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Weighted moduli of smoothness of k-monotone functions and applications

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Abstract

Let $\omega_\varphi^k(f,\delta)_{w,\mathbb{L}_q}$ be the Ditzian–Totik modulus with weight w, $\mathcal{M}^k$ be the cone of k-monotone functions on $(-1,1)$, *i.e.,* those functions whose kth divided differences are nonnegative for all selections of $k+1$ distinct points in $(-1,1)$, and denote $\mathcal{E}(X,\mathbb{P}_n)_{w,q} := \sup_{f\in X}\inf_{P\in\mathbb{P}_n}\|w(f-P)\|_{\mathbb{L}_q}$, where $\mathbb{P}_n$ is the set of algebraic polynomials of degree at most n. Additionally, let $w_{\alpha,\beta}(x) := (1+x)^\alpha(1-x)^\beta$ be the classical Jacobi weight, and denote by $\mathbb{S}_p^{\alpha,\beta}$ the class of all functions such that $\|w_{\alpha,\beta}f\|_{\mathbb{L}_p} = 1$.

In this paper, we determine the exact behavior (in terms of δ) of $\sup_{f\in\mathbb{S}_p^{\alpha,\beta}\cap\mathcal{M}^k}\omega_\varphi^k(f,\delta)_{w_{\alpha,\beta},\mathbb{L}_q}$ for $1\le p,q\le\infty$ (the interesting case being $q<p$ as expected) and $\alpha,\beta>-1/p$ (if $p<\infty$) or $\alpha,\beta\ge 0$ (if $p=\infty$). It is interesting to note that, in one case, the behavior is different for $\alpha=\beta=0$ and for $(\alpha,\beta)\ne(0,0)$. Several applications are given. For example, we determine the exact (in some sense) behavior of $\mathcal{E}(\mathcal{M}^k\cap\mathbb{S}_p^{\alpha,\beta},\mathbb{P}_n)_{w_{\alpha,\beta},\mathbb{L}_q}$ for $\alpha,\beta\ge 0$.

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1. Introduction and main results

Let $w_{\alpha,\beta}(x) := (1+x)^\alpha(1-x)^\beta$ be the (classical) Jacobi weight, $\|\cdot\|_p := \|\cdot\|_{\mathbb{L}_p[-1,1]}$,

$$\mathbb{L}_p^{\alpha,\beta} := \left\{ f : [-1,1] \mapsto \mathbb{R} \;\middle|\; \left\| w_{\alpha,\beta} f \right\|_p < \infty \right\},$$

and let $\mathbb{S}_p^{\alpha,\beta}$ be the unit sphere in $\mathbb{L}_p^{\alpha,\beta}$, *i.e.*, $f \in \mathbb{S}_p^{\alpha,\beta}$ iff $\left\| w_{\alpha,\beta} f \right\|_p = 1$. It is convenient to denote $J_p := (-1/p, \infty)$ if $p < \infty$, and $J_\infty := [0,\infty)$. Clearly, $1 \in \mathbb{L}_p^{\alpha,\beta}$ iff $\alpha, \beta \in J_p$. We note that more general than Jacobi weights can be considered, and many results in this paper are valid and/or can be modified to be valid for those general weights. However, we only consider Jacobi weights in order not to overcomplicate the proofs which are already rather technical, and since the estimates of rates of unweighted polynomial approximation that have matching converse results involve weighted moduli with classical Jacobi weights $w_{r/2,r/2} = \varphi^r$, $r \in \mathbb{N}$ (see [8,9] or (8.2) with $\alpha = \beta = 0$ for an example of such an estimate). Here, as usual, $\varphi(x) := w_{1/2,1/2} = (1-x^2)^{1/2}$.

Let

$$\Delta_h^k(f, x, [a,b]) := \begin{cases} \displaystyle\sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f(x - kh/2 + ih), & \text{if } x \pm kh/2 \in [a,b], \\ 0, & \text{otherwise}, \end{cases}$$

be the kth symmetric difference, $\Delta_h^k(f,x) := \Delta_h^k(f,x,[-1,1])$, and let

$$\overrightarrow{\Delta}_h^k(f,x) := \Delta_h^k(f, x + kh/2) \quad \text{and} \quad \overleftarrow{\Delta}_h^k(f,x) := \Delta_h^k(f, x - kh/2)$$

be the forward and backward kth differences, respectively. The weighted main part moduli and the weighted Ditzian–Totik (DT) moduli of smoothness (see [2, (8.1.2), (8.2.10) and Appendix B]) are defined, respectively, as

$$\Omega_\varphi^k(f,\delta)_{w,p} := \sup_{0<h\le\delta} \|w \Delta_{h\varphi}^k(f)\|_{\mathbb{L}_p[-1+2k^2h^2, 1-2k^2h^2]}$$

and

$$\omega_\varphi^k(f,\delta)_{w,p} := \Omega_\varphi^k(f,\delta)_{w,p} + \overrightarrow{\Omega}_\varphi^k(f,\delta)_{w,p} + \overleftarrow{\Omega}_\varphi^k(f,\delta)_{w,p}, \tag{1.1}$$

where

$$\overrightarrow{\Omega}_\varphi^k(f,\delta)_{w,p} := \sup_{0<h\le 2k^2\delta^2} \|w \overrightarrow{\Delta}_h^k(f)\|_{\mathbb{L}_p[-1,-1+2k^2\delta^2]}$$

and

$$\overleftarrow{\Omega}_\varphi^k(f,\delta)_{w,p} := \sup_{0<h\le 2k^2\delta^2} \|w \overleftarrow{\Delta}_h^k(f)\|_{\mathbb{L}_p[1-2k^2\delta^2,1]}.$$

If $\alpha = \beta = 0$, then $\omega_\varphi^k(f,\delta)_{1,p}$ is equivalent to the usual DT modulus $\omega_\varphi^k(f,\delta)_p = \sup_{0<h\le\delta} \|\Delta_{h\varphi}^k(f)\|_p$.

It is easy to see that $\Omega_\varphi^k(f,\delta)_{w_{\alpha,\beta},p} \le c \left\| w_{\alpha,\beta} f \right\|_p$ for all $\alpha, \beta \in \mathbb{R}$. (Throughout this paper, c denote positive constants that may be different even if they appear in the same line.) At the same time, moduli $\omega_\varphi^k(f,\delta)_{w_{\alpha,\beta},p}$ are usually defined with the restriction $\alpha,\beta \ge 0$ for all $p \le \infty$ and not just for $p=\infty$. The reason for this is that, on one hand, $\omega_\varphi^k(f,\delta)_{w_{\alpha,\beta},p} \le c \left\| w_{\alpha,\beta} f \right\|_p$ if $\alpha,\beta \ge 0$, and, on the other hand, if $\alpha < 0$ or $\beta < 0$, then there are functions f in $\mathbb{L}_p^{\alpha,\beta}$ for which $\omega_\varphi^k(f,\delta)_{w_{\alpha,\beta},p} = \infty$. Indeed, suppose that $p < \infty$ and that $\delta > 0$ is fixed. If $f(x) := (x+1-\varepsilon)^{-\alpha-1/p}\chi_{[-1+\varepsilon,-1+2\varepsilon]}(x)$ with $\alpha < 0$ and $0 < \varepsilon < 2k^2\delta^2$, then $\left\| w_{\alpha,\beta} f \right\|_p \le c$, $\left\| w_{\alpha,\beta} f(\cdot+\varepsilon) \right\|_p = \infty$, and $\left\| w_{\alpha,\beta} f(\cdot+i\varepsilon) \right\|_p = 0$, $2 \le i \le k$, and so $\overrightarrow{\Omega}_\varphi^k(f,\delta)_{w_{\alpha,\beta},p} = \infty$.

If $\alpha,\beta \ge 0$, then it is easy to see that, if $f \in \mathbb{L}_p^{\alpha,\beta}$, $1 \le p < \infty$, then $\lim_{\delta\to 0^+} \omega_\varphi^k(f,\delta)_{w_{\alpha,\beta},p} = 0$. In the case $p=\infty$, the fact that f is in $\mathbb{L}_\infty^{\alpha,\beta}$ implies that $\omega_\varphi^k(f,\delta)_{w_{\alpha,\beta},\infty}$ is bounded but it is not enough to guarantee its convergence to zero if $\alpha^2+\beta^2 \ne 0$ even if f is continuous on $(-1,1)$ (consider, for example, $f(x) = w_{\alpha,\beta}^{-1}(x)$). One can show (see *e.g.* [3, p. 287] for a similar proof) that, if $\alpha > 0$ and $\beta > 0$, then for $f \in \mathbb{C}(-1,1)$, $\lim_{\delta\to 0^+} \omega_\varphi^k(f,\delta)_{w_{\alpha,\beta},\infty} = 0$ iff $\lim_{x\to\pm 1} w_{\alpha,\beta}(x) f(x) = 0$.

One can easily show that, for $\alpha,\beta \in \mathbb{R}$,

$$\sup_{f\in\mathbb{S}_p^{\alpha,\beta}} \Omega_\varphi^k(f,\delta)_{w_{\alpha,\beta},q} \sim 1, \quad 1 \le q \le p \le \infty. \tag{1.2}$$

(Here and later in this paper, we write $F \sim G$ iff there exist positive constants c_1 and c_2 such that $c_1 F \le G \le c_2 F$. These constants are always independent of δ, n and x but may depend on k, α, β, p and q.) Indeed, since $\Omega_\varphi^k(f,\delta)_{w_{\alpha,\beta},q} \le c \left\| w_{\alpha,\beta} f \right\|_q$, Hölder's inequality implies the upper estimate. The lower estimate follows, for example, from the fact that, for $k \in \mathbb{N}$, $\alpha,\beta \in \mathbb{R}$, $0 < p, q \le \infty$, and $0 < \delta \le 1/(2k)$, the function

$$f_\delta(x) := \begin{cases} (-1)^i, & \text{if } x \in [k\delta i, k\delta(i+1/2)],\ 0 \le i \le \lfloor 1/(2k\delta) \rfloor, \\ 0, & \text{otherwise,} \end{cases}$$

satisfies $\left\| w_{\alpha,\beta} f_\delta \right\|_p \sim 1$ and $\Omega_\varphi^k(f_\delta,\delta)_{w_{\alpha,\beta},q} \ge c > 0$ (see Lemma 6.1 for details).

The restriction $q \le p$ in (1.2) is essential since

$$\sup_{f\in\mathbb{S}_p^{\alpha,\beta}} \Omega_\varphi^k(f,\delta)_{w_{\alpha,\beta},q} = \infty, \quad \text{if } p < q.$$

This, of course, is expected since $\mathbb{L}_p^{\alpha,\beta} \not\subset \mathbb{L}_q^{\alpha,\beta}$, if $p < q$, and follows, for example, from Corollary 6.5.

If $\alpha,\beta \ge 0$, then

$$\sup_{f\in\mathbb{S}_p^{\alpha,\beta}} \omega_\varphi^k(f,\delta)_{w_{\alpha,\beta},q} \sim 1, \quad 1 \le q \le p \le \infty. \tag{1.3}$$

This follows from (1.2) and the observation that, for $\alpha,\beta \ge 0$, $\overrightarrow{\Omega}_\varphi^k(f,\delta)_{w_{\alpha,\beta},q} \le c \left\| w_{\alpha,\beta} f \right\|_q$ and $\overleftarrow{\Omega}_\varphi^k(f,\delta)_{w_{\alpha,\beta},q} \le c \left\| w_{\alpha,\beta} f \right\|_q$.

In this paper, we show that if the suprema in (1.2) and (1.3) are taken over the subset of $\mathbb{S}_p^{\alpha,\beta}$ consisting of all k-monotone functions, then these quantities become significantly smaller. This will allow us to obtain the exact rates (in some sense) of polynomial approximation in the weighted $\mathbb{L}_q$-norm of k-monotone functions in $\mathbb{S}_p^{\alpha,\beta}$.

Recall that $f : I \to \mathbb{R}$ is said to be k-monotone on I if its kth divided differences $[x_0, \dots, x_k; f]$ are nonnegative for all selections of $k+1$ distinct points $x_0, \dots, x_k$ in I, and denote by $\mathcal{M}^k$ the set of all k-monotone functions on $(-1, 1)$. In particular, $\mathcal{M}^0$, $\mathcal{M}^1$ and $\mathcal{M}^2$ are the sets of all nonnegative, nondecreasing and convex functions on $(-1, 1)$, respectively. Note that if $f \in \mathcal{M}^k$, $k \ge 2$, then, for all $j \le k-2$, $f^{(j)}$ exists on $(-1, 1)$ and is in $\mathcal{M}^{k-j}$. In particular, $f^{(k-2)}$ exists, is convex, and therefore satisfies a Lipschitz condition on any closed subinterval of $(-1, 1)$, is absolutely continuous on that subinterval, is continuous on $(-1, 1)$, and has left and right (nondecreasing) derivatives, $f_-^{(k-1)}$ and $f_+^{(k-1)}$ on $(-1, 1)$. We also note that it is essential that $(-1, 1)$ and not $[-1, 1]$ is used in the definition of $\mathcal{M}^k$ since the set of all k-monotone functions on the closed interval $[-1, 1]$ contains only bounded functions (if $k \in \mathbb{N}$).

Our main result is

Theorem 1.1. *Let $k \in \mathbb{N}$, $1 \le q < p \le \infty$, $\alpha, \beta \in J_p$, and $0 < \delta < 1/4$. Then,*

$$\sup_{f \in \mathbb{S}_p^{\alpha,\beta} \cap \mathcal{M}^k} \omega_\varphi^k(f, \delta)_{w_{\alpha,\beta},q} \sim \begin{cases} \delta^{2/q-2/p}, & \text{if } k \ge 2 \text{ and } (k, q, p) \ne (2, 1, \infty), \\ \delta^2 |\ln \delta|, & \text{if } k = 2,\ q = 1,\ p = \infty, \text{ and } (\alpha, \beta) \ne (0, 0), \\ \delta^2, & \text{if } k = 2,\ q = 1,\ p = \infty, \text{ and } (\alpha, \beta) = (0, 0), \\ \delta^{2/q-2/p}, & \text{if } k = 1 \text{ and } p < 2q, \\ \delta^{1/q}, & \text{if } k = 1 \text{ and } p > 2q. \end{cases} \tag{1.4}$$

If $k = 1$ and $p = 2q$, then

$$c \frac{\delta^{1/q} |\ln \delta|^{1/(2q)}}{|\ln |\ln \delta||^{\lambda/(2q)}} \le \sup_{f \in \mathbb{S}_{2q}^{\alpha,\beta} \cap \mathcal{M}^1} \omega_\varphi^1(f, \delta)_{w_{\alpha,\beta},q} \le c \delta^{1/q} |\ln \delta|^{1/(2q)}, \quad \lambda > 1. \tag{1.5}$$

Remark 1.2. It is easy to see (and follows from Lemmas 2.1, 2.4 and Corollary 4.2) that, for $k \in \mathbb{N}$, $1 \le q \le p \le \infty$, $\alpha, \beta \in J_p$, and $f \in \mathbb{L}_p^{\alpha,\beta} \cap \mathcal{M}^k$,

$$\omega_\varphi^k(f, \delta)_{w_{\alpha,\beta},q} \le c \left\| w_{\alpha,\beta} f \right\|_p, \quad \delta > 0.$$

Hence, Theorem 1.1 needs to be proved only for "small" δ, and the restriction $\delta < 1/4$ is chosen for convenience only (to guarantee that none of the quantities in (1.4) and (1.5) are zero while keeping them simple).

In the case $\alpha = \beta = 0$, all upper estimates and several lower estimates of Theorem 1.1 were proved in [7], and so the upper estimates in (1.4) and (1.5) will only have to be established for $(\alpha, \beta) \ne (0, 0)$ in the current paper. We remark that the fact that the case $k = 2$, $q = 1$ and $p = \infty$ turned out to be anomalous for $(\alpha, \beta) \ne (0, 0)$ causes rather significant difficulties in the proof of Theorem 1.1 for $k \ge 2$, $q > 1$ and $p = \infty$, since the rather simple main approach from [7] can no longer be used. (Section 5 is devoted to overcoming these difficulties.) We also note that the restriction $\alpha, \beta \in J_p$ in Theorem 1.1 guarantees that the classes $\mathbb{S}_p^{\alpha,\beta} \cap \mathcal{M}^k$ contain constants and so are rather rich. Without this restriction, we would have to deal with various anomalous situations and vacuous statements of theorems. For example, $\mathbb{S}_p^{\alpha,\beta} \cap \mathcal{M}^1 = \emptyset$ if $\alpha, \beta \le -1/p$ since, in this case, it is clear that $\mathbb{L}_p^{\alpha,\beta} \cap \mathcal{M}^1$ contains only functions which are identically 0 on $(-1, 1)$. Similarly, it is possible to show that $\mathbb{S}_p^{\alpha,\beta} \cap \mathcal{M}^2 = \emptyset$ if $\alpha, \beta \le -1/p - 1$.

At the same time, putting restrictions on α and β in the statements of some of our theorems would be a red herring (Lemma 4.1, for example, is an illustration of this). Hence, an interested reader should keep in mind that even if a statement is given for all $\alpha, \beta \in \mathbb{R}$, it *may* happen that it only applies to trivial functions if $\alpha, \beta \notin J_p$.

It is convenient to denote

$$\Upsilon_\delta^{\alpha,\beta}(k,q,p) := \begin{cases} \delta^{2/q-2/p}, & \text{if } k \ge 2, \text{ and } (k,q,p) \ne (2,1,\infty), \\ \delta^2|\ln\delta|, & \text{if } k=2,\ q=1,\ p=\infty, \text{ and } (\alpha,\beta)\ne(0,0) \\ \delta^2, & \text{if } k=2,\ q=1,\ p=\infty, \text{ and } (\alpha,\beta)=(0,0), \\ \delta^{2/q-2/p}, & \text{if } k=1 \text{ and } p<2q, \\ \delta^{1/q}|\ln\delta|^{1/(2q)}, & \text{if } k=1 \text{ and } p=2q, \\ \delta^{1/q}, & \text{if } k=1 \text{ and } p>2q. \end{cases} \tag{1.6}$$

The following is an immediate corollary of Theorem 1.1.

Corollary 1.3. *Let $k \in \mathbb{N}$, $1 \le q < p \le \infty$, $\alpha, \beta \in J_p$, $f \in \mathcal{M}^k \cap \mathbb{L}_p^{\alpha,\beta}$ and $0 < \delta < 1/4$. Then,*

$$\omega_\varphi^k(f,\delta)_{w_{\alpha,\beta},q} \le c\,\Upsilon_\delta^{\alpha,\beta}(k,q,p)\left\|w_{\alpha,\beta} f\right\|_p, \tag{1.7}$$

where $\Upsilon_\delta^{\alpha,\beta}(k,q,p)$ which is defined in (1.6) is best possible in the sense that (1.7) is no longer valid if one increases (respectively, decreases) any of the powers of δ (respectively, $|\ln\delta|$) in its definition.

Remark 1.4. The restriction $q < p$ in the statement of Theorem 1.1 is essential since, if $p < q$, then Corollary 6.5 implies that

$$\sup_{f\in\mathbb{S}_p^{\alpha,\beta}\cap\mathcal{M}^k} \omega_\varphi^k(f,\delta)_{w_{\alpha,\beta},q} = \infty,$$

and, if $p = q$, then it is easy to see that

$$\sup_{f\in\mathbb{S}_p^{\alpha,\beta}\cap\mathcal{M}^k} \omega_\varphi^k(f,\delta)_{w_{\alpha,\beta},p} \sim 1.$$

Let $\mathbb{P}_n$ be the set of algebraic polynomials of degree at most n, and denote

$$E_n(f)_{w,q} := \inf_{P\in\mathbb{P}_n} \|w(f-P)\|_q$$

and

$$\mathcal{E}(X,\mathbb{P}_n)_{w,q} := \sup_{f\in X} E_n(f)_{w,q}.$$

It is rather well known that

$$\mathcal{E}(\mathbb{S}_p^{\alpha,\beta},\mathbb{P}_n)_{w_{\alpha,\beta},q} \sim 1, \quad 1 \le q \le p \le \infty.$$

(This also follows from (1.3), (7.1) and Remark 6.2.) At the same time, for the class of k-monotone functions from $\mathbb{S}_p^{\alpha,\beta}$, we have the following result.

Theorem 1.5. *Let $1 \le q < p \le \infty$, $k \in \mathbb{N}$, and $\alpha, \beta \ge 0$. Then, for any $n \in \mathbb{N}$,*

$$\mathcal{E}(\mathcal{M}^k \cap \mathbb{S}_p^{\alpha,\beta}, \mathbb{P}_n)_{w_{\alpha,\beta},q} \sim \begin{cases} n^{-2/q+2/p}, & \text{if } k \ge 2 \text{ and } (k,q,p) \ne (2,1,\infty), \\ n^{-2}, & \text{if } k = 2,\ q = 1,\ p = \infty, \text{ and } \alpha = \beta = 0, \\ n^{-\min\{2/q-2/p,1/q\}}, & \text{if } k = 1 \text{ and } p \ne 2q. \end{cases} \tag{1.8}$$

If $k = 2$, $q = 1$, $p = \infty$ and $(\alpha, \beta) \ne (0,0)$, then

$$cn^{-2} \le \mathcal{E}(\mathcal{M}^2 \cap \mathbb{S}_\infty^{\alpha,\beta}, \mathbb{P}_n)_{w_{\alpha,\beta},1} \le cn^{-2}\ln(n+1). \tag{1.9}$$

If $k = 1$ and $p = 2q$, then

$$cn^{-1/q} \le \mathcal{E}(\mathcal{M}^1 \cap \mathbb{S}_{2q}^{\alpha,\beta}, \mathbb{P}_n)_{w_{\alpha,\beta},q} \le cn^{-1/q}[\ln(n+1)]^{1/(2q)}. \tag{1.10}$$

Additionally, if $q > 1$, then for any $\varepsilon > 0$,

$$\limsup_{n\to\infty} n^{1/q}[\ln(n+1)]^{-1/(2q)+\varepsilon}\mathcal{E}(\mathcal{M}^1 \cap \mathbb{S}_{2q}^{\alpha,\beta}, \mathbb{P}_n)_{w_{\alpha,\beta},q} = \infty. \tag{1.11}$$

In the case $\alpha = \beta = 0$, (1.8) and the lower estimate in (1.10) were proved by Konovalov, Leviatan and Maiorov in [5, Theorem 1]. The upper estimate in (1.10) and (1.11) improve corresponding estimates in [5, Theorem 1] (considered there in the special case $\alpha = \beta = 0$).

We remark that it is an open problem if $\ln(n+1)$ in (1.9) can be replaced by a smaller quantity or removed altogether, and if $[\ln(n+1)]^{1/2}$ is necessary in (1.10) in the case $(k,q,p) = (1,1,2)$. Also, while it follows from (1.11) that, in the case $q > 1$, the quantity $[\ln(n+1)]^{1/(2q)}$ in (1.10) cannot be replaced by $[\ln(n+1)]^{1/(2q)-\varepsilon}$ with $\varepsilon > 0$, the precise behavior of $\mathcal{E}(\mathcal{M}^1 \cap \mathbb{S}_{2q}^{\alpha,\beta}, \mathbb{P}_n)_{w_{\alpha,\beta},q}$ is still unknown (see Section 7 for more details).

Finally, we mention that several other applications of Theorem 1.1 are given in Section 8.

2. "Truncated" k-monotone functions

For $k \ge 1$, we denote

$$\mathcal{M}_+^k := \left\{ f \in \mathcal{M}^k \;\middle|\; f(x) = 0, \text{ for all } x \in (-1,0] \right\}.$$

Note that, if $f \in \mathcal{M}_+^k$, then $f^{(i)}(0) = 0$, $0 \le i \le k-2$, and $f_-^{(k-1)}(0) = 0$.

In this section, we prove that it is sufficient to consider classes $\mathcal{M}_+^k$ instead of $\mathcal{M}^k$ in Theorem 1.1 (see Lemma 2.4). This will significantly simplify the proofs of upper estimates.

Lemma 2.1. *Let $k \in \mathbb{N}$, $1 \le p \le \infty$, $\alpha, \beta \in J_p$, and $f \in \mathcal{M}^k \cap \mathbb{L}_p^{\alpha,\beta}$. Then*

$$\left\| w_{\alpha,\beta} T_{k-1}(f) \right\|_p \le c \left\| w_{\alpha,\beta} f \right\|_p,$$

where

$$T_{k-1}(f,x) := f_-^{(k-1)}(0)x^{k-1}/(k-1)! + \sum_{i=0}^{k-2} f^{(i)}(0)x^i/i!. \tag{2.1}$$

Proof. It follows from [7, Lemma 3.7] that $\|T_{k-1}(f)\|_{\mathbb{L}_p[-1/2,1/2]} \le c\,\|f\|_{\mathbb{L}_p[-1/2,1/2]}$. Therefore, taking into account that $\left\|w_{\alpha,\beta}\right\|_p \sim 1$ and $w_{\alpha,\beta}(x) \sim 1$ on $[-1/2, 1/2]$, we have

$$\begin{aligned}\left\|w_{\alpha,\beta}T_{k-1}(f)\right\|_p &\le c\,\|T_{k-1}(f)\|_\infty \le c\,\|T_{k-1}(f)\|_{\mathbb{L}_p[-1/2,1/2]}\\ &\le c\,\|f\|_{\mathbb{L}_p[-1/2,1/2]} \le c\left\|w_{\alpha,\beta}f\right\|_p,\end{aligned}$$

where we used the fact that, for any $p_{k-1} \in \mathbb{P}_{k-1}$ and $I \subseteq J$,

$$\|p_{k-1}(f)\|_{\mathbb{L}_\infty(J)} \le c\,\|p_{k-1}(f)\|_{\mathbb{L}_p(I)}, \quad c = c\,(k, |I|, |I|/|J|),$$

which follows, for example, from [1, (4.2.10) and (4.2.14)]. □

The following lemma can be easily proved by induction.

Lemma 2.2. *Let $f \in \mathcal{M}^k$, $k \in \mathbb{N}$, be such that $f^{(i)}(0) = 0$, $0 \le i \le k-2$, and $f_-^{(k-1)}(0) = 0$. Then f is j-monotone on $[0, 1)$ and $(-1)^{k-j} f$ is j-monotone on $(-1, 0]$, for all $0 \le j \le k-1$.*

Corollary 2.3. *If $k \in \mathbb{N}$ and $f \in \mathcal{M}_+^k$, then $f \in \mathcal{M}_+^j$, for all $0 \le j \le k-1$.*

Lemma 2.4. *Let $k \in \mathbb{N}$, $1 \le q < p \le \infty$, $\alpha, \beta \in J_p$, and $\delta > 0$. Then*

$$\sup_{f\in\mathbb{S}_p^{\alpha,\beta}\cap\mathcal{M}^k} \omega_\varphi^k(f,\delta)_{w_{\alpha,\beta},q} \sim \sup_{f\in\mathbb{S}_p^{\alpha,\beta}\cap\mathcal{M}_+^k} \omega_\varphi^k(f,\delta)_{w_{\alpha,\beta},q} + \sup_{f\in\mathbb{S}_p^{\beta,\alpha}\cap\mathcal{M}_+^k} \omega_\varphi^k(f,\delta)_{w_{\beta,\alpha},q}.$$

Proof. First of all, it is clear that

$$\sup_{f\in\mathbb{S}_p^{\alpha,\beta}\cap\mathcal{M}^k} \omega_\varphi^k(f,\delta)_{w_{\alpha,\beta},q} = \sup_{f\in\mathbb{S}_p^{\beta,\alpha}\cap\mathcal{M}^k} \omega_\varphi^k(f,\delta)_{w_{\beta,\alpha},q}. \tag{2.2}$$

This immediately follows from the observation that $f(x) \in \mathbb{S}_p^{\alpha,\beta} \cap \mathcal{M}^k$ iff $(-1)^k f(-x) \in \mathbb{S}_p^{\beta,\alpha} \cap \mathcal{M}^k$.

Now, the estimate

$$\begin{aligned}2\sup_{f\in\mathbb{S}_p^{\alpha,\beta}\cap\mathcal{M}^k} \omega_\varphi^k(f,\delta)_{w_{\alpha,\beta},q} &= \sup_{f\in\mathbb{S}_p^{\alpha,\beta}\cap\mathcal{M}^k} \omega_\varphi^k(f,\delta)_{w_{\alpha,\beta},q} + \sup_{f\in\mathbb{S}_p^{\beta,\alpha}\cap\mathcal{M}^k} \omega_\varphi^k(f,\delta)_{w_{\beta,\alpha},q}\\ &\ge \sup_{f\in\mathbb{S}_p^{\alpha,\beta}\cap\mathcal{M}_+^k} \omega_\varphi^k(f,\delta)_{w_{\alpha,\beta},q} + \sup_{f\in\mathbb{S}_p^{\beta,\alpha}\cap\mathcal{M}_+^k} \omega_\varphi^k(f,\delta)_{w_{\beta,\alpha},q}\end{aligned}$$

is obvious since $\mathcal{M}_+^k \subset \mathcal{M}^k$. To prove the estimate in the opposite direction, suppose that k, α, β, δ, q and p satisfy all conditions of the theorem, and let f be an arbitrary function from $\mathcal{M}^k \cap \mathbb{S}_p^{\alpha,\beta}$. Denote

$$f_1(x) := (f(x) - T_{k-1}(f,x))\,\chi_{[0,1]}(x) \quad \text{and} \quad f_2(x) := (f(x) - T_{k-1}(f,x))\,\chi_{[-1,0]}(x),$$

where $T_{k-1}(f)$ is the Maclaurin polynomial of degree $\le k-1$ defined in (2.1). It is clear that $f_1(x)$ and $\tilde{f}_2(x) := (-1)^k f_2(-x)$ are both in $\mathcal{M}_+^k$. Taking into account that $f - T_{k-1}(f) = f_1 + f_2$, $|f_1| + |f_2| = |f_1 + f_2|$,

$$\left\|w_{\alpha,\beta}f_2\right\|_p = \left\|w_{\beta,\alpha}\tilde{f}_2\right\|_p \quad \text{and} \quad \omega_\varphi^k(f_2,\delta)_{w_{\alpha,\beta},q} = \omega_\varphi^k(\tilde{f}_2,\delta)_{w_{\beta,\alpha},q},$$

we have

$$\left\|w_{\alpha,\beta} f_1\right\|_p + \left\|w_{\beta,\alpha} \tilde{f}_2\right\|_p = \left\|w_{\alpha,\beta} f_1\right\|_p + \left\|w_{\alpha,\beta} f_2\right\|_p \le c \left\|w_{\alpha,\beta} \left(|f_1| + |f_2|\right)\right\|_p$$
$$= c \left\|w_{\alpha,\beta} \left(f - T_{k-1}(f)\right)\right\|_p \le c \left\|w_{\alpha,\beta} f\right\|_p \le c,$$

where the second last inequality follows from Lemma 2.1.

Now, if neither f_1 nor $\tilde{f}_2$ is identically equal to 0 on $(-1, 1)$, using the fact that

$$\left\|w_{\alpha,\beta} f_1\right\|_p^{-1} f_1 \in \mathbb{S}_p^{\alpha,\beta} \cap \mathcal{M}_+^k \quad \text{and} \quad \left\|w_{\beta,\alpha} \tilde{f}_2\right\|_p^{-1} \tilde{f}_2 \in \mathbb{S}_p^{\beta,\alpha} \cap \mathcal{M}_+^k$$

we have

$$\begin{aligned}
\omega_\varphi^k(f,\delta)_{w_{\alpha,\beta},q} &\le \omega_\varphi^k(f_1,\delta)_{w_{\alpha,\beta},q} + \omega_\varphi^k(f_2,\delta)_{w_{\alpha,\beta},q} = \omega_\varphi^k(f_1,\delta)_{w_{\alpha,\beta},q} + \omega_\varphi^k(\tilde{f}_2,\delta)_{w_{\beta,\alpha},q} \\
&= \left\|w_{\alpha,\beta} f_1\right\|_p \omega_\varphi^k\left(\left\|w_{\alpha,\beta} f_1\right\|_p^{-1} f_1, \delta\right)_{w_{\alpha,\beta},q} \\
&\quad + \left\|w_{\beta,\alpha} \tilde{f}_2\right\|_p \omega_\varphi^k\left(\left\|w_{\beta,\alpha} \tilde{f}_2\right\|_p^{-1} \tilde{f}_2, \delta\right)_{w_{\beta,\alpha},q} \\
&\le c \sup_{f\in\mathbb{S}_p^{\alpha,\beta}\cap\mathcal{M}_+^k} \omega_\varphi^k(f,\delta)_{w_{\alpha,\beta},q} + c \sup_{f\in\mathbb{S}_p^{\beta,\alpha}\cap\mathcal{M}_+^k} \omega_\varphi^k(f,\delta)_{w_{\beta,\alpha},q}.
\end{aligned}$$

If f_1 or $\tilde{f}_2$ is identically zero, the estimate is obvious. □

Lemma 2.5. *Let $k \in \mathbb{N}$, $1 \le q < p \le \infty$, $\alpha, \beta \in J_p$, $\gamma_1, \gamma_2 \in \mathbb{R}$, and $0 < \delta < 1/k$. Then*

$$\sup_{f\in\mathbb{S}_p^{\alpha,\beta}\cap\mathcal{M}^k} \omega_\varphi^k(f,\delta)_{w_{\alpha,\beta},q} \sim \sup_{f\in\mathbb{S}_p^{\gamma_1,\beta}\cap\mathcal{M}_+^k} \omega_\varphi^k(f,\delta)_{w_{\gamma_1,\beta},q} + \sup_{f\in\mathbb{S}_p^{\gamma_2,\alpha}\cap\mathcal{M}_+^k} \omega_\varphi^k(f,\delta)_{w_{\gamma_2,\alpha},q}.$$

Proof. The lemma immediately follows from Lemma 2.4 and the observation that

$$w_{\alpha,\beta}(x) \sim w_{\gamma_1,\beta}(x) \quad \text{and} \quad w_{\beta,\alpha}(x) \sim w_{\gamma_2,\alpha}(x), \quad -1/2 \le x \le 1,$$
$$\left\|w \Delta_{h\varphi}^k(f)\right\|_{\mathbb{L}_q(S)} = \left\|w \Delta_{h\varphi}^k(f)\right\|_{\mathbb{L}_q(S\cap[-1/2,1])}, \quad 0 < h \le 1/k,$$

and

$$\|wf\|_{\mathbb{L}_p(S)} = \|wf\|_{\mathbb{L}_p(S\cap[0,1])},$$

for any f which is identically 0 on $[-1, 0]$. □

3. Auxiliary results and upper estimates for $q = 1$

The proof of the following proposition is elementary and will be omitted.

Proposition 3.1. *Let $0 < \eta < 1$. Then the following holds.*

(a) *If $|\lambda| \le \sqrt{2\eta}$, then the function $x \mapsto x + \lambda\varphi(x)$ is increasing on $[-1+\eta, 1-\eta]$ and has the inverse $y \mapsto \psi(\lambda, y)$, where*

$$\psi(\lambda, y) := \frac{y - \lambda\sqrt{1 - y^2 + \lambda^2}}{1 + \lambda^2}. \tag{3.1}$$

(b) *If* $|\lambda| \le \sqrt{2\eta}$, *then*

$$\int_{-1+\eta}^{1-\eta} g(x) f(x+\lambda\varphi(x))\, dx$$

$$= \int_{-1+\eta+\lambda\sqrt{2\eta-\eta^2}}^{1-\eta+\lambda\sqrt{2\eta-\eta^2}} f(y)\, g(\psi(\lambda, y))\, \frac{\partial \psi(\lambda, y)}{\partial y}\, dy. \tag{3.2}$$

(c) *If* $|x| \le 1/\sqrt{4\lambda^2+1}$, *then* $\frac{1}{2} \le \frac{\partial(x+\lambda\varphi(x))}{\partial x} \le 2$. *In particular, if* $|\lambda| \le \sqrt{\eta/2}$, *then* $\frac{1}{2} \le \frac{\partial(x+\lambda\varphi(x))}{\partial x} \le 2$ *for* $x \in [-1+\eta, 1-\eta]$, *and hence* $\frac{1}{2} \le \frac{\partial\psi(\lambda,y)}{\partial y} \le 2$ *for* $y \in [-1+\eta+\lambda\sqrt{2\eta-\eta^2}, 1-\eta+\lambda\sqrt{2\eta-\eta^2}]$.

(d) *If* $|x| \le 1-\eta$, *then* $\varphi(x) \le \sqrt{2/\eta}(1-|x|)$.

(e) *If* $|\lambda| \le \sqrt{\eta}/2$ *and* $|x| \le 1-\eta$, *then* $(1-x)/4 \le 1-x+\lambda\varphi(x) \le 2(1-x)$ *and* $(1+x)/4 \le 1+x+\lambda\varphi(x) \le 2(1+x)$.

We are now ready to prove the main auxiliary theorem which will yield upper estimates in Theorem 1.1 for $q = 1$. In view of Lemma 2.5 we consider $f \in \mathcal{M}_+^k \cap \mathbb{L}_1^{\beta,\beta}$ noting that while we could consider $f \in \mathcal{M}_+^k \cap \mathbb{L}_1^{0,\beta}$, the symmetry makes things more convenient. We also note that it is possible to use the same approach in order to prove this theorem for $f \in \mathcal{M}^k \cap \mathbb{L}_1^{\alpha,\beta}$, but the estimates become more cumbersome. Finally, recall that $w_{\beta,\beta}(x) = \varphi^{2\beta}(x)$.

Theorem 3.2. *Let* $k \in \mathbb{N}$, $\beta \in \mathbb{R}$, $f \in \mathcal{M}_+^k \cap \mathbb{L}_1^{\beta,\beta}$, *and* $0 < \delta \le 1/(2k)$. *Then*

$$\omega_\varphi^k(f,\delta)_{w_{\beta,\beta},1} \le c \left\| w_{\beta,\beta} f \right\|_{\mathbb{L}_1[1-3k^2\delta^2,1]} + c \sup_{0<h\le\delta} h^k \left\| (1-y^2)^{-k/2} w_{\beta,\beta}(y) f(y) \right\|_{\mathbb{L}_1[0,1-2k^2h^2]}. \tag{3.3}$$

The following corollary immediately follows by Hölder's inequality and the fact that, for $1 \le p' \le \infty$ (with $1/p' + 1/p = 1$),

$$\left\| (1-y^2)^{-k/2} \right\|_{\mathbb{L}_{p'}[0,1-2k^2h^2]} \le c \begin{cases} h^{-k+2/p'}, & \text{if } kp' > 2, \\ |\ln h|^{1/p'}, & \text{if } kp' = 2, \\ 1, & \text{if } kp' < 2. \end{cases}$$

Corollary 3.3. *Let* $k \in \mathbb{N}$, $\beta \in \mathbb{R}$, $1 \le p \le \infty$, $f \in \mathcal{M}_+^k \cap \mathbb{L}_p^{\beta,\beta}$, *and* $0 < \delta \le 1/(2k)$. *Then*

$$\omega_\varphi^k(f,\delta)_{w_{\beta,\beta},1} \le c \left\| w_{\beta,\beta} f \right\|_p \begin{cases} \delta^{2-2/p}, & \text{if } k \ge 3, \text{ or } k = 2 \text{ and } 1 \le p < \infty, \\ & \text{or } k = 1 \text{ and } 1 \le p < 2, \\ \delta^2 |\ln \delta|, & \text{if } k = 2 \text{ and } p = \infty, \\ \delta\sqrt{|\ln \delta|}, & \text{if } k = 1 \text{ and } p = 2, \\ \delta, & \text{if } k = 1 \text{ and } 2 < p \le \infty. \end{cases} \tag{3.4}$$

Remark 3.4. If $\beta = 0$ and k is even, or if $\beta = -1/2$ and k is odd, then estimates (3.3) and (3.4) can be improved (see Remark 3.7 and [7, Theorem 3.2]). In fact, if $\beta = -1/2$ and $k = 1$, then we have $\omega_\varphi^1(f,\delta)_{w_{-1/2,-1/2},1} \le c\delta^{2-2/p} \left\| w_{\beta,\beta} f \right\|_p$, for all $1 \le p \le \infty$ and $f \in \mathcal{M}_+^1 \cap \mathbb{L}_p^{-1/2,-1/2}$, and not only for $1 \le p < 2$ as (3.4) implies. However, this is not too exciting since, on one hand,

$\beta = -1/2$ is in J_p only if $1 \le p < 2$ and, on the other hand, if $p \ge 2$ then the set $\mathcal{M}_+^1 \cap \mathbb{L}_p^{-1/2,-1/2}$ consists only of functions which are identically equal to 0 on $(-1, 1)$.

Remark 3.5. Corollary 3.3, together with Lemmas 2.4 and 2.5, implies the upper estimates in Theorem 1.1 in the case $q = 1$ (except for the case $\alpha = \beta = 0$ when $k = 2$ and $p = \infty$ which follows from [7]).

Now, if $f \in \mathcal{M}_+^k \cap \mathbb{L}_p^{\beta,\beta}$ is such that $f \equiv 0$ on $[0, 1 - A\delta^2]$, for some constant $0 < A \le \delta^{-2}$, then taking into account that

$$\sup_{0<h\le\delta} h^k \left\| (1-y^2)^{-k/2} \right\|_{\mathbb{L}_{p'}[1-A\delta^2, 1-2k^2h^2]} \le c(A,k,p)\delta^{2/p'},$$

we have another corollary of Theorem 3.2.

Corollary 3.6. *Let $k \in \mathbb{N}$, $\beta \in \mathbb{R}$, $1 \le p \le \infty$, $0 < \delta \le 1/(2k)$, and let $f \in \mathcal{M}_+^k \cap \mathbb{L}_p^{\beta,\beta}$ be such that $f(x) = 0$ for $x \in [0, 1 - A\delta^2]$, for some positive constant $A \le \delta^{-2}$. Then*

$$\omega_\varphi^k(f,\delta)_{w_{\beta,\beta},1} \le c\delta^{2-2/p} \left\| w_{\beta,\beta} f \right\|_p,$$

where c depends on A.

Proof of Theorem 3.2. Let $h \in (0, \delta]$ be fixed. Taking into account that $f \in \mathcal{M}_+^k$, $\Delta_{h\varphi(x)}^k(f,x) \ge 0$ and Proposition 3.1(b) with $\eta = 2k^2h^2$ and $\lambda_i := (i - k/2)h$, $0 \le i \le k$, we have

$$\begin{aligned}
&\|w_{\beta,\beta}\Delta_{h\varphi}^k f\|_{\mathbb{L}_1[-1+2k^2h^2, 1-2k^2h^2]} \\
&\quad = \sum_{i=0}^k \binom{k}{i} (-1)^{k-i} \int_{-1+2k^2h^2}^{1-2k^2h^2} w_{\beta,\beta}(x) f(x + \lambda_i\varphi(x))\,dx \\
&\quad = \sum_{i=0}^k \binom{k}{i} (-1)^{k-i} \int_{-1+2k^2h^2+(2i-k)kh^2\sqrt{1-k^2h^2}}^{1-2k^2h^2+(2i-k)kh^2\sqrt{1-k^2h^2}} w_{\beta,\beta}(\psi(\lambda_i,y)) f(y) \frac{\partial\psi(\lambda_i,y)}{\partial y}\,dy \\
&\quad = \sum_{i=0}^k \binom{k}{i} (-1)^{k-i} \left(\int_0^{1-2k^2h^2-k^2h^2\sqrt{1-k^2h^2}} + \int_{1-2k^2h^2-k^2h^2\sqrt{1-k^2h^2}}^{1-2k^2h^2+(2i-k)kh^2\sqrt{1-k^2h^2}} \right) \\
&\qquad \times w_{\beta,\beta}(\psi(\lambda_i,y)) f(y) \frac{\partial\psi(\lambda_i,y)}{\partial y}\,dy \\
&\quad =: \sum_{i=0}^k \binom{k}{i} (-1)^{k-i} (\mathcal{I}_c + \mathcal{I}_r).
\end{aligned} \tag{3.5}$$

It follows from Proposition 3.1(e) that

$$w_{\beta,\beta}(x) \sim w_{\beta,\beta}(x + \lambda\varphi(x)), \quad \text{for } |x| \le 1 - \eta \text{ and } |\lambda| \le \sqrt{\eta}/2. \tag{3.6}$$

In particular, this implies that

$$w_{\beta,\beta}(\psi(\lambda,y)) \sim w_{\beta,\beta}(y), \quad \text{for } y \in [-1+\eta+\lambda\sqrt{2\eta-\eta^2}, 1-\eta+\lambda\sqrt{2\eta-\eta^2}] \\ \text{and } |\lambda| \le \sqrt{\eta}/2.$$

Hence, noting also that Proposition 3.1(c) implies that $|\partial\psi(\lambda_i, y)/\partial y| \le 2$, for all $0 \le i \le k$, we have

$$\left|\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\mathcal{I}_r\right| \le c\int_{1-3k^2h^2}^{1-k^2h^2}\left|w_{\beta,\beta}(y)f(y)\right|\,dy \le c\left\|w_{\beta,\beta}f\right\|_{\mathbb{L}_1[1-3k^2\delta^2,1]}. \tag{3.7}$$

Now,

$$\left|\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\mathcal{I}_c\right| = \left|\int_0^{1-2k^2h^2-k^2h^2\sqrt{1-k^2h^2}} f(y)A_k(y,h)\,dy\right| \le \int_0^{1-2k^2h^2}|f(y)||A_k(y,h)|\,dy,$$

where

$$A_k(y,h) := \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}w_{\beta,\beta}(\psi(\lambda_i,y))\widetilde{\psi}(\lambda_i,y)$$

and

$$\widetilde{\psi}(\lambda_i,y) := \frac{\partial\psi(\lambda_i,y)}{\partial y} = \frac{\lambda_i y+\sqrt{1-y^2+\lambda_i^2}}{(1+\lambda_i^2)\sqrt{1-y^2+\lambda_i^2}}.$$

Suppose now that $y \in [0, 1-2k^2h^2]$ is fixed and, for convenience, denote $\vartheta := \varphi(y)$. Then $\vartheta \ge \sqrt{3}kh$.

Note that

$$A_k(y,h) = \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}g_y(\lambda_i/\vartheta) = \Delta^k_{h/\vartheta}(g_y,0),$$

where

$$g_y(t) := w_{\beta,\beta}(\psi(t\vartheta,y))\widetilde{\psi}(t\vartheta,y).$$

Recall that, if $g^{(m)}$ is continuous on $[x-m\mu/2, x+m\mu/2]$, then for some $\xi \in (x-m\mu/2, x+m\mu/2)$,

$$\Delta^m_\mu(g,x) = \mu^m g^{(m)}(\xi). \tag{3.8}$$

Hence,

$$|A_k(y,h)| = |\Delta^k_{h/\vartheta}(g_y,0)| \le h^k\vartheta^{-k}\left\|\frac{d^k}{dt^k}g_y(t)\right\|_{\mathbb{C}[-1/2,1/2]}. \tag{3.9}$$

We now note that

$$\varphi(\psi(t\vartheta,y)) = \vartheta\frac{ty+\sqrt{1+t^2}}{1+t^2\vartheta^2} = \frac{\vartheta}{\sqrt{1+t^2}-ty}$$

and

$$\widetilde{\psi}(t\vartheta,y) = \frac{ty+\sqrt{1+t^2}}{(1+t^2\vartheta^2)\sqrt{1+t^2}} = \frac{1}{\sqrt{1+t^2}-ty}\cdot\frac{1}{\sqrt{1+t^2}}$$

and, in particular,

$$\widetilde{\psi}(t\vartheta, y) = \frac{\varphi(\psi(t\vartheta, y))}{\vartheta\sqrt{1+t^2}}.$$

Therefore, recalling that $w_{\beta,\beta} = \varphi^{2\beta}$ we have

$$g_y(t) = \frac{\varphi^{2\beta+1}(\psi(t\vartheta, y))}{\vartheta\sqrt{1+t^2}} = \vartheta^{2\beta}(1+t^2)^{-\beta-1}\left(1 - \frac{ty}{\sqrt{1+t^2}}\right)^{-2\beta-1}.$$

Remark 3.7. If $G_y(t) := (g_y(t)+(-1)^k g_y(-t))/2$, then $A_k(y,h) = \Delta^k_{h/\vartheta}(G_y, 0)$. If $\beta = -1/2$ and k is odd, then G_y is identically equal to 0, and so $|A_k(y,h)| = 0$. Also, if $\beta = 0$ and k is even, then $G_y(t) = (1+t^2\vartheta^2)^{-1}$, and so $|G_y^{(k)}(t)| \le c\vartheta^k$ and $|A_k(y,h)| \le ch^k$. Hence, (3.3) can be improved in these cases.

Noting that $|t|y/\sqrt{1+t^2} < 1$, we have the following expansion into binomial series

$$\left(1 - \frac{ty}{\sqrt{1+t^2}}\right)^{-2\beta-1} = \sum_{i=0}^{\infty}\binom{-2\beta-1}{i}(-1)^i \frac{t^i y^i}{(1+t^2)^{i/2}},$$

and so

$$g_y(t) = \vartheta^{2\beta}\sum_{i=0}^{\infty}\binom{-2\beta-1}{i}(-1)^i \frac{t^i y^i}{(1+t^2)^{\beta+1+i/2}}.$$

The derivatives of this series are uniformly convergent on $[-1,1]$ (to take a simple interval) because it can be easily seen that, for $|t| \le 1$,

$$\left|\frac{d^k}{dt^k}\frac{t^i}{(1+t^2)^{\beta+1+i/2}}\right| \le c\sum_{j=0}^{k}\left|\left[\left(\frac{t}{\sqrt{1+t^2}}\right)^i\right]^{(j)}\right| \le c\sum_{j=0}^{\min\{i,k\}}(i+1)^j\left(\frac{|t|}{\sqrt{1+t^2}}\right)^{i-j}$$
$$\le c(i+1)^k 2^{-i/2}.$$

Hence, for $|t| \le 1$,

$$\left|\frac{d^k}{dt^k}g_y(t)\right| \le c\vartheta^{2\beta}\sum_{i=0}^{\infty}\left|\binom{-2\beta-1}{i}\right|(i+1)^k 2^{-i/2} \le c\vartheta^{2\beta}.$$

Estimate (3.9) now implies that

$$|A_k(y,h)| \le ch^k\vartheta^{2\beta-k},$$

and so

$$\left|\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\mathcal{I}_c\right| \le ch^k\int_0^{1-2k^2h^2}(1-y^2)^{\beta-k/2}|f(y)|\,dy. \tag{3.10}$$

Together with (3.5), inequalities (3.7) and (3.10) imply that

$$\Omega^k_\varphi(f,\delta)_{w_{\beta,\beta},1} = c\left\|w_{\beta,\beta}f\right\|_{\mathbb{L}_1[1-3k^2\delta^2,1]}$$
$$+ c\sup_{0<h\le\delta}h^k\left\|(1-y^2)^{-k/2}w_{\beta,\beta}(y)f(y)\right\|_{\mathbb{L}_1[0,1-2k^2h^2]}. \tag{3.11}$$

Finally, Lemma 4.1 (that we prove in Section 4 for all $q \ge 1$) with $q = 1$, together with (3.11), implies (3.3). □

4. Upper estimates for $q > 1$

Lemma 4.1. *Let $k \in \mathbb{N}$, $1 \le q < \infty$, $\alpha, \beta \in \mathbb{R}$, and $f \in \mathcal{M}_+^k \cap \mathbb{L}_q^{\alpha,\beta}$. Then*

$$\overleftarrow{\Omega}_{\varphi}^{k}(f,\delta)_{w_{\alpha,\beta},q} \le c \left\| w_{\alpha,\beta} f \right\|_{\mathbb{L}_q[1-2k^2\delta^2,1]}.$$

Proof. Corollary 2.3 implies that f is non-negative and non-decreasing on $[0, 1]$ and so, for any $0 < h \le 2k^2\delta^2$, we have

$$\begin{aligned}\|w_{\alpha,\beta} \overleftarrow{\Delta}_h^k(f)\|_{\mathbb{L}_q[1-2k^2\delta^2,1]}^q &\le c \int_{1-2k^2\delta^2}^{1} \sum_{i=0}^{k} \left[\binom{k}{i}\right]^q w_{\alpha,\beta}^q(x) |f(x-ih)|^q \, dx \\ &\le c \sum_{i=0}^{k} \int_{1-2k^2\delta^2}^{1} w_{\alpha,\beta}^q(x) |f(x)|^q \, dx \le c \left\| w_{\alpha,\beta} f \right\|_{\mathbb{L}_q[1-2k^2\delta^2,1]}^q,\end{aligned}$$

and it remains to take supremum over $h \in (0, 2k^2\delta^2]$. $\square$

By Hölder's inequality, the following corollary is an immediate consequence of Lemma 4.1.

Corollary 4.2. *Let $k \in \mathbb{N}$, $1 \le q < p \le \infty$, $\alpha, \beta \in \mathbb{R}$, and $f \in \mathcal{M}_+^k \cap \mathbb{L}_p^{\alpha,\beta}$. Then*

$$\overleftarrow{\Omega}_{\varphi}^{k}(f,\delta)_{w_{\alpha,\beta},q} \le c\delta^{2/q-2/p} \left\| w_{\alpha,\beta} f \right\|_{\mathbb{L}_p[1-2k^2\delta^2,1]}.$$

Lemma 4.3. *Let $1 < q < \infty$, $\alpha, \beta \in \mathbb{R}$, and let $f \in \mathbb{L}_q^{\alpha,\beta}$ be nonnegative on $[-1, 1]$. Then,*

$$\omega_{\varphi}^{1}(f,\delta)_{w_{\alpha,\beta},q} \le c\omega_{\varphi}^{1}(f^q,\delta)_{w_{q\alpha,q\beta},1}^{1/q}.$$

Remark 4.4. If $f \in \mathcal{M}^1 \cap \mathbb{L}_q^{\alpha,\beta}$, $1 < q < \infty$, is nonnegative on $[-1, 1]$, then $f^q \in \mathcal{M}^1 \cap \mathbb{L}_1^{q\alpha,q\beta}$.

Proof. Let $1 < q < \infty$, and let $f \in \mathbb{L}_q^{\alpha,\beta}$ be nonnegative on $[-1, 1]$. It was shown in the proof of [7, Lemma 3.4] (and is easy to see) that,

$$\left|\Delta_{\mu}^{1}(f,x)\right|^q \le \left|\Delta_{\mu}^{1}(f^q,x)\right|, \quad \mu > 0.$$

This implies

$$\begin{aligned}\Omega_{\varphi}^{1}(f,\delta)_{w_{\alpha,\beta},q}^q &= \sup_{0<h\le\delta} \int_{-1+2h^2}^{1-2h^2} \left| w_{\alpha,\beta}(x) \Delta_{h\varphi(x)}^{1}(f,x)\right|^q dx \\ &\le \sup_{0<h\le\delta} \int_{-1+2h^2}^{1-2h^2} w_{\alpha,\beta}^q(x) \left|\Delta_{h\varphi(x)}^{1}(f^q,x)\right| dx = \Omega_{\varphi}^{1}(f^q,\delta)_{w_{q\alpha,q\beta},1}\end{aligned}$$

and, similarly,

$$\begin{aligned}\overleftarrow{\Omega}_{\varphi}^{1}(f,\delta)_{w_{\alpha,\beta},q}^q &= \sup_{0<h\le 2\delta^2} \int_{1-2\delta^2}^{1} \left| w_{\alpha,\beta}(x) | \overleftarrow{\Delta}_{h}^{1}(f,x)\right|^q dx \\ &\le \sup_{0<h\le 2\delta^2} \int_{1-2\delta^2}^{1} w_{\alpha,\beta}^q(x) \left|\overleftarrow{\Delta}_{h}^{1}(f^q,x)\right| dx = \overleftarrow{\Omega}_{\varphi}^{1}(f^q,\delta)_{w_{q\alpha,q\beta},1},\end{aligned}$$

and, since $\overrightarrow{\Omega}_{\varphi}^{1}(f,\delta)_{w_{\alpha,\beta},q}$ can be estimated similarly, the proof is complete. $\square$

Lemma 4.5. *Let $1 < q < \infty$, $\alpha, \beta \in \mathbb{R}$, and let $f \in \mathcal{M}^2 \cap \mathbb{L}_q^{\alpha,\beta}$ be nonnegative on $[-1, 1]$. Then, $f^q \in \mathcal{M}^2 \cap \mathbb{L}_1^{q\alpha,q\beta}$, and*

$$\omega_\varphi^2(f, \delta)_{w_{\alpha,\beta},q} \le c\omega_\varphi^2(f^q, \delta)_{w_{q\alpha,q\beta},1}^{1/q}.$$

Proof. It was shown in the proof of [7, Lemma 3.5] that, for any nonnegative convex function f,

$$\left(\Delta_\mu^2(f, x)\right)^q \le 2^{q-1}\Delta_\mu^2(f^q, x), \quad \mu > 0,$$

and the rest of the proof is analogous to that of Lemma 4.3. □

Now, taking into account that, for a nonnegative f, $\left\|w_{q\alpha,q\beta} f^q\right\|_{p/q}^{1/q} = \left\|w_{\alpha,\beta} f\right\|_p$, and using Lemmas 4.3, 4.5 and Corollary 3.3 (with p/q instead of p) we get the following result.

Corollary 4.6. *Let $k = 1$ or $k = 2$, $\beta \in \mathbb{R}$, $1 < q < p \le \infty$, $f \in \mathcal{M}_+^k \cap \mathbb{L}_p^{\beta,\beta}$, and $0 < \delta \le 1/(2k)$. Then*

$$\omega_\varphi^k(f, \delta)_{w_{\beta,\beta},q} \le c\left\|w_{\beta,\beta} f\right\|_p \begin{cases} \delta^{2/q-2/p}, & \text{if } k = 2 \text{ and } p < \infty, \text{ or } k = 1 \text{ and } p < 2q, \\ \delta^{2/q}|\ln\delta|^{1/q}, & \text{if } k = 2 \text{ and } p = \infty, \\ \delta^{1/q}|\ln\delta|^{1/(2q)}, & \text{if } k = 1 \text{ and } p = 2q, \\ \delta^{1/q}, & \text{if } k = 1 \text{ and } p > 2q. \end{cases}$$

Lemmas 2.4 and 2.5 now imply upper estimates in Theorem 1.1 for $k = 1$ and $k = 2$ and $q > 1$ except for the case $(k, p) = (2, \infty)$, which will be dealt with separately in the next section.

We will now finish the proof of the upper estimates in the case $k \ge 3$. It follows from [2, Theorem 6.2.5] that

$$\Omega_\varphi^k(f, \delta)_{w_{\alpha,\beta},q} \le c\Omega_\varphi^2(f, \delta)_{w_{\alpha,\beta},q}, \quad k \ge 3. \tag{4.1}$$

Now, suppose that $f \in \mathcal{M}_+^k \cap \mathbb{L}_p^{\beta,\beta}$, $k \ge 3$. Corollary 2.3 implies that $f \in \mathcal{M}_+^2$, and so using Corollary 4.2 and (4.1) we have

$$\begin{aligned}\omega_\varphi^k(f, \delta)_{w_{\beta,\beta},q} &\le c\Omega_\varphi^2(f, \delta)_{w_{\beta,\beta},q} + \overleftarrow{\Omega}_\varphi^k(f, \delta)_{w_{\beta,\beta},q} \\ &\le c\omega_\varphi^2(f, \delta)_{w_{\beta,\beta},q} + \delta^{2/q-2/p}\left\|w_{\beta,\beta} f\right\|_p.\end{aligned}$$

We have already proved that

$$\omega_\varphi^2(f, \delta)_{w_{\beta,\beta},q} \le \delta^{2/q-2/p}\left\|w_{\beta,\beta} f\right\|_p, \quad f \in \mathcal{M}_+^2 \cap \mathbb{L}_p^{\beta,\beta}, \tag{4.2}$$

in the case $q > 1$ and $p < \infty$, and will prove it for $q > 1$ and $p = \infty$ in the next section, and so upper estimates of Theorem 1.1 for $k \ge 3$ and $q > 1$ now follow from Lemmas 2.4 and 2.5.

Hence, in order to finish the proof of all upper estimates in Theorem 1.1 it remains to prove (4.2) in the case $q > 1$ and $p = \infty$. This is done in Section 5 (see Lemma 5.3).

5. Improvement of estimates for convex functions if $q > 1$

For $n \in \mathbb{N}$, we define $t_i := \cos(i\pi/n)$, $0 \le i \le n$, and $I_i := [t_i, t_{i-1}]$, $1 \le i \le n$. Recall that $(t_i)_0^n$ is the so-called Chebyshev partition of $[-1, 1]$. Some of its properties are stated in the following proposition that can be verified by straightforward computations.

Proposition 5.1. *For each $n \in \mathbb{N}$, the following statements are valid.*

(a) *For $2 \le i \le n-1$ and $x \in I_i$, $2\varphi(x)/n \le |I_i| \le 5\varphi(x)/n$, and $2n^{-2} \le |I_1| = |I_n| \le 5n^{-2}$.*
(b) $|I_{j-1}|/3 \le |I_i| \le 3|I_{j-1}|$, $2 \le i \le n$.
(c) *For any $n \in \mathbb{N}$, $1 \le j \le n$ and $\lambda \le 1/n$, $t_j + \lambda\varphi(t_j) \le t_{j-1} - \lambda\varphi(t_{j-1})$.*

Lemma 5.2. *Let $0 < \delta < 1/100$, $\beta \in \mathbb{R}$, and let $f \in \mathcal{M}^2 \cap \mathbb{L}_q^{\beta,\beta}$, $1 < q < \infty$, be such that its restrictions to $[-1, -1+100\delta^2]$ and $[1-100\delta^2, 1]$ are linear polynomials. Then*

$$\Omega_\varphi^2(f,\delta)_{\varphi^{2\beta},q} \le c\delta^{1/q-1}\Omega_\varphi^2(f,\delta)_{\varphi^{2\beta-1+1/q},1}.$$

Proof. First, note that, for $0 < h \le \delta$, if $|x| \ge 1 - 85\delta^2$ then $|x| - h\varphi(x) \ge 1 - 100\delta^2$, and so $\Delta_{h\varphi(x)}^2(f,x) = 0$ if $x \in [-1, -1+85\delta^2] \cup [1-85\delta^2, 1]$. Therefore,

$$\begin{aligned}\Omega_\varphi^2(f,\delta)_{w_{\beta,\beta},q}^q &= \sup_{0<h\le\delta} \left\| w_{\beta,\beta}\Delta_{h\varphi}^2(f)\right\|_{\mathbb{L}_q[-1+8h^2,1-8h^2]}^q \\ &\le \sup_{0<h\le\delta} \left\| w_{\beta,\beta}\Delta_{h\varphi}^2(f)\right\|_{\mathbb{L}_q[-1+85\delta^2,1-85\delta^2]}^q.\end{aligned}$$

Now, note that, for each $m \in \mathbb{N}$ and $n \ge 2m+1$, if $\eta \ge 5m^2/n^2$, then $[-1+\eta, 1-\eta] \subset [t_{n-m}, t_m]$. Hence, if we let $n := \lfloor 1/\delta \rfloor$ then $[-1+85\delta^2, 1-85\delta^2] \subset [t_{n-4}, t_4] = \bigcup_{i=5}^{n-4} I_i$, and so

$$\Omega_\varphi^2(f,\delta)_{w_{\beta,\beta},q}^q \le \sup_{0<h\le\delta} \sum_{i=5}^{n-4} \int_{I_i} |w_{\beta,\beta}(x)\Delta_{h\varphi(x)}^2(f,x)|^q\,dx.$$

Since $h \le \delta \le 1/n$, Proposition 5.1(c) implies that if $x \in I_i$, then $x \pm h\varphi(x) \in \widetilde{I}_i := [t_{i+1}, t_{i-2}]$.

Now, for $5 \le i \le n-4$, let p_i be the linear polynomial interpolating f at the endpoints of $\widetilde{I}_i$, and let $g_i := f - p_i$. If $x_0 \in \widetilde{I}_i$ is such that $\|g_i\|_{\mathbb{C}(\widetilde{I}_i)} = |g_i(x_0)|$ (recall that convex functions are continuous in the interior of their domains), using the fact that g_i is convex (and so lies below its secant lines) and is 0 at the endpoints of $\widetilde{I}_i$, we get

$$\frac{1}{2}|\widetilde{I}_i|\,\|g_i\|_{\mathbb{C}(\widetilde{I}_i)} = \frac{1}{2}|\widetilde{I}_i||g_i(x_0)| \le \int_{\widetilde{I}_i} |g_i(x)|\,dx,$$

and so

$$\|f - p_i\|_{\mathbb{C}(\widetilde{I}_i)} \le 2|\widetilde{I}_i|^{-1}\,\|f - p_i\|_{\mathbb{L}_1(\widetilde{I}_i)}, \quad 5 \le i \le n-4.$$

Therefore, recalling that $w_{\beta,\beta} = \varphi^{2\beta}$ and using the fact that $w_{\beta,\beta}(x) \sim w_{\beta,\beta}(t_i)$, $x \in I_i$, and Proposition 5.1(a) we have

$$\begin{aligned}\Omega_\varphi^2(f,\delta)_{w_{\beta,\beta},q}^q &\le \sup_{0<h\le\delta} \sum_{i=5}^{n-4} \int_{I_i} \left|\varphi^{2\beta}(x)\Delta_{h\varphi(x)}^2(f-p_i,x)\right|^q\,dx \\ &\le c\sum_{i=5}^{n-4} \varphi^{2\beta q}(t_i)|I_i|\,\|f-p_i\|_{\mathbb{C}(\widetilde{I}_i)}^q \\ &\le c\sum_{i=5}^{n-4} \varphi^{2\beta q}(t_i)|I_i|^{1-q}\,\|f-p_i\|_{\mathbb{L}_1(\widetilde{I}_i)}^q\end{aligned}$$

$$\leq c\sum_{i=5}^{n-4} n^{q-1}\varphi^{2\beta q-q+1}(t_i)\,\|f-p_i\|^q_{\mathbb{L}_1(\widetilde{I}_i)}$$
$$\leq cn^{q-1}\left(\sum_{i=5}^{n-4}\varphi^{2\beta-1+1/q}(t_i)\,\|f-p_i\|_{\mathbb{L}_1(\widetilde{I}_i)}\right)^q,$$

where, in the last estimate, we used the inequality $\sum|a_i|^q \leq (\sum|a_i|)^q$.

It follows from [6, Theorem 1] that

$$\|f-p_i\|_{\mathbb{L}_1(\widetilde{I}_i)} \leq c\omega_2(f,|\widehat{I}_i|,\widehat{I}_i)_1, \quad 5\leq i\leq n-4,$$

where $\widehat{I}_i := [t_{i+2}, t_{i-3}]$ (since $\widetilde{I}_i$ is in the "interior" of $\widehat{I}_i$), and $\omega_2(f,\mu,I)$ is the usual second modulus on I. Proposition 5.1(a, b) implies that $n|\widehat{I}_i|/\varphi(x) \sim 1$, $x\in\widehat{I}_i$, and, in particular, $|\widehat{I}_i|/\varphi(x)\leq c_*/n$, for some absolute constant c_*. Now, [12, Lemma 7.2, p. 191] yields

$$\omega_2(f,\mu,[a,b])_1 \leq \frac{c}{\mu}\int_0^\mu\int_a^b |\Delta_h^2(f,x,[a,b])|\,dx\,dh,$$

and hence

$$\begin{aligned}\omega_2(f,|\widehat{I}_i|,\widehat{I}_i)_1 &\leq c\omega_2(f,|\widehat{I}_i|/(2c_*),\widehat{I}_i)_1\\ &\leq \frac{c}{|\widehat{I}_i|}\int_{\widehat{I}_i}\int_0^{|\widehat{I}_i|/(2c_*)}|\Delta_h^2(f,x,\widehat{I}_i)|\,dh\,dx\\ &\leq \frac{c}{|\widehat{I}_i|}\int_{\widehat{I}_i}\int_0^{|\widehat{I}_i|/(2c_*\varphi(x))}\varphi(x)|\Delta^2_{h\varphi(x)}(f,x,\widehat{I}_i)|\,dh\,dx\\ &\leq cn\int_{\widehat{I}_i}\int_0^{1/(2n)}|\Delta^2_{h\varphi(x)}(f,x)|\,dh\,dx.\end{aligned}$$

Therefore,

$$\begin{aligned}\Omega_\varphi^2(f,\delta)^q_{w_{\beta,\beta},q} &\leq cn^{q-1}\left(\sum_{i=5}^{n-4}\varphi^{2\beta-1+1/q}(t_i)n\int_{\widehat{I}_i}\int_0^{1/(2n)}|\Delta^2_{h\varphi(x)}(f,x)|\,dh\,dx\right)^q\\ &\leq cn^{2q-1}\left(\int_0^{1/(2n)}\sum_{i=5}^{n-4}\int_{\widehat{I}_i}\varphi^{2\beta-1+1/q}(x)|\Delta^2_{h\varphi(x)}(f,x)|\,dx\,dh\right)^q\\ &\leq cn^{q-1}\left(\sup_{0<h\leq 1/(2n)}\int_{t_{n-1}}^{t_1}\varphi^{2\beta-1+1/q}(x)|\Delta^2_{h\varphi(x)}(f,x)|\,dx\right)^q\\ &\leq cn^{q-1}\left(\sup_{0<h\leq 1/(2n)}\int_{-1+8h^2}^{1-8h^2}\varphi^{2\beta-1+1/q}(x)|\Delta^2_{h\varphi(x)}(f,x)|\,dx\right)^q\\ &\leq cn^{q-1}\Omega_\varphi^2(f,1/(2n))^q_{\varphi^{2\beta-1+1/q},1},\end{aligned}$$

and it remains to recall that $n=\lfloor 1/\delta\rfloor$ and so, in particular, $1/(2n)<\delta\leq 1/n$. $\square$

Lemma 5.3. *Let $\beta\in\mathbb{R}$, $1<q<\infty$ and $f\in\mathcal{M}_+^2\cap\mathbb{L}_\infty^{\beta,\beta}$. Then*

$$\omega_\varphi^2(f,\delta)_{w_{\beta,\beta},q} \leq c\delta^{2/q}\left\|w_{\beta,\beta}f\right\|_\infty.$$

Proof. Let $0 < \delta < 1/100$, denote $x_0 := 1 - 100\delta^2$, and define

$$f_1(x) := \begin{cases} f(x), & \text{if } x \le x_0, \\ f(x_0) + f'_+(x_0)(x - x_0), & \text{if } x_0 < x \le 1. \end{cases}$$

Clearly, $f_1 \in \mathcal{M}^2_+$ and, since $0 \le f_1(x) \le f(x)$, $x_0 \le x \le 1$, we conclude that $\|w_{\beta,\beta} f_1\|_\infty \le \|w_{\beta,\beta} f\|_\infty$. Also, $f_2 := f - f_1 \in \mathcal{M}^2_+$ is such that $f_2(x) = 0$ if $x \le x_0$ and $\|w_{\beta,\beta} f_2\|_\infty \le \|w_{\beta,\beta} f\|_\infty$, and so Lemma 4.5 and Corollary 3.6 imply that

$$\omega^2_\varphi(f_2, \delta)_{w_{\beta,\beta},q} \le c\omega^2_\varphi(f_2^q, \delta)^{1/q}_{w_{q\beta,q\beta},1} \le c\left(\delta^2 \left\|w_{q\beta,q\beta} f_2^q\right\|_\infty\right)^{1/q} \le c\delta^{2/q} \left\|w_{\beta,\beta} f\right\|_\infty .$$

Now, since $\overleftarrow{\Omega}^2_\varphi(f_1, \delta)_{w_{\beta,\beta},q} = 0$, by Lemma 5.2 and Theorem 3.2 we have

$$\begin{aligned}
\omega^2_\varphi(f_1, \delta)_{w_{\beta,\beta},q} &= \Omega^2_\varphi(f_1, \delta)_{w_{\beta,\beta},q} \le c\delta^{1/q-1} \Omega^2_\varphi(f_1, \delta)_{\varphi^{2\beta-1+1/q},1} \\
&\le c\delta^{1/q-1} \left\|\varphi^{2\beta-1+1/q} f_1\right\|_{\mathbb{L}_1[1-12\delta^2,1]} + c\delta^{1/q-1} \sup_{0<h\le\delta} h^2 \left\|\varphi^{2\beta-3+1/q} f_1\right\|_{\mathbb{L}_1[0,1-8h^2]} \\
&\le c\delta^{1/q-1} \left\|\varphi^{2\beta} f_1\right\|_\infty \left\|\varphi^{-1+1/q}\right\|_{\mathbb{L}_1[1-12\delta^2,1]} \\
&\quad + c\delta^{1/q-1} \left\|\varphi^{2\beta} f_1\right\|_\infty \sup_{0<h\le\delta} h^2 \left\|\varphi^{-3+1/q}\right\|_{\mathbb{L}_1[0,1-8h^2]} \\
&\le c\delta^{2/q} \left\|w_{\beta,\beta} f\right\|_\infty ,
\end{aligned}$$

where, in the last estimate, we used

$$\left\|\varphi^{-\gamma}\right\|_{\mathbb{L}_1[1-c\delta^2,1]} \le c\delta^{-\gamma+2}, \quad \text{if } \gamma < 2,$$

and

$$\left\|\varphi^{-\gamma}\right\|_{\mathbb{L}_1[0,1-ch^2]} \le ch^{-\gamma+2}, \quad \text{if } \gamma > 2. \quad \square$$

Together with Lemmas 2.4 and 2.5, this now completes the proof of the upper estimate in Theorem 1.1 in the case $k = 2$, $p = \infty$ and $q > 1$.

6. Lower estimates of moduli

The following lemma verifies the lower estimate in (1.2).

Lemma 6.1. *Let $k \in \mathbb{N}$, $\alpha, \beta \in \mathbb{R}$, $0 < p, q \le \infty$, and $0 < \delta \le 1/(2k)$. Then the function*

$$f_\delta(x) := \begin{cases} (-1)^i, & \text{if } x \in J_i,\ 0 \le i \le \lfloor 1/(2k\delta)\rfloor, \\ 0, & \text{otherwise}, \end{cases}$$

where $J_i := [k\delta i, k\delta(i + 1/2)]$, is such that $\left\|w_{\alpha,\beta} f_\delta\right\|_p \sim 1$, and

$$\Omega^k_\varphi(f_\delta, \delta)_{w_{\alpha,\beta},q} \ge c > 0.$$

Proof. Since $\bigcup_{i=0}^{\lfloor 1/(2k\delta)\rfloor} J_i \subset [0, 3/4]$,

$$\left\|w_{\alpha,\beta} f_\delta\right\|_p^p \sim \sum_{i=0}^{\lfloor 1/(2k\delta)\rfloor} |J_i| = (\lfloor 1/(2k\delta)\rfloor + 1)\, k\delta/2 \sim 1.$$

Now, note that, if $x \in J_i$ and $0 < h \le \delta$, then $x \pm kh\varphi(x)/2 \notin \cup_{j\neq i} J_j$, and so

$$\Omega_\varphi^k(f_\delta,\delta)^q_{w_{\alpha,\beta},q} \ge \sup_{0<h\le\delta} \sum_{i=0}^{\lfloor 1/(2k\delta)\rfloor} \int_{D_i} w^q_{\alpha,\beta}(x)\,dx \sim \sup_{0<h\le\delta} \sum_{i=0}^{\lfloor 1/(2k\delta)\rfloor} |D_i|,$$

where

$$D_i := \left\{ x \;\middle|\; x + (k/2-1)h\varphi(x) \le k\delta i \le x + kh\varphi(x)/2 \right\}.$$

Since $|D_i| \sim h$, $0 \le i \le \lfloor 1/(2k\delta)\rfloor$, we have

$$\Omega_\varphi^k(f_\delta,\delta)^q_{w_{\alpha,\beta},q} \ge c\delta\lfloor 1/(2k\delta)\rfloor \ge c. \quad \square$$

Remark 6.2. For each $n \in \mathbb{N}$, letting $k = 1$ and $\delta := 1/(4n)$ in Lemma 6.1, noting that f_δ is positive on $n+1$ intervals and negative on n intervals J_i, and that any polynomial of degree $\le n$ can have at most n sign changes on $[-1, 1]$, we conclude that

$$E_n(f_\delta)_{w_{\alpha,\beta},q} \ge c(n\delta)^{1/q} \ge c > 0.$$

This implies that, for any $\alpha, \beta \in \mathbb{R}$ and $0 < p, q \le \infty$,

$$\mathcal{E}(\mathbb{S}_p^{\alpha,\beta}, \mathbb{P}_n)_{w_{\alpha,\beta},q} \ge c > 0.$$

The following result verifies the lower estimate in (1.4) in the case $k = 1$ and $p > 2q$. Its proof is elementary and will be omitted.

Lemma 6.3. *If $f(x) = \chi_{[0,1]}(x)$, $\alpha \in \mathbb{R}$ and $\beta \in J_p$, then $f \in \mathcal{M}^1$, $\|w_{\alpha,\beta} f\|_p \sim 1$, and $\omega_\varphi^1(f,\delta)_{w_{\alpha,\beta},q} \sim \delta^{1/q}$, for any $0 < \delta < 1$.*

Lemma 6.4. *Let $k \in \mathbb{N}$, $0 < p, q \le \infty$, $\alpha \in \mathbb{R}$, $\beta \in J_p$, $\delta > 0$, and $0 < \varepsilon \le \min\{2k^2\delta^2, 1\}$. Then the function $f(x) := \lambda(x-1+\varepsilon)_+^{k-1}$, $\lambda := \varepsilon^{-k-\beta-1/p+1}$, is such that $f \in \mathcal{M}^k$, $\|w_{\alpha,\beta} f\|_p \sim 1$, and*

$$\omega_\varphi^k(f,\delta)_{w_{\alpha,\beta},q} \ge c\varepsilon^{1/q-1/p}.$$

Proof. It is straightforward to check that $\|w_{\alpha,\beta} f\|_p \sim 1$. Now, since $S_\varepsilon(h) := [1-\varepsilon, 1-\varepsilon+\min\{\varepsilon, h\}/2] \subset [1-2k^2\delta^2, 1]$ and $\overleftarrow{\Delta}_h^k(f, x) = f(x)$, $x \in S_\varepsilon(h)$, we have

$$\begin{aligned}
\overleftarrow{\Omega}_\varphi^k(f,\delta)^q_{w_{\alpha,\beta},q} &= \sup_{0<h\le 2k^2\delta^2} \left\| w_{\alpha,\beta} \overleftarrow{\Delta}_h^k(f) \right\|^q_{\mathbb{L}_q[1-2k^2\delta^2,1]} \\
&\ge \sup_{0<h\le 2k^2\delta^2} \int_{S_\varepsilon(h)} |w_{\alpha,\beta}(x) f(x)|^q\,dx \\
&\ge c \sup_{0<h\le 2k^2\delta^2} \int_{S_\varepsilon(h)} \varepsilon^{q\beta}\lambda^q (x-1+\varepsilon)^{kq-q}\,dx \\
&\ge c \sup_{0<h\le 2k^2\delta^2} \varepsilon^{q\beta}\lambda^q (\min\{\varepsilon, h\})^{kq-q+1} \\
&\ge c\lambda^q \varepsilon^{q\beta+kq-q+1}.
\end{aligned}$$

Therefore,

$$\omega_\varphi^k(f,\delta)_{w_{\alpha,\beta},q} \ge \overleftarrow{\Omega}_\varphi^k(f,\delta)_{w_{\alpha,\beta},q} \ge c\varepsilon^{1/q-1/p}.$$

If p and/or q are ∞, the proof is similar. $\square$

Since $\lim_{\varepsilon\to 0^+}\varepsilon^{1/q-1/p} = \infty$ if $p<q$, we immediately get the following corollary.

Corollary 6.5. *Let $k\in\mathbb{N}$, $\alpha,\beta\in\mathbb{R}$, $\delta>0$, and $0<p<q\le\infty$. Then, for any $A>0$, there exists $f\in\mathbb{S}_p^{\alpha,\beta}\cap\mathcal{M}^k$ such that*

$$\omega_\varphi^k(f,\delta)_{w_{\alpha,\beta},q} \ge A.$$

This corollary confirms that one cannot expect to get any useful upper estimates for the moduli ω_φ^k (even restricting classes to k-monotone function) if $p<q$.

Corollary 6.6. *Let $k\in\mathbb{N}$, $0<p,q\le\infty$, $\alpha\in\mathbb{R}$, $\beta\in J_p$, $0<\delta\le 1/(2k)$, and $\varepsilon := 2k^2\delta^2$. Then the function $f(x) := \lambda(x-1+\varepsilon)_+^{k-1}$, $\lambda := \varepsilon^{-k-\beta-1/p+1}$, is such that $f\in\mathcal{M}^k$, $\|w_{\alpha,\beta}f\|_p \sim 1$, and*

$$\omega_\varphi^k(f,\delta)_{w_{\alpha,\beta},q} \ge c\delta^{2/q-2/p}.$$

This corollary verifies the lower estimates in (1.4) in the cases $k\ge 2$ and $(k,q,p)\ne(2,1,\infty)$ (unless $\alpha=\beta=0$), and $k=1$ and $p<2q$.

The following lemma yields the lower estimate in (1.4) in the case $(k,q,p)=(2,1,\infty)$ and $(\alpha,\beta)\ne(0,0)$.

Lemma 6.7 (*Lower Estimate in the Case $k=2$, $q=1$ and $p=\infty$*)**.** *Let $\beta>0$ and $f(x) := (1-x)^{-\beta}$. Then $f\in\mathcal{M}^2\cap\mathbb{S}_\infty^{0,\beta}$ and, if $\delta<1/5$,*

$$\Omega_\varphi^2(f,\delta)_{w_{0,\beta},1} \ge c\delta^2|\ln\delta|.$$

Proof. It is obvious that $f\in\mathcal{M}^2\cap\mathbb{S}_\infty^{0,\beta}$. Using the fact that

$$\Delta_{h\varphi(x)}^2(f,x) = h^2\varphi^2(x)f''(\xi), \quad \text{for some } \xi\in(x-h\varphi(x), x+h\varphi(x)),$$

we have

$$\Omega_\varphi^2(f,\delta)_{w_{0,\beta},1} \ge c\int_0^{1-8\delta^2}(1-x)^\beta\delta^2\varphi^2(x)|f''(\xi_x)|\,dx,$$

where $\xi_x\in(x-\delta\varphi(x), x+\delta\varphi(x))$. Now, Proposition 3.1(e) implies that

$$1-\xi_k \sim 1-x\pm\delta\varphi(x)\sim 1-x,$$

and so $|f''(\xi_x)|\ge c(1-x)^{-\beta-2}$. Therefore,

$$\Omega_\varphi^2(f,\delta)_{w_{0,\beta},1} \ge c\delta^2\int_0^{1-8\delta^2}(1-x)^{-1}\,dx \ge c\delta^2|\ln\delta|. \quad \square$$

We conclude this section with the proof of the lower estimate in (1.5).

Lemma 6.8 (*Lower Estimate in the Case $k=1$ and $p=2q$*)**.** *Let $1\le q<\infty$, $p=2q$, $\beta>-1/p$, $0<\delta<1/4$, and $\lambda>1$. Then there exists a function $f\in\mathbb{S}_p^{\beta,\beta}\cap\mathcal{M}_+^1$ such* *that*

$$\Omega_\varphi^1(f,\delta)_{w_{\beta,\beta},q} \ge c\frac{\delta^{1/q}|\ln\delta|^{1/(2q)}}{|\ln|\ln\delta||^{\lambda/(2q)}}. \tag{6.1}$$

Proof. Let $n = 2^m$, where $m = \lfloor \log_2(1/\delta)\rfloor + 1$, and note that $1/n < \delta \le 2/n$.

Suppose that $(f_i)_1^n$ is a non-increasing sequence of real numbers such that $f_i = 0$, for $i > n/2$. Now, recalling that $t_i = \cos(i\pi/n)$, $0 \le i \le n$, and $I_i = [t_i, t_{i-1}]$, $1 \le i \le n$, define

$$f(x) := f_i, \quad t_i < x \le t_{i-1},\ 1 \le i \le n.$$

In other words, f is a non-decreasing piecewise constant spline with knots at t_i's which is identically equal to 0 on $[-1, 0]$, *i.e.*, $f \in \mathcal{M}_+^1$.

Now, using Proposition 5.1, the fact that $2i/n \le \varphi(t_i) \le 4i/n$, $1 \le i \le n/2$, and denoting $\sum := \sum_{i=1}^{n/2}$, we have

$$\begin{aligned}\|w_{\beta,\beta} f\|_p^p &= \sum\int_{I_i}\varphi^{2\beta p}(x)|f(x)|^p\,dx \le c\sum|I_i|\varphi^{2\beta p}(t_i)f_i^p\\ &\le cn^{-1}\sum\varphi^{2\beta p+1}(t_i)f_i^p \le cn^{-2\beta p-2}\sum i^{2\beta p+1}f_i^p.\end{aligned}$$

Now, let

$$\begin{aligned}D_i(h) &:= \left\{x \;\middle|\; x - h\varphi(x)/2 \le t_i \le x + h\varphi(x)/2\right\}\\ &= \left[\frac{t_i - (h/2)\sqrt{1-t_i^2+h^2/4}}{1+h^2/4}, \frac{t_i + (h/2)\sqrt{1-t_i^2+h^2/4}}{1+h^2/4}\right], \quad 1 \le i \le n-1.\end{aligned}$$

We note that intervals $D_i(h)$, $1 \le i \le n-1$, have the following properties:

(i) if $0 < h \le 1/n$, then $D_i(h) \cap D_{i-1}(h) = \emptyset$ for all $2 \le i \le n-1$;
(ii) if $0 < h \le 1/(2n)$, then $D_i(h) \subset [-1+2h^2, 1-2h^2]$ for all $1 \le i \le n-1$;
(iii) $|D_i(h)| \ge h\varphi(t_i)/2$, $1 \le i \le n-1$.

In order to verify (i), we suppose that $D_i(h) \cap D_{i-1}(h) \ne \emptyset$. Then there is $x \in [t_i, t_{i-1}]$ such that $x - h\varphi(x)/2 \le t_i$ and $x + h\varphi(x)/2 \ge t_{i-1}$. Then, $t_{i-1} - h\varphi(x)/2 \le x \le t_i + h\varphi(x)/2$, which implies $t_{i-1} - h\varphi(x)/2 \le t_i + h\varphi(x)/2$, and so

$$t_{i-1} - t_i \le h\varphi(x), \quad \text{for some } x \in [t_i, t_{i-1}].$$

At the same time, it is known that $|I_i| := t_{i-1} - t_i$ satisfies $\rho_n(x) \le |I_i|$, for any $1 \le i \le n$ and $x \in [t_i, t_{i-1}]$, where $\rho_n(x) := \sqrt{1-x^2}/n + 1/n^2$ (see *e.g.* [3], or this can be verified directly). Therefore,

$$h\varphi(x) \le \varphi(x)/n < \rho_n(x) \le t_{i-1} - t_i,$$

for any $x \in [t_i, t_{i-1}]$, which is a contradiction.

In order to verify (ii), we note that, in the case $i = 1$ (which implies (ii) for all $1 \le i \le n-1$), (ii) follows from the observation that, if $x = 1 - 2h^2$, then $x - h\varphi(x)/2 > t_1 = \cos(\pi/n)$. This inequality is equivalent to

$$\cos(\pi/n) < 1 - 2h^2 - h^2\sqrt{1-h^2} \iff 2h^2 + h^2\sqrt{1-h^2} < 2\sin^2(\pi/(2n)),$$

which is true since

$$(2h^2 + h^2\sqrt{1-h^2})/2 \le 3h^2/2 \le 3/(8n^2) \quad \text{and}$$
$$\sin^2(\pi/(2n)) \ge [(2/\pi)\pi/(2n)]^2 = 1/n^2.$$

Finally, (iii) immediately follows from

$$|D_i(h)| = \frac{h\sqrt{1-t_i^2+h^2/4}}{1+h^2/4} \ge \frac{h\varphi(t_i)}{1+h^2/4} \ge \frac{h\varphi(t_i)}{2}.$$

Therefore, letting $h := 1/(2n)$ we have

$$\begin{aligned}
\Omega_\varphi^1(f, 1/n)_{w_{\beta,\beta},q}^q &\ge \int_{-1+2h^2}^{1-2h^2} \varphi^{2\beta q}(x)\left(\Delta_{h\varphi}^1(f,x)\right)^q \, dx \\
&\ge \sum \int_{D_i(h)} \varphi^{2\beta q}(x)\left(\Delta_{h\varphi}^1(f,x)\right)^q \, dx \\
&\ge c\sum \int_{D_i(h)} \varphi^{2\beta q}(t_i)\,(f_i - f_{i+1})^q \, dx \ge c\sum h\varphi^{2\beta q+1}(t_i)\,(f_i - f_{i+1})^q \\
&\ge cn^{-2\beta q-2}\sum i^{2\beta q+1}\,(f_i - f_{i+1})^q\,.
\end{aligned}$$

Now, define

$$f_i := \begin{cases} 2^{2\beta(m-k)+2(m-k)/p}\zeta_k^{1/p}, & \text{if } 2^k \le i \le 2^{k+1}-1,\ 0 \le k \le m-2, \\ 0, & \text{if } i \ge 2^{m-1} \end{cases}$$

where (ζ_k) is a non-increasing sequence to be chosen later. Observe that $\left(2^{-2\beta k-2k/p}\right)_k$ is non-increasing since $\beta > -1/p$. Then,

$$\left\| w_{\beta,\beta} f \right\|_p^p \le c\sum_{k=0}^{m-2}\sum_{i=2^k}^{2^{k+1}-1} i^{2\beta p+1}2^{-2\beta kp-2k}\zeta_k \le c\sum_{k=0}^{m-2}\zeta_k$$

and

$$\begin{aligned}
\Omega_\varphi^1(f, 2^{-m})_{w_{\beta,\beta},q}^q &\ge c2^{-2\beta mq-2m}\sum_{k=0}^{m-2} 2^{2\beta kq+k} \\
&\quad \times \left(2^{2\beta(m-k)+2(m-k)/p}\zeta_k^{1/p} - 2^{2\beta(m-k-1)+2(m-k-1)/p}\zeta_{k+1}^{1/p}\right)^q \\
&\ge c2^{-m}\sum_{k=0}^{m-2}\left(\zeta_k^{1/p} - 2^{-2\beta-2/p}\zeta_{k+1}^{1/p}\right)^q \\
&\ge c2^{-m}\left(1-2^{-2\beta-2/p}\right)^q\sum_{k=0}^{m-2}\zeta_k^{1/2}.
\end{aligned}$$

Now, let $\zeta_k := (k+2)^{-1}(\ln(k+2))^{-\lambda}$, where $\lambda > 1$. Then,

$$\left\| w_{\beta,\beta} f \right\|_p^p \le c\sum_{k=0}^{\infty}(k+2)^{-1}(\ln(k+2))^{-\lambda} \le c$$

and

$$\Omega_\varphi^1(f, 2^{-m})_{w_{\beta,\beta},q}^q \ge c2^{-m} \sum_{k=0}^{m-2} (k+2)^{-1/2}(\ln(k+2))^{-\lambda/2} \ge c2^{-m} m^{1/2} (\ln m)^{-\lambda/2}.$$

Finally, recalling that $2^{-m} < \delta \le 2^{1-m}$ and replacing f with $g := \|w_{\beta,\beta} f\|_p^{-1} f$ we get a function in $\mathbb{S}_p^{\beta,\beta} \cap \mathcal{M}_+^1$ such that

$$\Omega_\varphi^1(g, \delta)_{w_{\beta,\beta},q} \ge \|w_{\beta,\beta} f\|_p^{-1} \Omega_\varphi^1(f, 2^{-m})_{w_{\beta,\beta},q} \ge c \frac{\delta^{1/q} |\ln \delta|^{1/(2q)}}{|\ln |\ln \delta||^{\lambda/(2q)}}. \quad \square$$

Remark 6.9. One can improve the estimate (6.1) slightly by letting

$$\zeta_k := (g_{m,\lambda}(c(k+1)))^{-1},$$

where

$$g_{m,\lambda}(x) := x(\ln x)(\ln\ln x) \cdots (\underbrace{\ln \cdots \ln}_{m} x)(\underbrace{\ln \cdots \ln}_{m+1} x)^\lambda,$$

with $m \in \mathbb{N}$, $\lambda > 1$ and a sufficiently large constant $c = c(m)$ that guarantees that $g_{m,\lambda}$ is well defined on $[c, \infty)$.

7. Proof of Theorem 1.5

It was proved by Luther and Russo [10, Corollary 2.2] that, for $\alpha, \beta \ge 0$, there exists $n_0 \in \mathbb{N}$ such that

$$E_n(f)_{w_{\alpha,\beta},q} \le c\omega_\varphi^k(f, n^{-1})_{w_{\alpha,\beta},q}, \quad n \ge n_0. \tag{7.1}$$

If $\alpha = \beta = 0$, then this is a well known Jackson type estimate that was proved by Ditzian and Totik in [2, Theorem 7.2.1]. Taking into account that, for $0 \le n < n_0$, $E_n(f)_{w_{\alpha,\beta},q} \le c\|w_{\alpha,\beta} f\|_q \le c\|w_{\alpha,\beta} f\|_p$, if $q \le p$, we immediately get the following corollary of Theorem 1.1 that implies all upper estimates in Theorem 1.5.

Corollary 7.1. *Let $1 \le q < p \le \infty$, $k \in \mathbb{N}$, $\alpha, \beta \ge 0$, and let $f \in \mathcal{M}^k \cap \mathbb{L}_p^{\alpha,\beta}$. Then, for any $n \in \mathbb{N}$,*

$$E_n(f)_{w_{\alpha,\beta},q}$$
$$\le c\|w_{\alpha,\beta} f\|_p \begin{cases} n^{-2/q+2/p}, & \text{if } k \ge 2, \text{ and } (k,q,p) \ne (2,1,\infty), \\ n^{-2}\ln(n+1), & \text{if } k = 2,\ q = 1,\ p = \infty, \text{ and } (\alpha,\beta) \ne (0,0), \\ n^{-2}, & \text{if } k = 2,\ q = 1,\ p = \infty, \text{ and } \alpha = \beta = 0, \\ n^{-2/q+2/p}, & \text{if } k = 1 \text{ and } p < 2q, \\ n^{-1/q}[\ln(n+1)]^{1/(2q)}, & \text{if } k = 1 \text{ and } p = 2q, \\ n^{-1/q}, & \text{if } k = 1 \text{ and } p > 2q. \end{cases} \tag{7.2}$$

A matching inverse result to (7.1) is given by (see [2, Theorem 8.2.4])

$$\omega_\varphi^k(f, \delta)_{w_{\alpha,\beta},q} \le c\delta^k \sum_{0 \le i < 1/\delta} (i+1)^{k-1} E_i(f)_{w_{\alpha,\beta},q}. \tag{7.3}$$

Since, for $\mu, \lambda \in \mathbb{R}$ and $0 < \delta < 1/4$,

$$\int_2^{1/\delta} x^\mu (\ln x)^\lambda \, dx \sim \begin{cases} 1, & \text{if } \mu < -1, \\ \delta^{-\mu-1} |\ln \delta|^\lambda, & \text{if } \mu > -1, \\ 1, & \text{if } \mu = -1, \lambda < -1, \\ |\ln \delta|^{1+\lambda}, & \text{if } \mu = -1, \lambda > -1, \\ \ln |\ln \delta|, & \text{if } \mu = -1, \lambda = -1, \end{cases}$$

estimate (7.3) implies, in particular, that if for a function $f \in \mathbb{S}_p^{\alpha,\beta} \cap \mathcal{M}^k$,

$$E_n(f)_{w_{\alpha,\beta},q} \le c(n+2)^{\mu-k+1} [\ln(n+2)]^\lambda, \quad n \in \mathbb{N}_0,$$

then

$$\omega_\varphi^k(f,\delta)_{w_{\alpha,\beta},q} \le c \begin{cases} \delta^k, & \text{if } \mu < -1, \\ \delta^{k-\mu-1} |\ln \delta|^\lambda, & \text{if } \mu > -1, \\ \delta^k, & \text{if } \mu = -1, \lambda < -1, \\ \delta^k |\ln \delta|^{1+\lambda}, & \text{if } \mu = -1, \lambda > -1, \\ \delta^k \ln |\ln \delta|, & \text{if } \mu = -1, \lambda = -1. \end{cases}$$

Together with lower estimates in Theorem 1.1 this implies that none of the powers of n in (7.2) can be decreased (except for some cases when $q = 1$ and $k \le 2$). This is made precise in Corollaries 9.4 and 9.5 which imply lower estimates in (1.8)–(1.10).

Whether or not powers of $\ln(n+1)$ in (7.2) can be decreased is more involved. In the case $k = 2$, $q = 1$, $p = \infty$ and $(\alpha, \beta) \ne (0, 0)$, we only know that

$$cn^{-2} \le \sup_{f \in \mathcal{M}^2 \cap \mathbb{S}_\infty^{\alpha,\beta}} E_n(f)_{w_{\alpha,\beta},1} \le cn^{-2} \ln(n+1)$$

(see Corollary 9.5 with $r = 0$ for the lower estimate), and so it is an open problem if $\ln(n+1)$ in this estimate can be replaced by $o(\ln(n+1))$ or removed altogether.

In the case $k = 1$ and $p = 2q$, if $E_n(f)_{w_{\alpha,\beta},q} \le c(n+2)^{-1/q} [\ln(n+2)]^\lambda$, $n \in \mathbb{N}_0$ (*i.e.*, $\mu = -1/q$), for any function $f \in \mathcal{M}^1 \cap \mathbb{S}_p^{\alpha,\beta}$, then

$$\omega_\varphi^1(f,\delta)_{w_{\alpha,\beta},q} \le c\delta^{1/q} |\ln \delta|^\lambda, \quad \text{if } q > 1.$$

Together with lower estimates of Theorem 1.1 this implies that, if $k = 1$ and $p/2 = q > 1$, then the quantity $n^{-1/q}[\ln(n+1)]^{1/(2q)}$ in (7.2) cannot be replaced by $n^{-1/q}[\ln(n+1)]^{1/(2q)-\varepsilon}$, for any $\varepsilon > 0$. Also, this yields (1.11).

If $k = 1$, $q = 1$ and $p = 2$, then we know that (see Corollary 9.4 with $k = 1$ for the lower estimate)

$$cn^{-1} \le \sup_{f \in \mathcal{M}^1 \cap \mathbb{S}_2^{\alpha,\beta}} E_n(f)_{w_{\alpha,\beta},1} \le cn^{-1} [\ln(n+1)]^{1/2},$$

and it is an open problem if $[\ln(n+1)]^{1/2}$ in this estimate is necessary.

8. Other applications

1. Let $1 \le p \le \infty$, $r \in \mathbb{N}$. Then

$$\mathbb{L}_{p,r}^{\alpha,\beta} := \left\{ f : [-1,1] \mapsto \mathbb{R} \;\middle|\; f^{(r-1)} \in \mathrm{AC}_{\mathrm{loc}}(-1,1) \text{ and } \left\| w_{\alpha,\beta} f^{(r)} \right\|_p < \infty \right\},$$

and for convenience denote $\mathbb{L}_{p,0}^{\alpha,\beta} := \mathbb{L}_p^{\alpha,\beta}$. Note that, if $\alpha = \beta = r/2$, then $\mathbb{L}_{p,r}^{r/2,r/2} = B_p^r$, the classes discussed in [8,9].

The following lemma is a generalization of [9, Lemma 3.4].

Lemma 8.1. *Let $1 \le p \le \infty$, $r \in \mathbb{N}_0$, $\alpha, \beta \in \mathbb{R}$ and let $f \in \mathbb{L}_{p,r+1}^{\alpha,\beta}$. Then $f \in \mathbb{L}_{p,r}^{\alpha-\gamma,\beta-\gamma}$, for any $\gamma < 1$ such that $\alpha - \gamma, \beta - \gamma \in J_p$.*

Proof. Given $f \in \mathbb{L}_{p,r+1}^{\alpha,\beta}$, taking into account that $\left\| w_{\alpha-\gamma,\beta-\gamma} \right\|_p < \infty$ and replacing $f(x)$ with $f(x) - x^r f^{(r)}(0)/r!$ we can assume that $f^{(r)}(0) = 0$. Now, if $p = \infty$, then

$$\begin{aligned}
\left\| w_{\alpha-\gamma,\beta-\gamma} f^{(r)} \right\|_\infty &\le \left\| w_{\alpha-\gamma,\beta-\gamma}(x) \int_0^x f^{(r+1)}(u)\, du \right\|_\infty \\
&\le \left\| w_{\alpha,\beta} f^{(r+1)} \right\|_\infty \left\| w_{\alpha-\gamma,\beta-\gamma}(x) \int_0^x w_{\alpha,\beta}^{-1}(u)\, du \right\|_\infty \le c \left\| w_{\alpha,\beta} f^{(r+1)} \right\|_\infty .
\end{aligned}$$

Similarly, if $p = 1$, then

$$\begin{aligned}
\left\| w_{\alpha-\gamma,\beta-\gamma} f^{(r)} \right\|_1 &= \int_{-1}^1 w_{\alpha-\gamma,\beta-\gamma}(x) \left| \int_0^x f^{(r+1)}(u)\, du \right| dx \\
&\le \int_{-1}^1 w_{\alpha-\gamma,\beta-\gamma}(x) \left| \int_0^x w_{\alpha,\beta}(u) |f^{(r+1)}(u)| w_{\alpha,\beta}^{-1}(u)\, du \right| dx \\
&\le \left\| w_{\alpha,\beta} f^{(r+1)} \right\|_1 \int_{-1}^1 w_{\alpha-\gamma,\beta-\gamma}(x) \max_{u\in[0,x]} w_{\alpha,\beta}^{-1}(u) dx \le c \left\| w_{\alpha,\beta} f^{(r+1)} \right\|_1 .
\end{aligned}$$

Suppose now that $1 < p < \infty$ and denote $p' := p/(p-1)$. Using Hölder's inequality we have

$$\begin{aligned}
\left\| w_{\alpha-\gamma,\beta-\gamma} f^{(r)} \right\|_p^p &= \int_{-1}^1 w_{\alpha-\gamma,\beta-\gamma}^p(x) \left| \int_0^x f^{(r+1)}(u)\, du \right|^p dx \\
&\le \int_{-1}^1 w_{\alpha-\gamma,\beta-\gamma}^p(x) \left| \left(\int_0^x w_{\alpha,\beta}^{-p'}(u)\, du \right)^{1/p'} \left(\int_0^x |w_{\alpha,\beta}(u) f^{(r+1)}(u)|^p\, du \right)^{1/p} \right|^p dx \\
&\le \left\| w_{\alpha,\beta} f^{(r+1)} \right\|_p^p \left(\int_{-1}^0 + \int_0^1 \right) w_{\alpha-\gamma,\beta-\gamma}^p(x) \left| \int_0^x w_{\alpha,\beta}^{-p'}(u)\, du \right|^{p/p'} dx \\
&=: \left\| w_{\alpha,\beta} f^{(r+1)} \right\|_p^p \cdot \left(I_{\alpha,\beta,\gamma}^- + I_{\alpha,\beta,\gamma}^+ \right).
\end{aligned}$$

We will now show that $I_{\alpha,\beta,\gamma}^+ \le c$ (the proof that the same estimate holds for $I_{\alpha,\beta,\gamma}^-$ is analogous). Indeed, if $\beta p' \ne 1$, then

$$\begin{aligned}
I_{\alpha,\beta,\gamma}^+ &\le c \int_0^1 (1-x)^{(\beta-\gamma)p} \left(\int_0^x (1-u)^{-\beta p'}\, du \right)^{p/p'} dx \\
&\le c \int_0^1 (1-x)^{(\beta-\gamma)p} \left(\max\{1, (1-x)^{-\beta p'+1}\} \right)^{p/p'} dx \\
&\le c \int_0^1 \max\left\{ (1-x)^{(\beta-\gamma)p}, (1-x)^{-\gamma p+p-1} \right\} dx \le c.
\end{aligned}$$

Finally, if $\beta p' = 1$ (and so $\beta = 1 - 1/p$), then

$$I^+_{\alpha,\beta,\gamma} \le c \int_0^1 (1-x)^{(\beta-\gamma)p} |\ln(1-x)|^{p/p'}\,dx \le c \int_0^1 (1-x)^{p(1-\gamma)-1} |\ln(1-x)|^{p-1}\,dx \le c.$$

This completes the proof. □

Remark 8.2. We actually proved that, if $f \in \mathbb{L}^{\alpha,\beta}_{p,r+1}$ is such that $f^{(r)}(0) = 0$, then

$$\left\| w_{\alpha-\gamma,\beta-\gamma} f^{(r)} \right\|_p \le c \left\| w_{\alpha,\beta} f^{(r+1)} \right\|_p$$

provided that $\gamma < 1$ and $\alpha - \gamma, \beta - \gamma \in J_p$.

Corollary 8.3. *Let* $1 \le p \le \infty$, $r \in \mathbb{N}_0$ *and* $\alpha, \beta \in J_p$. *Then*

$$\mathbb{L}^{\alpha+(r+1)/2,\beta+(r+1)/2}_{p,r+1} \subset \mathbb{L}^{\alpha+r/2,\beta+r/2}_{p,r}$$

and, in particular,

$$\mathbb{L}^{\alpha+r/2,\beta+r/2}_{p,r} \subset \mathbb{L}^{\alpha,\beta}_{p}.$$

It was shown in [8, Theorem 5.1] that, if $1 \le q \le \infty$, $0 < r < k$, and f is such that $f^{(r-1)}$ is locally absolutely continuous in $(-1, 1)$ and $w_{\alpha,\beta}\varphi^r f^{(r)} \in \mathbb{L}_q[-1, 1]$, $\alpha, \beta \ge 0$, then

$$\omega^k_\varphi(f, \delta)_{w_{\alpha,\beta},q} \le ct^r \omega^{k-r}_\varphi(f^{(r)}, \delta)_{w_{\alpha,\beta}\varphi^r,q}. \tag{8.1}$$

Taking into account that $w_{\alpha,\beta}\varphi^r = w_{\alpha+r/2,\beta+r/2}$, together with (7.1), this implies the following Jackson-type result for weighted polynomial approximation (see also [8, Theorem 5.2]).

Corollary 8.4. *If* $k \in \mathbb{N}$, $0 \le r \le k - 1$, $1 \le q \le \infty$, $\alpha, \beta \ge 0$, *and* $f \in \mathbb{L}^{\alpha+r/2,\beta+r/2}_{q,r}$, *then there exists* $n_0 \in \mathbb{N}$ *such that*

$$E_n(f)_{w_{\alpha,\beta},q} \le cn^{-r} \omega^{k-r}_\varphi(f^{(r)}, n^{-1})_{w_{\alpha+r/2,\beta+r/2},q}, \quad n \ge n_0. \tag{8.2}$$

Now, let $1 \le q < p \le \infty$, $k \in \mathbb{N}$, $1 \le r \le k - 1$, and let $f \in \mathcal{M}^k \cap \mathbb{L}^{\alpha+r/2,\beta+r/2}_{p,r}$. Using Corollary 1.3 and the fact that $f^{(r)} \in \mathcal{M}^{k-r}$, we conclude that, for $n \ge n_0$,

$$E_n(f)_{w_{\alpha,\beta},q} \le cn^{-r} \Upsilon^{\alpha+r/2,\beta+r/2}_{1/n}(k-r, q, p) \left\| w_{\alpha+r/2,\beta+r/2} f^{(r)} \right\|_p. \tag{8.3}$$

It is not hard to see that this estimate holds for $r - 1 \le n < n_0$ as well. Indeed, given a function $f \in \mathbb{L}^{\alpha+r/2,\beta+r/2}_{p,r}$, let $T_{r-1}(f)$ be its Maclaurin polynomial of degree $\le r-1$ (see (2.1)). Then, for $r - 1 \le n < n_0$, we have using Remark 8.2

$$E_n(f)_{w_{\alpha,\beta},q} \le \left\| w_{\alpha,\beta}(f - T_{r-1}(f)) \right\|_q \le c \left\| w_{\alpha,\beta}\varphi^r f^{(r)} \right\|_q \le c \left\| w_{\alpha+r/2,\beta+r/2} f^{(r)} \right\|_p, \quad q \le p.$$

Hence, the following is another corollary of Theorem 1.1.

Corollary 8.5. *Let* $1 \le q < p \le \infty$, $k \ge 2$, $1 \le r \le k-1$, $\alpha, \beta \ge 0$, *and let* $f \in \mathcal{M}^k \cap \mathbb{L}_{p,r}^{\alpha+r/2,\beta+r/2}$. *Then, for any* $n \ge r$,

$$E_n(f)_{w_{\alpha,\beta},q} \le c \left\| w_{\alpha+r/2,\beta+r/2} f^{(r)} \right\|_p \times \begin{cases} n^{-r-2/q+2/p}, & \text{if } k-r \ge 2 \text{ and } (k-r,q,p) \ne (2,1,\infty), \\ n^{-r-2} \ln(n+1), & \text{if } k-r = 2,\ q=1 \text{ and } p=\infty, \\ n^{-r-2/q+2/p}, & \text{if } k-r=1 \text{ and } p<2q, \\ n^{-r-1/q}[\ln(n+1)]^{1/(2q)}, & \text{if } k-r=1 \text{ and } p=2q, \\ n^{-r-1/q}, & \text{if } k-r=1 \text{ and } p>2q. \end{cases}$$

It follows from Corollaries 9.4 and 9.5 that estimates in Corollary 8.5 are exact in the sense that none of the powers of n can be decreased. Using the inverse theorem [9, Theorem 9.1] it is also possible to show that, in the case $\alpha = \beta = 0$, $k = r+1$ and $p/2 = q > 1$, the power $1/(2q)$ of $\ln(n+1)$ cannot be decreased.

2. Littlewood's inequality $\|g\|_q \le \|g\|_s^\theta \|g\|_p^{1-\theta}$, $1/q = \theta/s + (1-\theta)/p$, $1 \le s < q < p \le \infty$, implies that

$$\Omega_\varphi^k(f,\delta)_{w,q} \le \Omega_\varphi^k(f,\delta)_{w,s}^\theta \, \Omega_\varphi^k(f,\delta)_{w,p}^{1-\theta},$$

with similar inequalities holding for $\overrightarrow{\Omega}_\varphi^k$ and $\overleftarrow{\Omega}_\varphi^k$ as well. Therefore,

$$\begin{aligned} \omega_\varphi^k(f,\delta)_{w,q} &= \Omega_\varphi^k(f,\delta)_{w,q} + \overrightarrow{\Omega}_\varphi^k(f,\delta)_{w,q} + \overleftarrow{\Omega}_\varphi^k(f,\delta)_{w,q} \\ &\le \Omega_\varphi^k(f,\delta)_{w,s}^\theta \, \Omega_\varphi^k(f,\delta)_{w,p}^{1-\theta} + \overrightarrow{\Omega}_\varphi^k(f,\delta)_{w,s}^\theta \, \overrightarrow{\Omega}_\varphi^k(f,\delta)_{w,p}^{1-\theta} \\ &\quad + \overleftarrow{\Omega}_\varphi^k(f,\delta)_{w,s}^\theta \, \overleftarrow{\Omega}_\varphi^k(f,\delta)_{w,p}^{1-\theta} \\ &\le 3\, \omega_\varphi^k(f,\delta)_{w,s}^\theta \, \omega_\varphi^k(f,\delta)_{w,p}^{1-\theta}. \end{aligned}$$

Hence, using (8.2) and Theorem 1.1 we have the following estimates for $f \in \mathcal{M}^k \cap \mathbb{L}_{p,r}^{\alpha+r/2,\beta+r/2}$, $0 \le r \le k-1$:

$$\begin{aligned} E_n(f)_{w_{\alpha,\beta},q} &\le cn^{-r}\, \omega_\varphi^{k-r}(f^{(r)}, n^{-1})_{w_{\alpha+r/2,\beta+r/2},q} \\ &\le cn^{-r}\, \omega_\varphi^{k-r}(f^{(r)}, n^{-1})_{w_{\alpha+r/2,\beta+r/2},s}^\theta \, \omega_\varphi^{k-r}(f^{(r)}, n^{-1})_{w_{\alpha+r/2,\beta+r/2},p}^{1-\theta} \\ &\le cn^{-r} \left[\Upsilon_{1/n}^{\alpha+r/2,\beta+r/2}(k-r,s,p) \right]^\theta \\ &\quad \times \left\| w_{\alpha+r/2,\beta+r/2} f^{(r)} \right\|_p^\theta \, \omega_\varphi^{k-r}(f^{(r)}, n^{-1})_{w_{\alpha+r/2,\beta+r/2},p}^{1-\theta}. \end{aligned}$$

If s is such that $1 < s < q$ and $s \ne p/2$, then

$$\Upsilon_{1/n}^{\alpha+r/2,\beta+r/2}(k-r,s,p) = \begin{cases} n^{-2/s+2/p}, & \text{if } k-r \ge 2, \\ n^{-2/s+2/p}, & \text{if } k-r=1 \text{ and } p<2s, \\ n^{-1/s}, & \text{if } k-r=1 \text{ and } p>2s, \end{cases}$$

and so

$$\left[\Upsilon_{1/n}^{\alpha+r/2,\beta+r/2}(k-r,s,p)\right]^{\theta} = \begin{cases} n^{-2/q+2/p}, & \text{if } 0 \le r \le k-2, \text{ or } r=k-1 \text{ and } p<2s, \\ n^{-(p-q)/q(p-s)}, & \text{if } r=k-1 \text{ and } p>2s. \end{cases}$$

We now note that one can choose s so that $1 < s < q$ and $p < 2s$ iff $p < 2q$. Also, note that, for any $s > 1$, $\left[\Upsilon_{1/n}^{\alpha+(k-1)/2,\beta+(k-1)/2}(1,s,\infty)\right]^{\theta} = n^{-1/q}$.

Therefore, taking into account that, in the case $p < \infty$, $\omega_\varphi^{k-r}(f^{(r)}, n^{-1})_{w_{\alpha+r/2,\beta+r/2},p} \to 0$ as $n \to \infty$, and that $\omega_\varphi^{k-r}(f^{(r)}, n^{-1})_{w_{\alpha+r/2,\beta+r/2},\infty} \to 0$ as $n \to \infty$ provided that $f^{(r)}$ is continuous on $(-1,1)$ and $\lim_{x\pm 1} w_{\alpha+r/2,\beta+r/2}(x) f^{(r)}(x) = 0$, we have the following two corollaries of Theorem 1.1.

Corollary 8.6. *Let $k \in \mathbb{N}$, $1 < q < p < \infty$, $0 \le r \le k-1$, $\alpha, \beta \ge 0$, and let $f \in \mathcal{M}^k \cap \mathbb{L}_{p,r}^{\alpha+r/2,\beta+r/2}$. Then*

$$E_n(f)_{w_{\alpha,\beta},q} = o\left(n^{-r-2/q+2/p}\right), \quad n \to \infty,$$

where either $0 \le r \le k-2$, or $r = k-1$ and $p < 2q$.

Corollary 8.7. *Let $k \in \mathbb{N}$, $1 < q < \infty$, $0 \le r \le k-1$, $\alpha, \beta \ge 0$, and let $f \in \mathcal{M}^k$ be such that $f^{(r)}$ is continuous on $(-1,1)$ and $\lim_{x\pm 1} w_{\alpha+r/2,\beta+r/2}(x) f^{(r)}(x) = 0$. Then*

$$E_n(f)_{w_{\alpha,\beta},q} = o\left(n^{-r-\min\{k-r,2\}/q}\right), \quad n \to \infty.$$

9. Lower estimates of polynomial approximation

The following Remez-type inequality follows from [11, (7.16), (6.10)].

Theorem 9.1. *Let $1 \le p \le \infty$, and let w be a doubling weight in the case $1 \le p < \infty$ or an A^* weight in the case $p = \infty$. For every $\Lambda \le n$, there is a constant $C = C(\Lambda)$ such that, if $E \subset [-1,1]$ is an interval and $\int_E (1-x^2)^{-1/2} dx \le \Lambda/n$, then, for each $p_n \in \mathbb{P}_n$, we have*

$$\int_{-1}^{1} |p_n(x)|^p w(x)\, dx \le C \int_{[-1,1]\setminus E} |p_n(x)|^p w(x)\, dx, \quad \text{if } 1 \le p < \infty,$$

or

$$\|p_n w\|_{\mathbb{L}_\infty[-1,1]} \le C \|p_n w\|_{\mathbb{L}_\infty([-1,1]\setminus E)}, \quad \text{if } p = \infty.$$

We recall that w is a doubling weight if $\int_{2I\cap[-1,1]} w(x)\,dx \le L \int_I w(x)\,dx$, for all intervals $I \subset [-1,1]$ ($2I$ is the interval twice the length of I and with midpoint at the midpoint of I), and it is an A^* weight if, for all intervals $I \subset [-1,1]$ and $x \in I$, $w(x) \le L \int_I w(x)\,dx/|I|$.

Since $w_{\alpha,\beta}^p$, $\alpha, \beta > -1/p$, is a doubling weight, and $w_{\alpha,\beta}$, $\alpha, \beta \ge 0$, is an A^* weight, we immediately get the following corollary (see also [4]).

Corollary 9.2. *Let $1 \le p \le \infty$, and let $\alpha, \beta \in J_p$. For every $\Lambda \le n$, there is a constant $C = C(\Lambda)$ such that, if $E \subset [-1,1]$ is an interval and $\int_E (1-x^2)^{-1/2} dx \le \Lambda/n$, then, for each*

$p_n \in \mathbb{P}_n$, we have

$$\left\| p_n w_{\alpha,\beta} \right\|_{\mathbb{L}_p[-1,1]} \le C \left\| p_n w_{\alpha,\beta} \right\|_{\mathbb{L}_p([-1,1]\setminus E)} .$$

We are now ready to construct (truncated power) functions which will yield lower estimates. Note that, if $k \in \mathbb{N}$, $1 \le p \le \infty$, $0 \le r \le k-1$, $0 \le \xi < 1$, $\alpha \in \mathbb{R}$, $\beta \in J_p$ and $f(x) := (x-\xi)_+^{k-1}$, then

$$\left\| w_{\alpha,\beta} f^{(r)} \right\|_p \sim (1-\xi)^{\beta+k-r-1+1/p}. \tag{9.1}$$

Lemma 9.3. *Let $1 \le q \le \infty$, $k \in \mathbb{N}$, $\alpha, \beta \ge 0$, $n \ge 2k$, $0 \le \xi \le 1 - 2k^2 n^{-2}$ and let $f(x) := (x-\xi)_+^{k-1}$. Then*

$$E_n(f)_{w_{\alpha,\beta},q} \ge c n^{-k+1-1/q} (1-\xi)^{\beta+(k-1)/2+1/(2q)},$$

for some constant c independent of n.

Proof. We only provide the proof for the case $q < \infty$. If $q = \infty$, it is obvious what modifications are needed. It is convenient to denote $\theta_n := k\varphi(\xi)/(2n)$. Then, in particular, $\theta_n \le 1/4$ and $\xi \pm 2\theta_n \in [-1,1]$. Now, let p_n be an arbitrary polynomial from $\mathbb{P}_n$, define

$$\tilde{f}_n(x) := \Delta^k_{\varphi(\xi)/n}(f,x), \qquad f_n(x) := \tilde{f}_n((1-\theta_n)x),$$
$$\tilde{q}_n(x) := \Delta^k_{\varphi(\xi)/n}(p_n,x), \qquad q_n(x) := \tilde{q}_n((1-\theta_n)x),$$

and note that $\tilde{q}_n$ is a polynomial of degree $\le n$ on $J_n := [-1+\theta_n, 1-\theta_n]$, and hence q_n is a polynomial of degree $\le n$ on $[-1,1]$. We also note that $\tilde{f}_n(x) = 0$, for $x \notin \tilde{I}_n := \left[\xi - \theta_n, \xi + \theta_n\right] \subset J_n$, and hence $f_n(x) = 0$, for $x \notin I_n := [(\xi-\theta_n)/(1-\theta_n), (\xi+\theta_n)/(1-\theta_n)] \subset [-1,1]$.

Now,

$$\begin{aligned}
\left\| w_{\alpha,\beta}(f_n - q_n) \right\|_q^q &= \int_{-1}^1 w^q_{\alpha,\beta}(x) |\tilde{f}_n((1-\theta_n)x) - \tilde{q}_n((1-\theta_n)x)|^q \, dx \\
&\le c \int_{-1+\theta_n}^{1-\theta_n} w^q_{\alpha,\beta}(x/(1-\theta_n)) |\tilde{f}_n(x) - \tilde{q}_n(x)|^q \, dx \\
&\le c \int_{-1+\theta_n}^{1-\theta_n} w^q_{\alpha,\beta}(x/(1-\theta_n)) \\
&\quad \times \sum_{i=0}^k |f(x-\theta_n + i\varphi(\xi)/n) - p_n(x-\theta_n+i\varphi(\xi)/n)|^q \, dx \\
&\le c \sum_{i=0}^k \int_{-1+i\varphi(\xi)/n}^{1-2\theta_n+i\varphi(\xi)/n} w^q_{\alpha,\beta}\left((y+\theta_n - i\varphi(\xi)/n)/(1-\theta_n)\right) |f(y) - p_n(y)|^q \, dy \\
&\le c \left\| w_{\alpha,\beta}(f-p_n) \right\|_q^q ,
\end{aligned}$$

since $w_{\alpha,\beta}\left((y+\theta_n - i\varphi(\xi)/n)/(1-\theta_n)\right) \le c w_{\alpha,\beta}(y)$.

It is straightforward to check that $\int_{I_n} (1-x^2)^{-1/2} \, dx \le c(k)/n$, and so Corollary 9.2 implies that

$$\left\| w_{\alpha,\beta} q_n \right\|_q \le c \left\| w_{\alpha,\beta} q_n \right\|_{\mathbb{L}_q([-1,1]\setminus I_n)} .$$

Therefore, recalling that $f_n(x) = 0$, $x \in [-1, 1] \setminus I_n$, we have

$$\begin{aligned}\left\|w_{\alpha,\beta} f_n\right\|_q &\le \left\|w_{\alpha,\beta}(f_n - q_n)\right\|_q + \left\|w_{\alpha,\beta} q_n\right\|_q \\ &\le \left\|w_{\alpha,\beta}(f_n - q_n)\right\|_q + c\left\|w_{\alpha,\beta}(f_n - q_n)\right\|_{\mathbb{L}_q([-1,1]\setminus I_n)} \\ &\le c\left\|w_{\alpha,\beta}(f_n - q_n)\right\|_q \\ &\le c\left\|w_{\alpha,\beta}(f - p_n)\right\|_q .\end{aligned}$$

Now, noting that $\tilde{f}_n(x) = f(x + \theta_n) = (x + \theta_n - \xi)^{k-1}$, if $x \in [\xi - \theta_n, \xi - \theta_n + \varphi(\xi)/n]$ we have

$$\begin{aligned}\left\|w_{\alpha,\beta} f_n\right\|_q^q &\ge c\int_{-1+\theta_n}^{1-\theta_n} w_{\alpha,\beta}^q(x/(1-\theta_n))|\tilde{f}_n(x)|^q\,dx \\ &\ge c\int_{\xi-\theta_n}^{\xi-\theta_n+\varphi(\xi)/n} (1-\theta_n - x)^{\beta q}(x+\theta_n-\xi)^{(k-1)q}\,dx \\ &\ge c\int_{\xi}^{\xi+\varphi(\xi)/n} (1-y)^{\beta q}(y-\xi)^{(k-1)q}\,dy \\ &\ge cn^{-(k-1)q-1}(1-\xi)^{\beta q+(k-1)q/2+1/2},\end{aligned}$$

and so $\left\|w_{\alpha,\beta} f_n\right\|_q \ge cn^{-k+1-1/q}(1-\xi)^{\beta+(k-1)/2+1/(2q)}$.

Hence, for any $p_n \in \mathbb{P}_n$,

$$\left\|w_{\alpha,\beta}(f - p_n)\right\|_q \ge cn^{-k+1-1/q}(1-\xi)^{\beta+(k-1)/2+1/(2q)},$$

and the proof is complete. □

The following two corollaries provide all lower estimates in Theorem 1.5 and show that none of the powers of n in Corollary 8.5 can be decreased.

Corollary 9.4. *Let $1 \le p, q \le \infty$, $k \in \mathbb{N}$, and $\alpha, \beta \ge 0$. Then, there exists a function $f \in \mathcal{M}^k \cap \mathbb{L}_{p,k-1}^{\alpha+(k-1)/2,\beta+(k-1)/2}$ such that, for each $n \in \mathbb{N}$,*

$$E_n(f)_{w_{\alpha,\beta},q} \ge cn^{-k-1/q+1}\left\|w_{\alpha+(k-1)/2,\beta+(k-1)/2} f^{(k-1)}\right\|_p, \tag{9.2}$$

for some constant c independent of n.

Proof. We let $f(x) := x_+^{k-1}$ and note that $f \in \mathcal{M}^k$. Now, (9.1) implies that $\left\|w_{\alpha+(k-1)/2,\beta+(k-1)/2} f^{(k-1)}\right\|_p \sim 1$, and Lemma 9.3 implies $E_n(f)_{w_{\alpha,\beta},q} \ge cn^{-k-1/q+1}$, for $n \ge 2k$. For $1 \le n < 2k$, (9.2) follows from $E_n(f)_{w_{\alpha,\beta},q} \ge E_{2k}(f)_{w_{\alpha,\beta},q} \ge c$. □

It follows from Corollary 8.7 that there does not exist $f \in \mathbb{C}^{k-1}(-1, 1) \cap \mathcal{M}^k$ which is independent of n, satisfies $\lim_{x\pm1} w_{\alpha+r/2,\beta+r/2}(x) f^{(r)}(x) = 0$, and for which (9.2) holds.

Corollary 9.5. *Let $1 \le p, q \le \infty$, $k \in \mathbb{N}$, $0 \le r \le k-1$, $\alpha, \beta \ge 0$, and $n \in \mathbb{N}$. Then, there exists a function $f_n \in \mathcal{M}^k \cap \mathbb{L}_{p,r}^{\alpha+r/2,\beta+r/2}$ such that*

$$E_n(f_n)_{w_{\alpha,\beta},q} \ge cn^{-r-2/q+2/p}\left\|w_{\alpha+r/2,\beta+r/2} f_n^{(r)}\right\|_p,$$

for some constant c independent of n.

Proof. For $1 \le n < 2k$, the statement is clearly true, for example, for $f_n(x) = x_+^{k-1}$. If $n \ge 2k$, we let $\xi_n = 1 - 2k^2n^{-2}$ and $f_n(x) := (x - \xi_n)_+^{k-1}$. Then $f_n \in \mathcal{M}^k$, Lemma 9.3 implies that $E_n(f_n)_{w_{\alpha,\beta},q} \ge cn^{-2\beta-2k+2-2/q}$, and (9.1) yields $\left\| w_{\alpha+r/2,\beta+r/2} f_n^{(r)} \right\|_p \sim n^{-2\beta-2k-2/p+2+r}$. Therefore, $E_n(f_n)_{w_{\alpha,\beta},q} / \left\| w_{\alpha+r/2,\beta+r/2} f_n^{(r)} \right\|_p \ge cn^{-r-2/q+2/p}$. $\square$

It is interesting to note that Corollary 8.6 implies that f_n in Corollary 9.5 cannot be replaced by a function which is independent of n.

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