, f', x_j, x_{j-2})\, du \right| \\
&\le Ch_j \omega_2(f', h_j, \tilde{I}_j).
\end{aligned}
$$

Now together with the estimate for $x \in \tilde{I}_j$,

$$
|L'(x, f - \bar{L}; x_j)| \le Ch_j^{-1} \|f - \bar{L}\|_{\tilde{I}_j} \le C\omega_2(f', h_j, \tilde{I}_j),
$$

the following inequalities complete the proof of (66):

$$
\begin{aligned}
|f'(x) - L'(x, f; x_j)| &\le |f'(x) - \bar{L}'(x)| + C\omega_2(f', h_j, \tilde{I}_j) \\
&= |f'(x) - L(x, f'; x_j, x_{j-2})| + C\omega_2(f', h_j, \tilde{I}_j) \\
&\le C\omega_2(f', h_j, \tilde{I}_j), \qquad x \in \tilde{I}_j.
\end{aligned}
$$

To prove (6) we use the following equations:

$$
S''(x) = 2[x_n, x_{n-1}, x_{n-2}; f] + \sum_{\substack{i=2,n-1 \\ x_i \in \mathrm{I} \cup \mathrm{III}}} A_i\{-2\chi_i(x)\} + \sum_{\substack{i=1,n-2 \\ x_i \in \mathrm{II} \cup \mathrm{III}}} 2B_i \chi_i(x)
$$

and

$$S''(x) = 2[x_j, x_{j-1}, x_{j-2}; f] \quad \text{or} \quad S''(x) = 2[x_{j+1}, x_j, x_{j-1}; f] \qquad \text{if} \quad x \in I_j.$$

It follows from (38), (50), (51), and (63) that

$$\begin{aligned}
(67)\quad |p_n''(x) - S''(x)| &\le \sum_{\substack{i=\overline{2,n-1}\\ x_i \in \mathrm{I} \cup \mathrm{III}}} |A_i| \{h_i |\sigma''_{i_1,n_1}(x)| + |2\chi_i(x) - R''_{i_1,n_1}(x)|\} \\
&\quad + \sum_{\substack{i=\overline{1,n-2}\\ x_i \in \mathrm{II} \cup \mathrm{III}}} |B_i| \{h_{i+1} |\sigma''_{i_1,n_1}(x)| + |2\chi_i(x) - \bar{R}''_{i_1,n_1}(x)|\} \\
&\le C \sum_{i=1}^{n-1} \omega(f'', \Delta_n(x)) \psi_i^{15} \\
&\le C\omega(f'', \Delta_n(x)).
\end{aligned}$$

Now (6) follows from the following estimate for $x \in I_j$ (see also Lemma 1.4.2 of [11]):

$$\begin{aligned}
(68)\quad & |f''(x) - 2[x_j, x_{j-1}, x_{j-2}; f]| \\
&= \left| 2 \int_0^1 \int_0^{t_1} \{f''(x) - f''(x_j + (x_{j-1} - x_j)t_1 + (x_{j-2} - x_{j-1})t_2)\}\, dt_2\, dt_1 \right| \\
&\le \omega(f'', h_j + h_{j-1}, I_j) \\
&\le C\omega(f'', \Delta_n(x)).
\end{aligned}$$

Thus Theorem 1 is proved for all $n \ge 2$.

5. Proof of Theorem 2

To prove inequalities (9)–(11) it is sufficient to estimate $|f^{(v)}(x) - S^{(v)}(x)|$ and $|p_n^{(v)}(x) - S^{(v)}(x)|$ in terms of the $\bar{\omega}_\phi^{3-v}$ modulus with $v = 0$, 1, and 2, respectively.

First, let us note the following:

> For the interval $[x_i, x_{i-2}]$ we denote $\xi_i := x_i$ if $|x_{i-2}| < |x_i|$ and $\xi_i := x_{i-2}$ otherwise. Then for any $y \in [x_i, x_{i-2}]$ and $0 < h \le n^{-1}$ the inequality $\rho(\xi_i, h) \le \rho(y, h)$ is valid.

If $h = \sqrt{1 - x^2}\tilde{h} + \tilde{h}^2 = \rho(x, \tilde{h})$, then $0 < h \le \Delta_n(x) \Leftrightarrow 0 < \tilde{h} \le n^{-1}$. Using (57) and the above we get, for a fixed $x \in [x_i, x_{i-2}]$,

$$\begin{aligned}
|f(x) - L(x, f; x_i)| &\le 10^5 \omega_3(f, \Delta_n(x); [x_i, x_{i-2}]) \\
&\le 10^5 \omega_3(f, 15\Delta_n(\xi_i); [x_i, x_{i-2}]) \\
&\le C\omega_3(f, \Delta_n(\xi_i); [x_i, x_{i-2}]) \\
&= C \sup_{0 < h \le \Delta_n(\xi_i)} \|\bar{\Delta}_h^3(f, y, [x_i, x_{i-2}])\|_{C[x_i, x_{i-2}]} \\
&\le C \sup_{0 < h \le \Delta_n(\xi_i)} \|\bar{\Delta}_h^3(f, y)\|_{C[x_i, x_{i-2}]} \\
&= C \sup_{0 < \tilde{h} \le n^{-1}} \|\bar{\Delta}^3_{\rho(\xi_i, \tilde{h})}(f, y)\|_{C[x_i, x_{i-2}]} \\
&\le C |\bar{\Delta}^3_{\rho(\xi_i, \tilde{h}_0)}(f, \zeta_i)|
\end{aligned}$$

for some $0 < \tilde{h}_0 \le n^{-1}$ and $\zeta_i \in [x_i, x_{i-2}]$. (Actually, using the compactness argument the last inequality can be replaced by an equality.)

Now using inequality $\rho(\xi_i, \tilde{h}_0) \le \rho(\zeta_i, \tilde{h}_0)$, continuity of $\rho(\zeta_i, h)$, and the fact that $\rho(\zeta_i, h) \to 0$ as $h \to 0$ we can conclude that a number $\tilde{h}_1$, $0 < \tilde{h}_1 \le \tilde{h}_0 \le n^{-1}$, exists such that $\rho(\xi_i, \tilde{h}_0) = \rho(\zeta_i, \tilde{h}_1)$. Thus

$$\begin{aligned}|f(x) - L(x, f; x_i)| &\le C|\bar{\Delta}^3_{\rho(\zeta_i, \tilde{h}_1)}(f, \zeta_i)| \\ &\le C \sup_{0 < \tilde{h} \le n^{-1}} \|\bar{\Delta}^3_{\rho(y, \tilde{h})}(f, y)\|_{C[x_i, x_{i-2}]} \\ &\le C\bar{\omega}^3_\phi(f, n^{-1}).\end{aligned}$$

This implies

$$|f(x) - S(x)| \le C\bar{\omega}^3_\phi(f, n^{-1}), \qquad x \in I.$$

Now we can apply the same considerations as in (64) to estimate $|p_n(x) - S(x)|$. For this we need the estimates of the coefficients A_i which appeared in the constructions of $S(x)$ and $p_n(x)$ in terms of the "nonuniform" moduli $\bar{\omega}^3_\phi$.

Using the same method as above we have the following estimate of $|A_i|$:

$$\begin{aligned}|A_i| &= |[x_{i+1}, x_i, x_{i-1}, x_{i-2}; f](x_{i-2} - x_{i+1})| \\ &= \left| \frac{f(x_i) - L(x_i, f; x_{i+1}, x_{i-1}, x_{i-2})}{(x_i - x_{i+1})(x_i - x_{i-1})(x_i - x_{i-2})}(x_{i-2} - x_{i+1}) \right| \\ &\le Ch_i^{-2}\omega_3(f, \Delta_n(x_i); [x_{i+1}, x_{i-1}]) \\ &\le Ch_i^{-2}\bar{\omega}^3_\phi(f, n^{-1}).\end{aligned}$$

Moreover, if $f \in C^1(I)$, then

$$\begin{aligned}|A_i| &\le Ch_i^{-2}\omega_3(f, \Delta_n(x_i); [x_{i+1}, x_{i-1}]) \\ &\le Ch_i^{-2}(\Delta_n(x_i))\omega_2(f', \Delta_n(x_i); [x_{i+1}, x_{i-1}]) \\ &\le Ch_i^{-1}\bar{\omega}^2_\phi(f', n^{-1}).\end{aligned}$$

Similarly, if $f \in C^2(I)$, then

$$\begin{aligned}|A_i| &\le Ch_i^{-2}(\Delta_n(x_i))^2\omega(f'', \Delta_n(x_i); [x_{i+1}, x_{i-1}]) \\ &\le C\bar{\omega}_\phi(f'', n^{-1}).\end{aligned}$$

Now analogously to (64) we have

$$|p_n(x) - S(x)| \le C\bar{\omega}^3_\phi(f, n^{-1}) \sum_{i=1}^{n-1} \psi_i^{16} \le C\bar{\omega}^3_\phi(f, n^{-1}).$$

This completes the proof of the estimate (9).

Analogously to (65) and (67) in the cases $f \in C^1(I)$ and $f \in C^2(I)$, we have the estimates

$$|p_n'(x) - S'(x)| \le C\bar{\omega}^2_\phi(f', n^{-1}) \sum_{i=1}^{n-1} \psi_i^{17} \le C\bar{\omega}^2_\phi(f', n^{-1})$$

and

$$|p_n''(x) - S''(x)| \le C\bar{\omega}_\phi(f'', n^{-1}) \sum_{i=1}^{n-1} \psi_i^{18} \le C\bar{\omega}_\phi(f'', n^{-1}),$$

respectively.

Now inequalities (66) and (68) imply, for $x \in I_j$,

$$|f'(x) - S'(x)| \le C\omega_2(f', \Delta_n(x_j); \tilde{I}_j) \le C\bar{\omega}_\phi^2(f', n^{-1})$$

and

$$|f''(x) - S''(x)| \le C\omega(f'', \Delta_n(x_j); I_j) \le C\bar{\omega}_\phi(f'', n^{-1})$$

in the cases $f \in C^1(I)$ and $f \in C^2(I)$, respectively. Thus inequalities (10) and (11) are also proved. ■

6. Proof of Theorem 3

We need the well-known Jensen inequality, that is,

$$|a_1 b_1 + \cdots + a_n b_n|^p \le a_1 |b_1|^p + \cdots + a_n |b_n|^p,$$

where $a_i \ge 0$, $i = \overline{1, n}$, and $\sum_{i=1}^n a_i = 1$, $p \ge 1$, and also the following lemma.

Lemma 3 (see Lemma 2.5 of [8]). *Let* $\{z_i\colon -1 = z_0 < z_1 < \cdots < z_{n+1} = 1\}$ *be a partition of the interval* $[-1, 1]$ *into* $n + 1$ *subintervals and let* $r \ge 1$ *be an integer. Using the notation* $\delta_i = z_{i+1} - z_{i-1}$, $i = \overline{1, n}$, $d_n = \max\{\delta_i\colon 1 \le i \le n\}$, *then*

$$\left\{\tfrac{1}{2} \sum_{i=1}^n (\omega_r(f, z_i; 2h))^p \delta_i\right\}^{1/p} \le 2^{1/p + 2(r+1)} \tau_r\left(f; h + \frac{d_n}{r}\right)_p.$$

Now let us estimate $\|f - S\|_p$:

$$\begin{aligned}
\|f - S\|_p &= \left\{\tfrac{1}{2} \int_{-1}^1 |f(x) - S(x)|^p \, dx\right\}^{1/p} \\
&= \left\{\tfrac{1}{2} \sum_{j=1}^n \int_{I_j} |f(x) - S(x)|^p \, dx\right\}^{1/p} \\
&\le \left\{\tfrac{1}{2} \sum_{j=1}^n \int_{I_j} C^p(\omega_3(f, x_j; h_{j-1} + h_j + h_{j+1}))^p \, dx\right\}^{1/p} \\
&\le C\left\{\tfrac{1}{2} \sum_{j=1}^n (\omega_3(f, x_j; h_{j-1} + h_j + h_{j+1}))^p h_j \, dx\right\}^{1/p} \\
&\le C\tau_3(f; n^{-1})_p,
\end{aligned}$$

where $h_0 := h_{n+1} := 0$.

Using the estimate $\sum_{j=1}^{n-1} \psi_j^{16} \le C$ (see the proof of Lemma 2 in [4], for example) and the Jensen inequality we get

$$\begin{aligned}
\|S - p_n\|_p &= \left\{\tfrac{1}{2}\int_{-1}^{1} |p_n(x) - S(x)|^p\,dx\right\}^{1/p} \\
&\le C\left\{\tfrac{1}{2}\int_{-1}^{1}\left(\sum_{j=1}^{n} \omega_3(f, x_j; h_{j-1} + h_j + h_{j+1})\psi_j^{16}\right)^p dx\right\}^{1/p} \\
&\le C\left\{\tfrac{1}{2}\int_{-1}^{1}\sum_{j=1}^{n} (\omega_3(f, x_j; h_{j-1} + h_j + h_{j+1}))^p\psi_j^{16}\,dx\right\}^{1/p} \\
&\le C\left\{\tfrac{1}{2}\sum_{j=1}^{n} (\omega_3(f, x_j; h_{j-1} + h_j + h_{j+1}))^p \int_0^\infty \left(\frac{h_j}{t + h_j}\right)^{16} dt\right\}^{1/p} \\
&\le C\left\{\tfrac{1}{2}\sum_{j=1}^{n} (\omega_3(f, x_j; h_{j-1} + h_j + h_{j+1}))^p h_j\right\}^{1/p} \\
&\le C\tau_3(f, n^{-1})_p.
\end{aligned}$$

Thus, using Minkowski's inequality we have

$$\|f - p_n\|_p \le \|f - S\|_p + \|S - p_n\|_p \le C\tau_3(f; n^{-1})_p.$$

The proof of (16) is complete. The proofs of (17) and (18) are analogous. ■

Note that it is possible to relax some of the conditions put on function f in Theorems 1–4. This is connected with the fact that a convex function on the interval $[-1, 1]$ (at least in terms of divided differences) is continuous on the open interval $(-1, 1)$ and has left and right derivatives at every point of this interval (see, for example, Sections 11 and 72 of [7]).

Final Remark

After this paper was submitted, the author became aware that Y. Hu, D. Leviatan, and X. M. Yu had obtained the uniform estimate for convex polynomial approximation in terms of $\omega_3(f, n^{-1})$. Their paper "Convex Polynomial and Spline Approximation in $C[-1, 1]$" appeared in *Constructive Approximation*, **10**(1):31–64.

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