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On equivalence of moduli of smoothness of splines in $\mathbb{L}_p$, $0<p<1$

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Abstract

It is shown that, if $n, r \in \mathbb{N}$, $k \in \mathbb{N}_0$, $1 \leqslant v \leqslant r$, $\mathbf{t}_n := \left(\cos \frac{(n-i)\pi}{n}\right)_{i=0}^{n}$ is the Chebyshev partition of $[-1, 1]$, and s is a piecewise polynomial of degree $\leqslant r$ on $\mathbf{t}_n$ such that $s \in \mathbb{C}^{v-1}[-1, 1]$, then for any $0 < p < 1$ and $t > 0$,

$$\omega_{k+v}^{\varphi}(s, t)_p \leqslant c t^v \omega_{k,v}^{\varphi}(s^{(v)}, t)_p,$$

where ω_{k+v}^{φ} and $\omega_{k,v}^{\varphi}$ denote the Ditzian–Totik $(k+v)$th modulus of smoothness and kth modulus with the weight φ^v, respectively. In particular, in the case $k=0$, $\omega_v^{\varphi}(s, t)_p \leqslant c(r, p) t^v \left\| \varphi^v s^{(v)} \right\|_p$. It is known that these inequalities are no longer valid for a general f in place of s if $0 < p < 1$ even if it is assumed that $f \in \mathbb{C}^{\infty}[-1, 1]$.

This implies, in particular, that if a piecewise polynomial s of degree $\leqslant r$ on $\mathbf{t}_n$ is such that $s \in \mathbb{C}^m[-1, 1]$, $0 \leqslant m \leqslant r-1$, then for any $1 \leqslant k \leqslant r+1$, $1 \leqslant v \leqslant \min\{k, m+1\}$ and $0 < p < 1$,

$$n^{-v} \omega_{k-v,v}^{\varphi}\left(s^{(v)}, n^{-1}\right)_p \sim \omega_k^{\varphi}\left(s, n^{-1}\right)_p.$$

Similar results for quasi-uniform partitions and classical moduli of smoothness are also obtained.

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1. Introduction and main results

Let $\mathcal{S}_r(\mathbf{z}_n)$ be the space of all piecewise polynomial functions of degree r (order $r+1$) with the knots $\mathbf{z}_n := (z_i)_{i=0}^n$, $-1 =: z_0 < z_1 < \cdots < z_{n-1} < z_n := 1$. In other words, $s \in \mathcal{S}_r(\mathbf{z}_n)$ if, on each interval (z_i, z_{i+1}), $0 \leqslant i \leqslant n-1$, s is in Π_r, where Π_r denotes the space of algebraic polynomials of degree $\leqslant r$.

As usual, $\mathbb{L}_p(J)$, $0 < p \leqslant \infty$, denotes the space of all measurable functions f on J such that $\|f\|_{\mathbb{L}_p(J)} < \infty$, where $\|f\|_{\mathbb{L}_p(J)} := \left(\int_J |f(x)|^p\, dx\right)^{1/p}$ if $p < \infty$, and $\|f\|_{\mathbb{L}_\infty(J)} :=$ ess $\sup_{x\in J} |f(x)|$. We also denote $\|f\|_p := \|f\|_{\mathbb{L}_p[-1,1]}$. It is well known that $\|\cdot\|_{\mathbb{L}_p(J)}$ is a norm (and $\mathbb{L}_p(J)$ is a Banach space) if $1 \leqslant p \leqslant \infty$, and that it is a quasi-norm if $0 < p < 1$.

For $k \in \mathbb{N}_0$, let

$$\Delta_h^k(f, x, J) := \begin{cases} \sum_{i=0}^{k} \binom{k}{i}(-1)^{k-i} f(x - kh/2 + ih) & \text{if } x \pm kh/2 \in J, \\ 0 & \text{otherwise} \end{cases}$$

be the kth symmetric difference, and $\Delta_h^k(f, x) := \Delta_h^k(f, x, [-1, 1])$. The kth modulus of smoothness of a function $f \in \mathbb{L}_p(J)$ is defined by

$$\omega_k(f, t, J)_p := \sup_{0<h\leqslant t} \|\Delta_h^k(f, \cdot, J)\|_{\mathbb{L}_p(J)},$$

and we also denote

$$\omega_k(f, J)_p := \omega_k(f, |J|, J)_p \quad \text{and} \quad \omega_k(f, t)_p := \omega_k(f, t, [-1, 1])_p.$$

Note that $\Delta_h^0(f, x, J) := f(x)$ and, hence, $\omega_0(f, t, J)_p := \|f\|_{\mathbb{L}_p(J)}$.

The weighted Ditzian–Totik kth modulus of smoothness of a function $f \in \mathbb{L}_p[-1, 1]$, $0 < p \leqslant \infty$, is defined by

$$\omega_{k,v}^{\varphi}(f, t)_p := \sup_{0<h\leqslant t} \left\| \varphi(\cdot)^v \Delta_{h\varphi(\cdot)}^k (f, \cdot) \right\|_p,$$

where $\varphi(x) := \sqrt{1 - x^2}$. If $v = 0$, then

$$\omega_k^{\varphi}(f, t)_p := \omega_{k,0}^{\varphi}(f, t)_p = \sup_{0<h\leqslant t} \|\Delta_{h\varphi(\cdot)}^k (f, \cdot)\|_p$$

is the usual Ditzian–Totik modulus. Also, note that $\omega_{0,v}^{\varphi}(f, t)_p := \left\|\varphi^v f\right\|_p$.

For a partition $\mathbf{z}_n := \{z_0, \ldots, z_n \mid -1 =: z_0 < z_1 < \cdots < z_n := 1\}$ of the interval $[-1, 1]$, denote the *scale of the partition* $\mathbf{z}_n$ by $\vartheta := \vartheta(\mathbf{z}_n) := \max_{0\leqslant j\leqslant n-1} |J_{j\pm1}|/|J_j|$, where $J_j := [z_j, z_{j+1}]$ with $z_j := -1$, $j < 0$, and $z_j := 1$, $j > n$, and $|J| :=$ meas J.

We say that A is equivalent to B and write $A \sim B$ if there exists a positive constant c such that $c^{-1} A \leqslant B \leqslant cA$. We refer to this constant c as an *equivalence constant*.

Theorems 1.1–1.3 are the main results of this paper. Note that all of them were proved in [2] in the case $1 \leqslant p \leqslant \infty$, and the purpose of this note is to provide proofs (which turn out to be rather different) in the case $0 < p < 1$.

Theorem 1.1 (*Local estimates*). *Let $s \in \mathcal{S}_r(\mathbf{z}_n) \cap \mathbb{C}^m[-1, 1]$, $r \in \mathbb{N}$, $0 \leqslant m \leqslant r-1$, and $J = [z_{\mu_1}, z_{\mu_2}]$ with $\mu_2 - \mu_1 \leqslant c_0$ for some constant c_0. Then, for any $1 \leqslant k \leqslant r+1$ and $0 < p \leqslant \infty$,* *we have*

$$|J|^{\nu}\omega_{k-\nu}(s^{(\nu)}, J)_p \sim \omega_k(s, J)_p, \quad 1 \leqslant \nu \leqslant \min\{k, m+1\}.$$

Equivalence constants above depend only on r, ϑ, c_0 *and* p *as* $p \to 0$.

Theorem 1.1 is a consequence of Corollary 2.3 and [2, Theorem 1.1].

Suppose that $\delta_{\max} := \delta_{\max}(\mathbf{z}_n) := \max_{0\leqslant j\leqslant n-1}|J_j|$ and $\delta_{\min} := \delta_{\min}(\mathbf{z}_n) := \min_{0\leqslant j\leqslant n-1} |J_j|$. We say that $\mathbf{z}_n$ is Δ-*quasi-uniform* if $\Delta := \delta_{\max}/\delta_{\min}$ is bounded by a constant independent of n, and denote such partition by $\mathbf{u}_n^{\Delta}$. Note that the 1-quasi-uniform partition $\mathbf{u}_n := \mathbf{u}_n^1$ is just the uniform partition of $[-1, 1]$ into n subintervals of equal lengths. If $\mathbf{z}_n = \mathbf{u}_n^{\Delta}$, then clearly $2/(n\Delta) \leqslant \delta_{\min} \leqslant 2/n \leqslant \delta_{\max} \leqslant 2\Delta/n$, and $\vartheta(\mathbf{z}_n) \leqslant \Delta$. Therefore, $\delta_{\min} \sim \delta_{\max} \sim n^{-1}$ with equivalence constants depending only on Δ.

Theorem 1.2 (*Quasi-uniform partition*). *Let* $\mathbf{u}_n^{\Delta}$, $n \in \mathbb{N}$, *be a* Δ-*quasi-uniform partition of* $[-1, 1]$, *and let* $s \in \mathcal{S}_r(\mathbf{u}_n^{\Delta}) \cap \mathbb{C}^m[-1, 1]$, $r \in \mathbb{N}$, $0 \leqslant m \leqslant r-1$. *Then, for any* $1 \leqslant k \leqslant r+1$ *and* $0 < p \leqslant \infty$, *we have*

$$n^{-\nu}\omega_{k-\nu}(s^{(\nu)}, n^{-1})_p \sim \omega_k(s, n^{-1})_p, \quad 1 \leqslant \nu \leqslant \min\{k, m+1\}. \tag{1.1}$$

Equivalence constants above depend only on r, Δ *and* p *as* $p \to 0$.

Theorem 1.2 follows from Theorem 2.4 and [2, Theorem 1.4].

We say that $\mathbf{z}_n$ is a Chebyshev partition (and z_i's are Chebyshev knots) if $\mathbf{z}_n = \mathbf{t}_n := (t_i)_{i=0}^n$, where $t_i := \cos\frac{(n-i)\pi}{n}$, $0 \leqslant i \leqslant n$. The following result immediately follows from Theorem 2.5 and [2, Theorem 1.8].

Theorem 1.3 (*Chebyshev partition*). *Let* $s \in \mathcal{S}_r(\mathbf{t}_n) \cap \mathbb{C}^m[-1, 1]$, $r \in \mathbb{N}$, $0 \leqslant m \leqslant r-1$. *Then, for any* $1 \leqslant k \leqslant r+1$, $1 \leqslant \nu \leqslant \min\{k, m+1\}$ *and* $0 < p \leqslant \infty$, *we have*

$$n^{-\nu}\omega^{\varphi}_{k-\nu,\nu}(s^{(\nu)}, n^{-1})_p \sim \omega^{\varphi}_k(s, n^{-1})_p. \tag{1.2}$$

Equivalence constants above depend only on r *and* p *as* $p \to 0$.

Throughout this paper, $c(\gamma_1, \gamma_2, \ldots)$ denote positive constants which depend only on the parameters $\gamma_1, \gamma_2, \ldots$ (note that $c(p, \ldots)$ depends on p only as $p \to 0$) and which may be different on different occurrences.

2. Auxiliary results and proofs

The following lemma is a well-known fact about relationships among various (quasi)norms of algebraic polynomials, and will be frequently used in our proofs.

Lemma 2.1. *For any polynomial* $q_r \in \Pi_r$, $0 < p \leqslant \infty$, *and intervals* I *and* J *such that* $I \subseteq J$, *we have*

$$|J|^{1/p}\|q_r\|_{\mathbb{L}_\infty(J)} \sim \|q_r\|_{\mathbb{L}_p(J)} \leqslant c\,(r, |J|/|I|, p)\,\|q_r\|_{\mathbb{L}_p(I)},$$

where equivalence constants depend only on r *and* p *as* $p \to 0$.

2.1. Relationships between $\omega_{k+\nu}(s,J)_p$ and $\omega_k(s^{(\nu)},J)_p$ for $s\in\mathcal{S}_r(\mathbf{z}_n)$

Theorem 2.2. *Let $r\in\mathbb{N}$, $k\in\mathbb{N}_0$, $s\in\mathcal{S}_r(\mathbf{z}_n)$ and $J=[z_{\mu_1},z_{\mu_2}]$ with $\mu_2-\mu_1\leqslant c_0$ for some constant c_0. If s is* continuous *on J, then for any $0<p\leqslant\infty$,*

$$\omega_{k+1}(s,J)_p\leqslant c(r,k,\vartheta,c_0,p)|J|\omega_k(s',J)_p. \tag{2.1}$$

Note that this theorem is no longer true without the assumption that s is continuous (a step function is a trivial counterexample). Also, it is well known that the inequality

$$\omega_{k+1}(f,t)_p\leqslant c(k)t\omega_k(f',t)_p$$

is true with an arbitrary f from the Sobolev space $\mathbb{W}^1(\mathbb{L}_p)$ if $1\leqslant p\leqslant\infty$, and that it is not true in general if $0<p<1$ even if f is assumed to be in $\mathbb{C}^\infty$ (see Remark 2.6).

Corollary 2.3. *Let $r\in\mathbb{N}$, $k\in\mathbb{N}_0$, $1\leqslant\nu\leqslant r$, $s\in\mathcal{S}_r(\mathbf{z}_n)\cap\mathbb{C}^{\nu-1}(J)$, where $J=[z_{\mu_1},z_{\mu_2}]$ with $\mu_2-\mu_1\leqslant c_0$ for some constant c_0. Then, for any $0<p\leqslant\infty$,*

$$\omega_{k+\nu}(s,J)_p\leqslant c(r,k,\vartheta,c_0,p)|J|^\nu\omega_k(s^{(\nu)},J)_p. \tag{2.2}$$

In particular, in the case $k=0$,

$$\omega_\nu(s,J)_p\leqslant c(r,\vartheta,c_0,p)|J|^\nu\left\|s^{(\nu)}\right\|_{\mathbb{L}_p(J)}. \tag{2.3}$$

Proof of Theorem 2.2. Let $k\in\mathbb{N}_0$, $x\in J$ and $0<h\leqslant|J|$ be such that $x\pm(k+1)h/2\in J$, and suppose that $q\in\Pi_k$ is such that $q(\xi)=s(\xi)$ for some $\xi\in J$ (for example, $\xi=z_{\mu_1}$) and $\left\|s'-q'\right\|_{\mathbb{L}_p(J)}\leqslant c\omega_k(s',J)_p$ (such q exists by Whitney's theorem, and this inequality is trivial if $k=0$). We also assume that $J_\alpha\subset J$ is such that $\left\|s'-q'\right\|_{\mathbb{L}_\infty(J_\alpha)}=\max_{\mu_1\leqslant j\leqslant\mu_2-1}\left\|s'-q'\right\|_{\mathbb{L}_\infty(J_j)}$ $=\left\|s'-q'\right\|_{\mathbb{L}_\infty(J)}$.

Then, for any $x\in J$, using Lemma 2.1 we have

$$\begin{aligned}\left|\Delta_h^{k+1}(s,x,J)\right| &= \left|\Delta_h^{k+1}(s-q,x,J)\right|\leqslant 2^{k+1}\|s-q\|_{\mathbb{L}_\infty(J)}\\ &= 2^{k+1}\left\|\int_\xi^x\left(s'(t)-q'(t)\right)dt\right\|_{\mathbb{L}_\infty(J)}\leqslant 2^{k+1}|J|\left\|s'-q'\right\|_{\mathbb{L}_\infty(J)}\\ &= 2^{k+1}|J|\left\|s'-q'\right\|_{\mathbb{L}_\infty(J_\alpha)}\leqslant c(r,k,p)|J||J_\alpha|^{-1/p}\left\|s'-q'\right\|_{\mathbb{L}_p(J_\alpha)}\\ &\leqslant c(r,k,\vartheta,c_0,p)|J|^{1-1/p}\left\|s'-q'\right\|_{\mathbb{L}_p(J)}\\ &\leqslant c(r,k,\vartheta,c_0,p)|J|^{1-1/p}\omega_k(s',J)_p,\end{aligned}$$

which implies (2.1). $\square$

2.2. Relationships between $\omega_{k+\nu}(s,n^{-1})_p$ and $\omega_k(s^{(\nu)},n^{-1})_p$ for $s\in\mathcal{S}(\mathbf{u}_n^\Delta)$

The following theorem is a global analog of Corollary 2.3. Its proof uses Corollary 2.3 and is exactly the same (with obvious modifications) as the proof of Theorem 1.4 in [2]. Hence, we omit this proof.

Theorem 2.4. *Let* $\mathbf{u}_n^\Delta$, $n \in \mathbb{N}$, *be a* Δ*-quasi-uniform partition of* $[-1, 1]$, *and let* $s \in \mathcal{S}(\mathbf{u}_n^\Delta) \cap \mathbb{C}^{\nu-1}[-1, 1]$, $r \in \mathbb{N}$, $1 \leqslant \nu \leqslant r$. *Then, for any* $k \in \mathbb{N}_0$ and $0 < p \leqslant \infty$,

$$\omega_{k+\nu}(s, n^{-1})_p \leqslant c(r, k, \Delta, p) n^{-\nu} \omega_k(s^{(\nu)}, n^{-1})_p.$$

2.3. Relationships between $\omega_{k+\nu}^\varphi(s, n^{-1})_p$ *and* $\omega_{k,\nu}^\varphi(s^{(\nu)}, n^{-1})_p$ *for* $s \in \mathcal{S}_r(\mathbf{t}_n)$

Recall that $\mathbf{t}_n := (t_i)_{i=0}^n := \left(\cos \frac{(n-i)\pi}{n}\right)_{i=0}^n$ denotes a Chebyshev partition, $J_j := [t_j, t_{j+1}]$, $0 \leqslant j \leqslant n-1$, and denote

$$\mathfrak{D}_\delta := \{x \mid 1 - \delta\varphi(x)/2 \geqslant |x|\} \setminus \{\pm 1\} = \left\{ x \,\middle|\, |x| \leqslant \frac{4-\delta^2}{4+\delta^2} \right\}.$$

Observe that $\Delta_{h\varphi(x)}^k(f, x)$ is defined to be identically 0 if $x \notin \mathfrak{D}_{kh}$. For $x \in J_j \cap \mathfrak{D}_{mh}$ and $0 < h \leqslant n^{-1}$, we have (see e.g. [2])

$$\left\{x + \left(i - \frac{m}{2}\right) h\varphi(x)\right\}_{i=0}^m \subset \mathfrak{I}_{j,m} := \left[t_{j-3m}, t_{j+4+3m}\right]$$

(recall that $t_i := -1$ for $i < 0$, and $t_i := 1$ for $i > n$).

Theorem 2.5. *Let* $n, r \in \mathbb{N}$, $k \in \mathbb{N}_0$, $1 \leqslant \nu \leqslant r$, *and let* $\mathbf{t}_n$ *be the Chebyshev partition of* $[-1, 1]$. *If* $s \in \mathcal{S}_r(\mathbf{t}_n) \cap \mathbb{C}^{\nu-1}[-1, 1]$, *then for any* $0 < p \leqslant \infty$ *and* $t > 0$, *we have*

$$\omega_{k+\nu}^\varphi(s, t)_p \leqslant c(r, k, p) t^\nu \omega_{k,\nu}^\varphi(s^{(\nu)}, t)_p. \tag{2.4}$$

In particular, in the case $k = 0$,

$$\omega_\nu^\varphi(s, t)_p \leqslant c(r, p) t^\nu \left\| \varphi^\nu s^{(\nu)} \right\|_p. \tag{2.5}$$

Remark 2.6. It was shown in [2] that (2.4) is valid for all $f \in \mathbb{W}^\nu(\mathbb{L}_p)$ in place of s if $1 \leqslant p \leqslant \infty$ and $k \in \mathbb{N}$. Note that this inequality is no longer valid for a general f if $0 < p < 1$ even if we assume that $f \in \mathbb{C}^\infty[-1, 1]$. For example, suppose that $f_\varepsilon : [-1, 1] \to \mathbb{R}$ is such that

$$f_\varepsilon(x) := \begin{cases} \dfrac{1}{(\nu-2)!} \int_0^x (x-t)^{\nu-2} e^{-\varepsilon/t}\, dt & \text{if } 0 < x \leqslant 1, \\ 0 & \text{if } -1 \leqslant x \leqslant 0 \end{cases}$$

in the case $\nu \geqslant 2$, and

$$f_\varepsilon(x) := \begin{cases} e^{-\varepsilon/x} & \text{if } 0 < x \leqslant 1, \\ 0 & \text{if } -1 \leqslant x \leqslant 0 \end{cases}$$

in the case $\nu = 1$. Then,

$$f_\varepsilon^{(\nu-1)}(x) = \begin{cases} e^{-\varepsilon/x} & \text{if } 0 < x \leqslant 1, \\ 0 & \text{if } -1 \leqslant x \leqslant 0, \end{cases}$$

and so $f_\varepsilon \in \mathbb{C}^\infty[-1, 1]$, and $\omega_{k,\nu}^\varphi(f_\varepsilon^{(\nu)}, t)_p \leqslant c\omega_k(f_\varepsilon^{(\nu)}, t)_p \leqslant c \left\| f_\varepsilon^{(\nu)} \right\|_p \to 0$ as $\varepsilon \to 0^+$. At the same time, straightforward (but tedious) computations show that $\omega_{k+\nu}(f_\varepsilon, t)_p \geqslant c\omega_{k+\nu}^\varphi(f_\varepsilon, t)_p \geqslant ct^{\nu-1+1/p}$ for sufficiently small $t > 0$ and $\varepsilon > 0$.

Proof of Theorem 2.5. Suppose that $n \geqslant (k+v)/2$. For each $0 \leqslant j \leqslant n-1$, let $q_j \in \Pi_{k+v-1}$ be such that $\|s-q_j\|_{\mathbb{L}_p(\mathfrak{I}_j)} \leqslant c\omega_{k+v}(s,\mathfrak{I}_j)_p$ (q_j exists by Whitney's inequality), where $\mathfrak{I}_j := \mathfrak{I}_{j,k+v}$. Then,

$$\begin{aligned}
\omega^{\varphi}_{k+v}(s,n^{-1})^p_p &= \sup_{0<h\leqslant n^{-1}} \left\|\Delta^{k+v}_{h\varphi(\cdot)}(s,\cdot,[-1,1])\right\|^p_{\mathbb{L}_p[-1,1]} \\
&= \sup_{0<h\leqslant n^{-1}} \sum_{j=0}^{n-1} \int_{J_j} \left|\Delta^{k+v}_{h\varphi(x)}(s-q_j,x,[-1,1])\right|^p dx \\
&\leqslant c\sum_{j=0}^{n-1} \|s-q_j\|^p_{\mathbb{L}_p(\mathfrak{I}_j)},
\end{aligned}$$

where the last inequality follows by the same argument as was used in the proof of Theorem 6.1 of [2]. Therefore, using the inequality $\omega_{k+v}(f,\lambda t,J)_p \leqslant c(1+\lambda)^{k+v-1+\max\{1,1/p\}}\omega_{k+v}(f,t,J)_p$, we have

$$\begin{aligned}
\omega^{\varphi}_{k+v}(s,n^{-1})^p_p &\leqslant c\sum_{j=0}^{n-1} \omega_{k+v}(s,\mathfrak{I}_j)^p_p \leqslant c\sum_{j=0}^{n-1} \omega_{k+v}(s,h_j,\mathfrak{I}_j)^p_p \\
&\leqslant c\sum_{j=0}^{n-1} h_j^{-1}\int_0^{h_j}\int_{\mathfrak{I}_j} \left|\Delta^{k+v}_h(s,x,\mathfrak{I}_j)\right|^p dx\,dh, \qquad (2.6)
\end{aligned}$$

where $h_j := \frac{1}{2(k+v)}\min_{J_i\subset\mathfrak{I}_j}|J_i|$ (note that $h_j \sim |\mathfrak{I}_j|$ with an equivalence constant depending only on k and v). Now, using the identity

$$\Delta^{k+v}_h(f,x) = \int_{-h/2}^{h/2}\cdots\int_{-h/2}^{h/2} \Delta^k_h(f^{(v)},x+u_1+\cdots+u_v)\,du_1\ldots du_v,$$

and assuming for a moment that $0 \leqslant j \leqslant n-1$ and $0 < h \leqslant h_j$ are fixed, we have

$$\begin{aligned}
\int_{\mathfrak{I}_j}\left|\Delta^{k+v}_h(s,x,\mathfrak{I}_j)\right|^p dx &\leqslant |\mathfrak{I}_j|h^{pv}\left\|\Delta^k_h(s^{(v)},\cdot)\right\|^p_{\mathbb{L}_\infty(\{x\,:\,x\pm kh/2\in\mathfrak{I}_j\})} \\
&\leqslant 2|\mathfrak{I}_j|h^{pv}\left|\Delta^k_h(s^{(v)},x_0)\right|^p, \qquad (2.7)
\end{aligned}$$

for some x_0 such that $x_0 \pm kh/2 \in \mathfrak{I}_j$.

We now consider the cases $k \geqslant 1$ and $k=0$ separately.

Case $k \geqslant 1$: We have the following two possibilities:

(i) for any t_i such that $t_i \in \mathfrak{I}_j$, $t_i \notin (x_0-(k+1)h/2, x_0+(k+1)h/2)$. Then, we define $\mathcal{I}_{x_0,h} := [x_0-h/2, x_0+h/2]$;

(ii) for some $t_\mu \in \mathfrak{I}_j$, $t_\mu \in (x_0-(k+1)h/2, x_0+(k+1)h/2)$ (note that there can only be at most one such t_μ since $(k+1)h \leqslant (k+1)h_j \leqslant \frac{1}{2}\min_{J_i\subset\mathfrak{I}_j}|J_i|$). Let $0 \leqslant i_\mu \leqslant 2k+1$ be such that $t_\mu \in [x_0-(k-i_\mu+1)h/2, x_0-(k-i_\mu)h/2]$, and define $\mathcal{I}_{x_0,h} := [x_0-h/2, x_0]$ if i_μ is odd, and $\mathcal{I}_{x_0,h} := [x_0, x_0+h/2]$ if i_μ is even.

Then the restriction of $\Delta^k_h(s^{(v)},x)$ to $\mathcal{I}_{x_0,h}$ is a polynomial of degree $\leqslant r-v$ in x, and hence by Lemma 2.1 we have

$$\left|\Delta^k_h(s^{(v)},x_0)\right| \leqslant \left\|\Delta^k_h(s^{(v)},\cdot)\right\|_{\mathbb{L}_\infty(\mathcal{I}_{x_0,h})} \leqslant c|\mathcal{I}_{x_0,h}|^{-1/p}\left\|\Delta^k_h(s^{(v)},\cdot)\right\|_{\mathbb{L}_p(\mathcal{I}_{x_0,h})}.$$

Together with the inequalities (2.6) and (2.7) and taking into account that $\mathcal{I}_{x_0,h} \subset \mathfrak{I}_j$ this implies

$$\begin{aligned}
\omega^{\varphi}_{k+v}(s, n^{-1})^p_p &\leqslant c \sum_{j=0}^{n-1} \int_0^{h_j} h^{pv-1} \left\| \Delta^k_h(s^{(v)}, \cdot) \right\|^p_{\mathbb{L}_p(\mathcal{I}_{x_0,h})} dh \\
&\leqslant c \sum_{j=0}^{n-1} \int_0^{h_j} h^{pv-1} \int_{\mathfrak{I}_j} \left| \Delta^k_h(s^{(v)}, x) \right|^p dx\, dh \\
&\leqslant c \sum_{j=0}^{n-1} \int_{\mathfrak{I}_j} \int_0^{h_j/\varphi(x)} \varphi(x)^{pv} h^{pv-1} \left| \Delta^k_{h\varphi(x)}(s^{(v)}, x) \right|^p dh\, dx. \qquad (2.8)
\end{aligned}$$

Now, note that $h_j/\varphi(x) \sim n^{-1}$ for all $x \in \mathfrak{I}_j \setminus (J_0 \cup J_{n-1})$. If $x \in (J_0 \cup J_{n-1}) \cap \mathfrak{D}_{kh}$, then $4kh/(4 + k^2h^2) \leqslant \varphi(x) \leqslant \sin(\pi n^{-1})$ which can only happen if $h \leqslant (8/k)n^{-1}$. Therefore,

$$\begin{aligned}
\omega^{\varphi}_{k+v}(f, n^{-1})^p_p &\leqslant c \sum_{j=0}^{n-1} \int_{\mathfrak{I}_j} \int_0^{cn^{-1}} h^{pv-1} \left| \varphi(x)^v \Delta^k_{h\varphi(x)}(s^{(v)}, x) \right|^p dh\, dx \\
&\leqslant c \int_0^{cn^{-1}} h^{pv-1} \left\| \varphi^v \Delta^k_{h\varphi}(s^{(v)}, \cdot) \right\|^p_p dh \leqslant c n^{-pv} \omega^{\varphi}_{k,v}(s^{(v)}, cn^{-1})^p_p.
\end{aligned}$$

Case $k = 0$: In this case, (2.6), (2.7) and Lemma 2.1 imply

$$\begin{aligned}
\omega^{\varphi}_{v}(s, n^{-1})^p_p &\leqslant c \sum_{j=0}^{n-1} h_j^{pv+1} \left| s^{(v)}(x_0) \right|^p \leqslant c \sum_{j=0}^{n-1} h_j^{pv} \left\| s^{(v)} \right\|^p_{\mathbb{L}_p(\mathfrak{I}_j)} \\
&\leqslant c \sum_{j=0}^{n-1} h_j^{pv} \left(\int_{\mathfrak{I}_j \setminus (J_0 \cup J_{n-1})} + \int_{\mathfrak{I}_j \cap (J_0 \cup J_{n-1})} \right) \left| s^{(v)}(x) \right|^p dx,
\end{aligned}$$

and taking into account that $h_j/\varphi(x) \sim n^{-1}$ for all $x \in \mathfrak{I}_j \setminus (J_0 \cup J_{n-1})$, the fact that there are only $\leqslant c(v)$ indices j such that $\mathfrak{I}_j \cap (J_0 \cup J_{n-1}) \neq \emptyset$, and that for these j, $h_j \sim |J_0| = |J_{n-1}|$, we get

$$\begin{aligned}
\omega^{\varphi}_{v}(s, n^{-1})^p_p &\leqslant c n^{-pv} \sum_{j=0}^{n-1} \int_{\mathfrak{I}_j \setminus (J_0 \cup J_{n-1})} \left| \varphi(x)^v s^{(v)}(x) \right|^p dx \\
&\quad + c|J_0|^{pv} \left\| s^{(v)} \right\|^p_{\mathbb{L}_p(J_0)} + c|J_{n-1}|^{pv} \left\| s^{(v)} \right\|^p_{\mathbb{L}_p(J_{n-1})}.
\end{aligned}$$

We now use Lemma 2.1, the fact that $s^{(v)}$ is a polynomial of degree $\leqslant r - v$ on $J_{n-1} = [\cos(\pi/n), 1]$, and the estimate $\varphi(x) \geqslant \sin(\pi/(2n)) \geqslant 1/n$ for $\cos(\pi/n) \leqslant x \leqslant \cos(\pi/(2n))$ to conclude

$$\begin{aligned}
|J_{n-1}|^v \left\| s^{(v)} \right\|_{\mathbb{L}_p(J_{n-1})} &= 2^v \sin^{2v}\left(\frac{\pi}{2n}\right) \left\| s^{(v)} \right\|_{\mathbb{L}_p[\cos(\pi/n),1]} \\
&\leqslant c n^{-2v} \left\| s^{(v)} \right\|_{\mathbb{L}_p[\cos(\pi/n),\cos(\pi/(2n))]} \\
&\leqslant c n^{-v} \left\| \varphi^v s^{(v)} \right\|_{\mathbb{L}_p[\cos(\pi/n),\cos(\pi/(2n))]} \\
&\leqslant c n^{-v} \left\| \varphi^v s^{(v)} \right\|_{\mathbb{L}_p(J_{n-1})}.
\end{aligned}$$

Similarly,

$$|J_0|^\nu \left\| s^{(\nu)} \right\|_{\mathbb{L}_p(J_0)} \leqslant c n^{-\nu} \left\| \varphi^\nu s^{(\nu)} \right\|_{\mathbb{L}_p(J_0)},$$

and therefore

$$\omega_\nu^\varphi(s, n^{-1})_p \leqslant c n^{-\nu} \left\| \varphi^\nu s^{(\nu)} \right\|_{\mathbb{L}_p[-1,1]}.$$

Hence the inequality

$$\omega_{k+\nu}^\varphi(s, n^{-1})_p \leqslant c n^{-\nu} \omega_{k,\nu}^\varphi(s^{(\nu)}, \tilde{c} n^{-1})_p$$

is proved for all $k \in \mathbb{N}_0$ and all $n \geqslant (k+\nu)/2$ (and without loss of generality we can assume that $\tilde{c} \geqslant 1$).

Now, given $0 < t \leqslant 2/(k+\nu)$ (for $t > 2/(k+\nu)$ we use the fact that $\omega_{k+\nu}^\varphi(s,t)_p = \omega_{k+\nu}^\varphi(s, 2/(k+\nu))_p$) we let $n \geqslant (k+\nu)/2$ be such that $\tilde{c} n^{-1} \leqslant t < 2\tilde{c} n^{-1}$ (there may be more than one n), and using the inequality $\omega_{k+\nu}^\varphi(f, \lambda t)_p \leqslant c(\lambda+1)^{k+\nu} \omega_{k+\nu}^\varphi(f,t)_p$ (see e.g. [1]), we obtain

$$\begin{aligned} \omega_{k+\nu}^\varphi(s,t)_p &\leqslant \omega_{k+\nu}^\varphi(s, 2\tilde{c} n^{-1})_p \leqslant c\omega_{k+\nu}^\varphi(s, n^{-1})_p \\ &\leqslant c n^{-\nu} \omega_{k,\nu}^\varphi(s^{(\nu)}, \tilde{c} n^{-1})_p \leqslant c t^\nu \omega_{k,\nu}^\varphi(s^{(\nu)}, t)_p, \end{aligned}$$

and the proof is now complete. □

References

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