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Uniform polynomial approximation with A^* weights having finitely many zeros

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ABSTRACT

We prove matching direct and inverse theorems for uniform polynomial approximation with A^* weights (a subclass of doubling weights suitable for approximation in the $\mathbb{L}_\infty$ norm) having finitely many zeros and not too "rapidly changing" away from these zeros. This class of weights is rather wide and, in particular, includes the classical Jacobi weights, generalized Jacobi weights and generalized Ditzian–Totik weights. Main part and complete weighted moduli of smoothness are introduced, their properties are investigated, and equivalence type results involving related realization functionals are discussed.

1. Introduction

Recall that a nonnegative integrable function w is a doubling weight (on $[-1,1]$) if there exists a positive constant L (a so-called doubling constant of w) such that

$$w(2I) \le Lw(I), \tag{1.1}$$

for any interval $I \subset [-1,1]$. Here, $2I$ denotes the interval of length $2|I|$ ($|I|$ is the length of I) with the same center as I, and $w(I) := \int_I w(u)du$. Note that it is convenient to assume that w is identically zero outside $[-1,1]$ which allows us to write $w(I)$ for any interval I that is not necessarily contained in $[-1,1]$. Let $\mathcal{D}W_L$ denote the set of all doubling weights on $[-1,1]$ with the doubling constant L, and $\mathcal{D}W := \cup_{L>0}\mathcal{D}W_L$, *i.e.*, $\mathcal{D}W$ is the set of all doubling weights.

It is easy to see that $w \in \mathcal{D}W_L$ if and only if there exists a constant $\kappa \ge 1$ such that, for any two adjacent intervals $I_1, I_2 \subset [-1,1]$ of equal length,

$$w(I_1) \le \kappa w(I_2). \tag{1.2}$$

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Clearly, κ and L depend on each other. In fact, if $w \in \mathcal{D}W_L$ then (1.2) holds with $\kappa = L^2$. Conversely, if (1.2) holds, then $w \in \mathcal{D}W_{1+\kappa}$.

Following [6,7], we say that w is an A^* weight (on $[-1,1]$) if there is a constant L^* (a so-called A^* constant of w) such that, for all intervals $I \subset [-1,1]$ and $x \in I$, we have

$$w(x) \le \frac{L^*}{|I|} w(I). \tag{1.3}$$

Throughout this paper, $A^*_{L^*}$ denotes the set of all A^* weights on $[-1,1]$ with the A^* constant L^*. We also let $A^* := \cup_{L>0} A^*_{L^*}$, *i.e.*, A^* is the set of all A^* weights. Note that any A^* weight is doubling, *i.e.*, $A^*_{L^*} \subset \mathcal{D}W_L$, where L depends only on L^*. This was proved in [7] and is an immediate consequence of the fact (see [7, Theorem 6.1]) that if $w \in A^*_{L^*}$ then, for some l depending only on L^* (for example, $l = 2L^*$ will do), $w(I_1) \ge (|I_1|/|I_2|)^l w(I_2)$, for all intervals $I_1, I_2 \subset [-1,1]$ such that $I_1 \subset I_2$. Indeed, for any $I \subset [-1,1]$, this implies $w(I) \ge \left(|I|/|2I \cap [-1,1]|\right)^l w(2I) \ge 2^{-l} w(2I)$, which shows that $w \in \mathcal{D}W_{2^l}$.

Moreover, it is known and is not difficult to check (see [7, pp. 58 and 68]) that all A^* weights are A_∞ weights. Here, A_∞ is the union of all Muckenhoupt A_p weights and can be defined as the set of all weights w such that, for any $0 < \alpha < 1$, there is $0 < \beta < 1$ so that $w(E) \ge \beta w(I)$, for all intervals $I \subset [-1,1]$ and all measurable subsets $E \subset I$ with $|E| \ge \alpha |I|$ (see *e.g.* [10, Chapter V]).

Clearly, any A^* weight on $[-1,1]$ is bounded since if $w \in A^*_{L^*}$, then $w(x) \le L^* w[-1,1]/2$, $x \in [-1,1]$. (We slightly abuse the notation and write $w[a,b]$ instead of $w([a,b])$ throughout this paper.) At the same time, not every bounded doubling weight is an A^* weight (for example, the doubling weight constructed in [2] is bounded and is not in A_∞, and so it is not an A^* weight either).

Throughout this paper, we use the standard notation $\|f\|_I := \|f\|_{\mathbb{L}_\infty(I)} := \operatorname{ess\,sup}_{u\in I} |f(u)|$ and $\|f\| := \|f\|_{[-1,1]}$. Also,

$$E_n(f, I)_w := \inf_{q \in \Pi_n} \|w(f-q)\|_I ,$$

where Π_n is the space of algebraic polynomials of degree $\le n-1$.

The following theorem is due to G. Mastroianni and V. Totik [8, Theorem 1.4] and is the main motivation for the present paper (see also [5–7]).

Theorem A. *(See [8, Theorem 1.4].) Let $r \in \mathbb{N}$, $M \ge 3$, $-1 = z_1 < \cdots < z_M = 1$, and let w be a bounded generalized Jacobi weight*

$$w_{\mathcal{J}}(x) := \prod_{j=1}^{M} |x - z_j|^{\gamma_j} \quad \text{with } \gamma_j \ge 0,\ 1 \le j \le M. \tag{1.4}$$

Then there is a constant c depending only on r and the weight w such that, for any f,

$$E_n(f, [-1,1])_{w_{\mathcal{J}}} \le c\, \omega_\varphi^r(f, 1/n)^*_{w_{\mathcal{J}}},$$

and

$$\omega_\varphi^r(f, 1/n)^*_{w_{\mathcal{J}}} \le c n^{-r} \sum_{k=1}^{n} k^{r-1} E_k(f, [-1,1])_{w_{\mathcal{J}}},$$

where

$$\omega_\varphi^r(f, t)^*_{w_{\mathcal{J}}} := \sum_{j=1}^{M-1} \sup_{0<h\le t} \left\| w_{\mathcal{J}}(\cdot) \Delta^r_{h\varphi(\cdot)}(f, \cdot, J_{j,h}) \right\| + \sum_{j=1}^{M} E_r(f, I_{j,t})_{w_{\mathcal{J}}}$$

with $I_{1,h} = [-1, -1+h^2]$, $I_{M,h} = [1-h^2, 1]$, $J_{1,h} = [-1+h^2, z_2 - h]$, $J_{M-1,h} = [z_{M-1}+h, 1-h^2]$, *and* $I_{j,h} = [z_j - h, z_j + h]$ *and* $J_{j,h} = [z_j + h, z_{j+1} - h]$ *for* $1 < j < M-1$, *and the* r*th symmetric difference is defined in (3.3).*

The purpose of the present paper is to prove an analog of Theorem A for more general weights (namely, for A^* weights having finitely many zeros inside $[-1,1]$ and not too "rapidly changing" away from these zeros), and give a more natural and transparent (in our opinion) definition of the modulus of smoothness ω_φ^r. Our recent paper [4] deals with approximation in the weighted $\mathbb{L}_p$, $p < \infty$, (quasi)norm and a certain class of doubling weights having finitely many zeros and singularities. Approximation in the weighted $\mathbb{L}_\infty$ norm considered in the current paper is similar in some sense, but it also presents some challenges that have to be dealt with, and our present proofs are different from those in both [8] and [4]. The main results of the present paper are Theorem 6.1 (direct result), Theorem 7.1 (inverse result) and Theorem 8.1 (equivalence of the modulus and an appropriate realization functional). Finally, we mention that Theorem A is a corollary of our results taking into account that $w_{\mathcal{J}} \in \mathcal{W}^*(\mathcal{Z})$, $\mathcal{Z} \in \mathbb{Z}_M$ (see Remark 3.3), and

$$\begin{aligned}
&\omega_\varphi^r\left(f, \max\left\{(1-z_2^2)^{-1/2}, (1-z_{M-1}^2)^{-1/2}\right\}, 1/2, t\right)_{w_{\mathcal{J}}} \\
&\quad \le \omega_\varphi^r(f,t)^*_{w_{\mathcal{J}}} \le M \cdot \omega_\varphi^r\left(f, 1/2, \max\left\{(1-z_2^2)^{-1/2}, (1-z_{M-1}^2)^{-1/2}\right\}, t\right)_{w_{\mathcal{J}}}, \quad 0 < t \le 1,
\end{aligned}$$

where $\mathcal{W}^*(\mathcal{Z})$ and $\omega_\varphi^r(f, A, B, t)_w$ are defined in Definition 3.1 and (3.5), respectively.

2. Some properties of A^* weights

Note that, for any interval $I \subset [-1,1]$ and $x \in I$, if (1.3) holds for $I_1 := I \cap [-1,x]$ and $I_2 := I \cap [x,1]$, then it also holds for I since $|I_1| + |I_2| = |I|$ and $w(I_1) + w(I_2) = w(I)$. Therefore, $w \in A^*_{L^*}$ if and only if, for all intervals $[a,b] \subset [-1,1]$,

$$\max\{w(a), w(b)\} \le \frac{L^*}{b-a} w[a,b]. \tag{2.1}$$

Lemma 2.1. *Let* $w \in A^*_{L^*}$, $\xi \in [-1,1]$, *and let* $w_1(x) := f(|x-\xi|)$, *where* $f : [0,2] \mapsto \mathbb{R}_+$ *is nondecreasing and such that* $f(2x) \le K f(x)$, *for some* $K > 0$ *and all* $0 \le x \le 1$. *Then,* $\widetilde{w} := w w_1 \in A^*_L$ *with the constant* L *depending only on* K *and* L^*.

Proof. Suppose that $I \subset [-1,1]$ and d is one of the endpoints of I. We need to show that $\widetilde{w}(d) \le L\widetilde{w}(I)/|I|$.

Case 1: $\xi \notin int(I)$.

Then, w_1 is monotone on I, and so either $w_1(d) \le w_1(u)$ or $w_1(d) \ge w_1(u)$, for $u \in I$. In the former case, we immediately have

$$\widetilde{w}(d) = w(d)w_1(d) \le \frac{L^*}{|I|}\int_I w_1(d)w(u)du \le \frac{L^*}{|I|}\widetilde{w}(I).$$

Suppose now that $w_1(d) \ge w_1(u)$, for $u \in I$. This means that d is the endpoint of I furthest from ξ. Let ζ be the midpoint of I, and let $J := [d, \zeta]$ (as usual, if $x < y$, then $[y,x] := [x,y]$). Then, $w_1(\zeta) \le w_1(u)$, for all $u \in J$. Also, since $|d-\xi|/2 \le |\zeta - \xi|$ and $|d-\xi| \le 2$, we conclude that

$$w_1(d) = f(|d-\xi|) \le K f(|d-\xi|/2) \le K f(|\zeta - \xi|) = K w_1(\zeta).$$

Therefore, $w_1(d) \le K w_1(u)$, for all $u \in J$, and so

$$\begin{aligned}\widetilde{w}(d) = w(d)w_1(d) &\le \frac{L^*}{|J|}\int_J w_1(d)w(u)du \le \frac{L^*K}{|J|}\int_J w_1(u)w(u)du \le \frac{L^*K}{|J|}\widetilde{w}(I)\\ &= \frac{2L^*K}{|I|}\widetilde{w}(I).\end{aligned} \tag{2.2}$$

Case 2: $\xi \in int(I)$.

If $|d-\xi| \ge |I|/4$, then using (2.2) for $I' := [d, \xi]$, we have

$$\widetilde{w}(d) \le \frac{2L^*K}{|I'|}\widetilde{w}(I') \le \frac{8L^*K}{|I|}\widetilde{w}(I).$$

We now assume that $|d-\xi| < |I|/4$. Let d' be the point symmetric to d about ξ, *i.e.*, $\xi = (d+d')/2$, and let $I'' := I \setminus [d, d')$. Then $|I''| = |I| - 2|d-\xi| \ge |I|/2$, and $w_1(d) = w_1(d') \le w_1(u)$, for all $u \in I''$. Hence, taking into account that w is doubling with the doubling constant depending only on L^*, we have

$$\widetilde{w}(d) = w(d)w_1(d) \le \frac{L^*w_1(d)}{|I|}\int_I w(u)du \le \frac{cw_1(d)}{|I|}\int_{I''} w(u)du \le \frac{c}{|I|}\int_{I''} w_1(u)w(u)du \le \frac{c}{|I|}\widetilde{w}(I).$$

This completes the proof. □

Corollary 2.2. *Suppose that $w \in A^*_{L^*}$, $M \in \mathbb{N}$ and, for each $1 \le i \le M$, $z_i \in [-1,1]$, $\gamma_i \ge 0$ and $\Gamma_i \in \mathbb{R}$ (if $\gamma_i > 0$) or $\Gamma_i \le 0$ (if $\gamma_i = 0$). Then*

$$\widetilde{w}(x) := w(x)\prod_{i=1}^{M} |x-z_i|^{\gamma_i}\left(\ln\frac{e}{|x-z_i|}\right)^{\Gamma_i} \tag{2.3}$$

is an A^* weight with the A^* constant depending only on γ_i's, Γ_i's and L^*.

We remark that, with $w \sim 1$, the weights $\widetilde{w}$ in (2.3) are sometimes called "generalized Ditzian–Totik weights".

Proof of Corollary 2.2. Denote

$$f_{\gamma,\Gamma}(x) := \begin{cases}(1-\ln x)^{\Gamma}, & \text{if } \gamma = 0 \text{ and } \Gamma \le 0,\\ x^{\gamma}(\Psi - \ln x)^{\Gamma}, & \text{if } \gamma > 0 \text{ and } \Gamma \in \mathbb{R},\end{cases}$$

where $\Psi := 1 + \max\{0, \Gamma\}/\gamma$. It is easy to check that $f_{\gamma,\Gamma}$ is nonnegative and nondecreasing on $[0,2]$, and satisfies $\sup_{x\in[0,1]} |f_{\gamma,\Gamma}(2x)/f_{\gamma,\Gamma}(x)| < \infty$. Hence, Lemma 2.1 implies that the weight

$$\widehat{w}(x) := w(x)\prod_{i=1}^{M} f_{\gamma_i,\Gamma_i}(|x-z_i|)$$

is an A^* weight with the A^* constant depending only on γ_i's, Γ_i's, and L^*.

Finally, it remains to notice that, if $\gamma > 0$ and $\Gamma \in \mathbb{R}$, then $f_{\gamma,\Gamma}(x) \sim x^{\gamma}(1-\ln x)^{\Gamma}$ on $[0,2]$ with equivalence constants depending only on γ and Γ, and so $\widetilde{w} \sim \widehat{w}$ on $[-1,1]$. Clearly, this implies that $\widetilde{w} \in A^*$. □

Remark 2.3. It follows from Corollary 2.2 that, for any A^* weight w and any $\mu \ge 0$, $w\varphi^\mu$ is also an A^* weight, where $\varphi(x) := \sqrt{1-x^2}$.

For $n \in \mathbb{N}$, following *e.g.* [8], we denote

$$w_n(x) := \rho_n(x)^{-1} \int\limits_{x-\rho_n(x)}^{x+\rho_n(x)} w(u)du,$$

where $\rho_n(x) := n^{-1}\varphi(x) + n^{-2}$ (recall that w is assumed to be 0 outside $[-1,1]$). Note that, for any $w \in A^*_{L^*}$ and $x \in [-1,1]$,

$$\begin{aligned} w(x) &\le \frac{L^*}{|[x-\rho_n(x), x+\rho_n(x)] \cap [-1,1]|} \int\limits_{[x-\rho_n(x), x+\rho_n(x)]\cap[-1,1]} w(u)du \\ &\le \frac{L^*}{\rho_n(x)} \int\limits_{x-\rho_n(x)}^{x+\rho_n(x)} w(u)du = L^* w_n(x). \end{aligned} \tag{2.4}$$

Lemma 2.4. *Let $w \in A^*_{L^*}$ and $n \in \mathbb{N}$. Then $w_n \in A^*_L$ with L depending only on L^*.*

Proof. Suppose that $n \in \mathbb{N}$ is fixed. Let I be a subinterval of $[-1,1]$, and suppose that $x \in I$ is the left endpoint of I (the case for the right endpoint is analogous). If $[x, x+\rho_n(x)] \subset I$, using the fact that w is doubling, we have

$$\begin{aligned} w_n(x) &= \rho_n(x)^{-1} \int\limits_{x-\rho_n(x)}^{x+\rho_n(x)} w(u)du \le c\rho_n(x)^{-1} \int\limits_{x}^{x+\rho_n(x)} w(u)du \\ &\le c\rho_n(x)^{-1} \int\limits_{x}^{x+\rho_n(x)} \frac{L^*}{|I|} \int\limits_I w(v)dv\,du \le \frac{c}{|I|} \int\limits_I w(v)dv \le \frac{c}{|I|} \int\limits_I w_n(v)dv. \end{aligned}$$

Recall now that, if $|x-u| \le K\rho_n(x)$, then $w_n(x) \sim w_n(u)$ (see *e.g.* [8, (2.3)]). This implies that, if x is the left endpoint of I and $x+\rho_n(x) \notin I$, then $I \subset [x, x+\rho_n(x)]$, and so $w_n(u) \sim w_n(x)$, for all $u \in I$. Hence, in this case,

$$w_n(x) \sim \frac{1}{|I|} \int\limits_I w_n(u)du.$$

Therefore, (2.1) implies that w_n is an A^* weight. □

3. Special A^* weights and associated moduli of smoothness

Let

$$\rho(h,x) := h\varphi(x) + h^2$$

(note that $\rho(1/n, x) = \rho_n(x)$), and

$$\mathbb{Z}_M := \left\{ (z_j)_{j=1}^M \;\middle|\; -1 \le z_1 < \cdots < z_{M-1} < z_M \le 1 \right\},\ M \in \mathbb{N}.$$

For $\mathcal{Z} \in \mathbb{Z}_M$, it is convenient to denote

$$\mathcal{Z}^j_{A,h} := \mathcal{Z}^j_{A,h}(\mathcal{Z}) := \left\{ x \in [-1,1] \;\middle|\; |x - z_j| \le A\rho(h, z_j) \right\}, \quad 1 \le j \le M,$$
$$\mathcal{Z}_{A,h} := \mathcal{Z}_{A,h}(\mathcal{Z}) := \cup_{j=1}^M \mathcal{Z}^j_{A,h},$$

and

$$\mathcal{I}_{A,h} := \mathcal{I}_{A,h}(\mathcal{Z}) := \left([-1,1] \setminus \mathcal{Z}_{A,h}\right)^{cl} = \left\{ x \in [-1,1] \;\middle|\; |x - z_j| \ge A\rho(h, z_j), \text{ for all } 1 \le j \le M \right\}.$$

Also,

$$\delta(\mathcal{Z}) := \operatorname{pmin}\left\{ |z_j - z_{j-1}| \;\middle|\; 1 \le j \le M+1 \right\},$$

where $z_0 := -1$, $z_{M+1} := 1$ and $\operatorname{pmin}(S)$ is the smallest *positive* number from the finite set S of nonnegative reals. Note that $\delta(\mathcal{Z}) \le 2$, for any $\mathcal{Z} \in \mathbb{Z}_M$.

The following definition is an analog of [4, Definition 2.1] for A^* weights.

Definition 3.1. Let $\mathcal{Z} \in \mathbb{Z}_M$. We say that w is an A^* weight from the class $\mathcal{W}^*(\mathcal{Z})$ (and write $w \in \mathcal{W}^*(\mathcal{Z})$) if

(i) $w \in A^*$,

and

(ii) for any $\varepsilon > 0$ and $x, y \in [-1,1]$ such that $|x-y| \le \rho(\varepsilon, x)$ and $\operatorname{dist}([x,y], z_j) \ge \rho(\varepsilon, z_j)$ for all $1 \le j \le M$, the following inequalities are satisfied

$$c_* w(y) \le w(x) \le c_*^{-1} w(y), \tag{3.1}$$

where the constant c_* depends only on w, and does not depend on x, y and ε.

Clearly, there are non-A^* weights satisfying condition (ii) in Definition 3.1. For instance, the non-doubling weight

$$w(x) := \begin{cases} -x, & \text{if } x < 0, \\ x^2, & \text{if } x \ge 0, \end{cases}$$

is one such example for $\mathcal{Z} := \{0\}$.

Remark 3.2. A weight from the class $\mathcal{W}^*(\mathcal{Z})$ may have zeros only at the points in $\mathcal{Z}$. At the same time, it is not required to have zeros at those points.

Remark 3.3. It follows from [4, Example 2.7] and Corollary 2.2 that the following weights belong to $\mathcal{W}^*(\mathcal{Z})$ with $\mathcal{Z} = (z_j)_{j=1}^M$, $-1 \le z_1 < \cdots < z_{M-1} < z_M \le 1$:

- bounded classical Jacobi weights: $w(x) = (1+x)^\alpha (1-x)^\beta$, $\alpha, \beta \ge 0$, with $M = 2$, $z_1 = -1$ and $z_2 = 1$,
- bounded generalized Jacobi weights (1.4),
- bounded generalized Ditzian–Totik weights (2.3) with $w \equiv 1$.

The following lemma immediately follows from [4, Lemma 2.3] taking into account the fact that any A^* weight is doubling.

Lemma 3.4. *Let w be an A^* weight and $\mathcal{Z} \in \mathbb{Z}_M$. The following conditions are equivalent.*

(i) $w \in \mathcal{W}^*(\mathcal{Z})$.
(ii) *For any $n \in \mathbb{N}$ and x, y such that $[x, y] \subset \mathcal{I}_{1,1/n}$ and $|x - y| \le \rho_n(x)$, inequalities (3.1) are satisfied with the constant c_* depending only on w.*
(iii) *For some $N \in \mathbb{N}$ that depends only on w, and any $n \ge N$ and x, y such that $[x, y] \subset \mathcal{I}_{1,1/n}$ and $|x - y| \le \rho_n(x)$, inequalities (3.1) are satisfied with the constant c_* depending only on w.*
(iv) *For any $n \in \mathbb{N}$, $A, B > 0$, and x, y such that $[x, y] \subset \mathcal{I}_{A,1/n}$ and $|x - y| \le B\rho_n(x)$, inequalities (3.1) are satisfied with the constant c_* depending only on w, A and B.*
(v) *For any $n \in \mathbb{N}$ and $A > 0$,*

$$w(x) \sim w_n(x), \quad x \in \mathcal{I}_{A,1/n},$$

where the equivalence constants depend only on w and A, and are independent of x and n.

For $r \in \mathbb{N}$, $t > 0$ and $\mathcal{Z} \in \mathbb{Z}_M$, *the main part weighted modulus of smoothness* is defined as

$$\Omega_\varphi^r(f, A, t)_w := \Omega_\varphi^r(f, A, t; \mathcal{Z})_w := \sup_{0<h\le t} \left\| w(\cdot)\Delta_{h\varphi(\cdot)}^r(f, \cdot, \mathcal{I}_{A,h}) \right\|, \tag{3.2}$$

where

$$\Delta_h^r(f, x, J) := \begin{cases} \displaystyle\sum_{i=0}^r \binom{r}{i}(-1)^{r-i} f(x - rh/2 + ih), & \text{if } [x - rh/2, x + rh/2] \subset J, \\ 0, & \text{otherwise} \end{cases} \tag{3.3}$$

is the rth symmetric difference.

Note that if we denote

$$\mathfrak{D}(A, h, r) := \left\{ x \;\middle|\; [x - rh\varphi(x)/2, x + rh\varphi(x)/2] \subset \mathcal{I}_{A,h} \right\} \tag{3.4}$$

then

$$\Omega_\varphi^r(f, A, t)_w = \sup_{0<h\le t} \left\| w(\cdot)\Delta_{h\varphi(\cdot)}^r(f, \cdot, \mathbb{R}) \right\|_{\mathfrak{D}(A,h,r)}.$$

The *weighted Ditzian–Totik modulus of smoothness* (see [1]) is

$$\omega_\varphi^r(f, t)_w := \sup_{0<h\le t} \left\| w(\cdot)\Delta_{h\varphi(\cdot)}^r(f, \cdot, [-1, 1]) \right\|.$$

For $A, B, t > 0$, we define the *complete weighted modulus of smoothness* as

$$\omega_\varphi^r(f, A, B, t)_w := \omega_\varphi^r(f, A, B, t; \mathcal{Z})_w := \Omega_\varphi^r(f, A, t; \mathcal{Z})_w + \sum_{j=1}^M E_r(f, \mathcal{Z}_{B,t}^j)_w. \tag{3.5}$$

We will also need the following auxiliary quantity ("restricted main part modulus"):

$$\Omega_\varphi^r(f,t)_{S,w} := \sup_{0<h\le t} \left\| w(\cdot)\Delta_{h\varphi(\cdot)}^r(f,\cdot,S)\right\|, \tag{3.6}$$

where S is some subset (a union of intervals) of $[-1,1]$ that does not depend on h.

4. Properties of main part and complete weighted moduli

Proposition 4.1. *For any weight function w and a set $\mathcal{Z} \in \mathbb{Z}_M$, the moduli defined in (3.2), (3.5) and (3.6) have the following properties:*

(i) $\Omega_\varphi^r(f,A,t)_w = \Omega_\varphi^r(f,A,\sqrt{2/A})_w$ *for any* $t \ge \sqrt{2/A}$*;*
(ii) $\omega_\varphi^r(f,A,B,t)_w = \omega_\varphi^r(f,A,B,t_0)_w \ge ME_r(f,[-1,1])_w$ *for any* $t \ge t_0 := \max\{\sqrt{2/A}, \sqrt{2/B}\}$*;*
(iii) $\Omega_\varphi^r(f,A,t_1)_w \le \Omega_\varphi^r(f,A,t_2)_w$ *and* $\omega_\varphi^r(f,A,B,t_1)_w \le \omega_\varphi^r(f,A,B,t_2)_w$ *if* $0 < t_1 \le t_2$*;*
(iv) $\Omega_\varphi^r(f,A_1,t)_w \ge \Omega_\varphi^r(f,A_2,t)_w$ *and* $\omega_\varphi^r(f,A_1,B,t)_w \ge \omega_\varphi^r(f,A_2,B,t)_w$ *if* $A_1 \le A_2$*;*
(v) $\omega_\varphi^r(f,A,B_1,t)_w \le \omega_\varphi^r(f,A,B_2,t)_w$ *if* $B_1 \le B_2$*;*
(vi) $\Omega_\varphi^r(f,c_*t)_{\mathcal{I}_{A,t},w} \le \Omega_\varphi^r(f,A/\max\{c_*,c_*^2\},c_*t)_w$ *for any* $t > 0$ *and* $c_* > 0$*.*

Proof. Properties (i) and (ii) immediately follow from the observation that, if $h \ge \sqrt{2/C}$, then $C\rho(h,z_j) \ge 2$. Properties (iii) and (v) follow from the definition and the fact that $\mathcal{Z}_{B_1,t_1}^j \subset \mathcal{Z}_{B_2,t_2}^j$ if $t_1 \le t_2$ and $B_1 \le B_2$. Property (iv) is a consequence of the inclusion $\mathcal{I}_{A_2,h} \subset \mathcal{I}_{A_1,h}$ if $A_1 \le A_2$. Property (vi) follows from the observation that, for $c_* > 0$ and $0 < h \le c_*t$, since $\rho(h,z_j)/\max\{c_*,c_*^2\} \le \rho(t,z_j)$, then $\mathcal{I}_{A,t} \subset \mathcal{I}_{A/\max\{c_*,c_*^2\},h}$. $\square$

We need an auxiliary lemma that is used in the proofs of several results below.

Lemma 4.2. *Suppose that $\mathcal{Z} \in \mathbb{Z}_M$ and $w \in \mathcal{W}^*(\mathcal{Z})$. If $A,h > 0$, $r \in \mathbb{N}$ and $x \in [-1,1]$ are such that*

$$[x - rh\varphi(x)/2, x + rh\varphi(x)/2] \subset \mathcal{I}_{A,h} \quad (\text{i.e., } x \in \mathfrak{D}(A,h,r)),$$

then, for any $y \in [x - rh\varphi(x)/2, x + rh\varphi(x)/2]$,

$$w(y) \sim w(x) \sim w_n(x),$$

where $n := \lceil 1/h \rceil$, and the equivalence constants depend only on r, A and the weight w.

Proof. First we note that, if $h > \sqrt{2/A}$, then $A\rho(h,z_j) > 2$, and so $\mathcal{I}_{A,h} = \emptyset$. Hence, we can assume that $0 < h \le \sqrt{2/A}$. Now, if $n = \lceil 1/h \rceil$, then $n \in \mathbb{N}$, $n^{-1} \le h < (n-1)^{-1}$ and $\mathcal{I}_{A,h} \subset \mathcal{I}_{A,1/n}$. Moreover, if $n \ge 2$, then $(n-1)^{-1} \le 2/n$ and so $\rho(h,x) \le 4\rho_n(x)$ and, if $n = 1$, then

$$\rho(h,x) \le \rho(\sqrt{2/A},x) \le \max\{\sqrt{2/A}, 2/A\}\rho_n(x).$$

Hence, if $y \in [x - rh\varphi(x)/2, x + rh\varphi(x)/2]$, then $[x,y] \subset \mathcal{I}_{A,1/n}$ and

$$|x - y| \le rh\varphi(x)/2 \le r\rho(h,x)/2 \le (r/2)\max\{4, \sqrt{2/A}, 2/A\}\rho_n(x).$$

Therefore, Lemma 3.4(iv) implies that $w(y) \sim w(x)$, and Lemma 3.4(v) yields the equivalence $w(x) \sim w_n(x)$. $\square$

In the following lemma and in the sequel, we use the usual notation

$$\mathbb{L}_\infty^w := \left\{ f : [-1,1] \mapsto \mathbb{R} \;\middle|\; \|wf\| < \infty \right\}.$$

Lemma 4.3. *If $\mathcal{Z} \in \mathbb{Z}_M$, $w \in \mathcal{W}^*(\mathcal{Z})$, $f \in \mathbb{L}_\infty^w$, $r \in \mathbb{N}$, and $A, B, t > 0$, then*

$$\omega_\varphi^r(f, A, B, t)_w \le c\, \|wf\|,$$

where c depends only on r, A and the weight w.

Proof. First of all, it is clear that

$$\sum_{j=1}^M E_r(f, \mathcal{Z}_{B,t}^j)_w \le \sum_{j=1}^M \|wf\|_{\mathcal{Z}_{B,t}^j} \le M\, \|wf\|.$$

We now let $h \in (0, t]$ and x be such that $[x - rh\varphi(x)/2, x + rh\varphi(x)/2] \subset \mathcal{I}_{A,h}$, and denote $y_i(x) := x + (i - r/2)h\varphi(x)$. Then, Lemma 4.2 implies that $w(y_i(x)) \sim w(x)$, $0 \le i \le r$, and so

$$\begin{aligned} w(x) \left| \Delta_{h\varphi(x)}^r(f, x, \mathcal{I}_{A,h}) \right| &\le w(x) \sum_{i=0}^r \binom{r}{i} |f(y_i(x))| \le 2^r w(x) \max_{0 \le i \le r} |f(y_i(x))| \\ &\le c \max_{0 \le i \le r} |w(y_i(x)) f(y_i(x))|. \end{aligned}$$

This yields $\Omega_\varphi^r(f, A, t)_w \le c\, \|wf\|$, which completes the proof of the lemma. □

Taking into account that $\omega_\varphi^r(f, A, B, t)_w = \omega_\varphi^r(f - q, A, B, t)_w$, for any $q \in \Pi_r$, we immediately get the following corollary.

Corollary 4.4. *If $\mathcal{Z} \in \mathbb{Z}_M$, $w \in \mathcal{W}^*(\mathcal{Z})$, $f \in \mathbb{L}_\infty^w$, $r \in \mathbb{N}$, and $A, B, t > 0$, then*

$$\omega_\varphi^r(f, A, B, t)_w \le c E_r(f, [-1,1])_w,$$

where c depends only on r, A and the weight w.

Lemma 4.5. *If $\mathcal{Z} \in \mathbb{Z}_M$, $w \in \mathcal{W}^*(\mathcal{Z})$, $f \in \mathbb{L}_\infty^w$, $r \in \mathbb{N}$, and $A, t > 0$, then*

$$\Omega_\varphi^r(f, A, 2t)_w \le c \Omega_\varphi^r(f, \sqrt{2}A, \sqrt{2}t)_w, \tag{4.1}$$

where c depends only on r, A and the weight w.

Now, Proposition 4.1(iii and iv) and Lemma 4.5 imply the following result.

Corollary 4.6. *If $\mathcal{Z} \in \mathbb{Z}_M$, $w \in \mathcal{W}^*(\mathcal{Z})$, $f \in \mathbb{L}_\infty^w$, $r \in \mathbb{N}$, and $A, t > 0$, then*

$$\Omega_\varphi^r(f, A, t)_w \sim \Omega_\varphi^r(f, \sqrt{2}A, t)_w,$$

and so

$$\Omega_\varphi^r(f, A, t)_w \sim \Omega_\varphi^r(f, 1, t)_w,$$

where the equivalence constants depend only on r, A and the weight w.

Moreover,

$$\Omega_\varphi^r(f,1,t)_w \le \Omega_\varphi^r(f,1,2t)_w \le c\Omega_\varphi^r(f,1,t)_w,$$

where c depends only on r and the weight w.

Proof of Lemma 4.5. Recall a rather well-known identity (see [9, (5) on p. 42], for example)

$$\Delta_{2h}^r(f,x) = \sum_{i_1=0}^{1}\cdots\sum_{i_r=0}^{1}\Delta_h^r\left(f, x+[i_1+\cdots+i_r-r/2]h\right). \tag{4.2}$$

Now, we fix $h\in(0,t]$, and let x be a fixed number such that $[x-rh\varphi(x), x+rh\varphi(x)]\subset \mathcal{I}_{A,2h}$ (*i.e.*, $x\in\mathfrak{D}(A,2h,r)$). We have

$$\begin{aligned}\left|\Delta_{2h\varphi(x)}^r(f,x,\mathcal{I}_{A,2h})\right| &\le \sum_{i_1=0}^{1}\cdots\sum_{i_r=0}^{1}\left|\Delta_{h\varphi(x)}^r\left(f,x+[i_1+\cdots+i_r-r/2]h\varphi(x)\right)\right|\\ &\le 2^r\left|\Delta_{h\varphi(x)}^r(f,y)\right| =: 2^rF,\end{aligned}$$

where $y := x+\gamma h\varphi(x)$, and γ is such that $\gamma+r/2\in\{0,1,\dots,r\}$ (and so $|\gamma|\le r/2$) and

$$\left|\Delta_{h\varphi(x)}^r(f,y)\right| = \max_{0\le m\le r}\left|\Delta_{h\varphi(x)}^r\left(f,x+[m-r/2]h\varphi(x)\right)\right|.$$

Note that Lemma 4.2 implies that $w(x)\sim w(y)$. Also, since $x\pm rh\varphi(x)\in[-1,1]$, we have $|x|\le(1-r^2h^2)/(1+r^2h^2)$, which implies

$$\frac{|x|}{\varphi(x)} \le \frac{1-r^2h^2}{2rh},$$

and so

$$\left[\frac{\varphi(y)}{\varphi(x)}\right]^2 = 1-\gamma^2h^2-2\gamma h\frac{x}{\varphi(x)} \ge 1-\gamma^2h^2-2|\gamma|h\frac{|x|}{\varphi(x)} \ge \frac12+\frac{r^2h^2}{4}\ge\frac12.$$

Therefore, $\varphi(x)\le\sqrt{2}\varphi(y)$, and

$$w(x)F \le cw(y)\left|\Delta_{h^*\varphi(y)}^r(f,y)\right|,$$

where

$$0<h^* := \frac{h\varphi(x)}{\varphi(y)} \le \sqrt{2}h \le \sqrt{2}t.$$

We now note that $\rho(2h,z_j)\ge\sqrt{2}\rho(h^*,z_j)$ which implies $\mathcal{I}_{A,2h}\subset\mathcal{I}_{\sqrt{2}A,h^*}$, and so

$$\begin{aligned}&[y-rh^*\varphi(y)/2, y+rh^*\varphi(y)/2] = [x+(\gamma-r/2)h\varphi(x), x+(\gamma+r/2)h\varphi(x)]\\ &\quad\subset[x-rh\varphi(x), x+rh\varphi(x)]\subset\mathcal{I}_{A,2h}\subset\mathcal{I}_{\sqrt{2}A,h^*}.\end{aligned}$$

Therefore, $\Delta^r_{h^*\varphi(y)}(f,y) = \Delta^r_{h^*\varphi(y)}\left(f, y, \mathcal{I}_{\sqrt{2}A,h^*}\right)$, and so we have

$$w(x)F \le c \sup_{0<h^*\le\sqrt{2}t} \operatorname{ess\,sup}_y w(y)\left|\Delta^r_{h^*\varphi(y)}\left(f,y,\mathcal{I}_{\sqrt{2}A,h^*}\right)\right| \le c\Omega^r_\varphi(f,\sqrt{2}A,\sqrt{2}t)_w$$

for almost all $x \in \mathfrak{D}(A,2h,r)$. The lemma is now proved. □

Lemma 4.7. *Let $\mathcal{Z} \in \mathbb{Z}_M$, $w \in \mathcal{W}^*(\mathcal{Z})$, $r \in \mathbb{N}$, $z \in \mathcal{Z}$, $z \ne 1$, $0 < \varepsilon < \delta(\mathcal{Z})/2$, $I := [z+\varepsilon/2, z+\varepsilon]$, and let $J := [z+\varepsilon, z+\varepsilon+\delta]$ with δ such that $0 < \delta \le \varepsilon/(2r)$. Then, for any $h \in [\delta, \varepsilon/(2r)]$ and any polynomial $q \in \Pi_r$, we have*

$$\|w(f-q)\|_J \le c\,\|w(\cdot)\Delta^r_h(f,\cdot,I\cup J)\|_{I\cup J} + c\,\|w(f-q)\|_I\,. \tag{4.3}$$

Additionally,

$$\|w(f-q)\|_J \le c\Omega^r_\varphi(f,54\delta t/\varepsilon)_{I\cup J,w} + c\,\|w(f-q)\|_I\,, \tag{4.4}$$

where $0 < t < 1$ is such that $\varepsilon = \rho(t,z)$, and all constants c depend only on r and the weight w.

Remark 4.8. By symmetry, the statement of the lemma is also valid for $I := [z-\varepsilon, z-\varepsilon/2]$ and $J := [z-\varepsilon-\delta, z-\varepsilon]$, where $z \in \mathcal{Z}$ is such that $z \ne -1$.

Remark 4.9. The condition $\varepsilon < \delta(\mathcal{Z})/2$ guarantees that I is "far" from all other points in $\mathcal{Z}$. In particular, $[z+\varepsilon/2, z+2\varepsilon] \cap (\mathcal{Z} \cup \{\pm 1\}) = \emptyset$.

Proof of Lemma 4.7. Denoting for convenience $g := f - q$ and taking into account that $\Delta^r_h(g,x,\mathbb{R}) = \Delta^r_h(f,x,\mathbb{R})$ we have

$$g(x+rh/2) = \Delta^r_h(f,x,\mathbb{R}) - \sum_{i=0}^{r-1}\binom{r}{i}(-1)^{r-i}g(x-rh/2+ih).$$

We now fix $h \in [\delta, \varepsilon/(2r)]$, and note that, for any x such that $x + rh/2 \in J$, we have $[x - rh/2, x + (r-2)h/2] \subset I$, and so

$$\begin{aligned}\|g\|_J &\le \|\Delta^r_h(f,\cdot,\mathbb{R})\|_{[z+\varepsilon-rh/2,z+\varepsilon+\delta-rh/2]} + (2^r-1)\,\|g\|_I\\ &\le \|\Delta^r_h(f,\cdot,I\cup J)\|_{I\cup J} + (2^r-1)\,\|g\|_I\,.\end{aligned}$$

Suppose now that $0 < t < 1$ is such that $\varepsilon = \rho(t,z)$, let $n \in \mathbb{N}$ be such that $n := \lfloor 1/t \rfloor$ and pick A so that $\varepsilon = A\rho_n(z)$. Note that $\rho_{n+1}(z) < \varepsilon \le \rho_n(z)$ and $1/4 < A \le 1$. Hence, $\operatorname{dist}(I\cup J, z) = \varepsilon/2 \ge \rho_n(z)/8$. Suppose now that $\widetilde{z} \in \mathcal{Z}$ is such that $\widetilde{z} > z$ and $(z,\widetilde{z}) \cap \mathcal{Z} = \emptyset$, *i.e.*, $\widetilde{z}$ is the "next" point from $\mathcal{Z}$ to the right of z (if there is no such $\widetilde{z}$ then there is nothing to do, and the next paragraph can be skipped).

We will now show that $\widetilde{d} := \operatorname{dist}(I\cup J, \widetilde{z}) \ge \rho_n(\widetilde{z})/20$. Indeed, $\widetilde{d} = \widetilde{z} - z - (\varepsilon+\delta) \ge \delta(\mathcal{Z}) - 3\varepsilon/2 \ge \varepsilon/2$. If $\varepsilon > \rho_n(\widetilde{z})/10$, then we are done, and so we suppose that $\varepsilon \le \rho_n(\widetilde{z})/10$. Recall (see *e.g.* [4, p. 27]) the well-known fact that

$$\rho_n(u)^2 \le 4\rho_n(v)(|u-v| + \rho_n(v)), \quad \text{for all } u,v \in [-1,1]. \tag{4.5}$$

This implies

$$|\widetilde{z} - z| \ge \frac{\rho_n(\widetilde{z})^2}{4\rho_n(z)} - \rho_n(z) \ge \frac{5A}{2}\rho_n(\widetilde{z}) - \frac{\varepsilon}{A} \ge \left(\frac{5A}{2} - \frac{1}{10A}\right)\rho_n(\widetilde{z}) \ge (9/40)\rho_n(\widetilde{z}).$$

Also, $|\widetilde{z} - z| = \widetilde{d} + \varepsilon + \delta \le \widetilde{d} + 3\varepsilon/2 \le 4\widetilde{d}$, which implies $\widetilde{d} \ge (9/160)\rho_n(\widetilde{z}) \ge \rho_n(\widetilde{z})/20$ as needed. Therefore, we can conclude that

$$I \cup J \subset \mathcal{I}_{1/20,1/n}.$$

Now, using (4.5) we conclude that, if $u \in I \cup J$, then $|u - z| \le 3\varepsilon/2 \le 3\rho_n(z)/2$, and so

$$\rho_n(u)^2 \le 4\rho_n(z)(|u - z| + \rho_n(z)) \le 10\rho_n(z)^2$$

and

$$\rho_n(z)^2 \le 4\rho_n(u)(|u - z| + \rho_n(u)) \le 4\rho_n(u)(3\rho_n(z)/2 + \rho_n(u)).$$

This implies that, for any $u \in I \cup J$,

$$\rho_n(u)/4 \le \rho_n(z) \le 7\rho_n(u).$$

Hence, for any $u, v \in I \cup J$,

$$|u - v| \le \varepsilon \le \rho_n(z) \le 7\rho_n(u).$$

It now follows from Lemma 3.4(iv) that $w(u) \sim w(v)$, for any $u, v \in I \cup J$, and so

$$\|wg\|_J \le c \|w(\cdot)\Delta_h^r(f, \cdot, I \cup J)\|_{I\cup J} + c \|wg\|_I ,$$

and (4.3) is proved.

In order to prove (4.4), we note that, for any $x \in I \cup J$,

$$1 - |x| \ge \varepsilon/2 = \rho(t, z)/2 \ge t^2/2,$$

which implies $\varphi(x) \ge t/\sqrt{2}$, and so, with $h := \delta$, we have

$$0 < \frac{h}{\varphi(x)} \le \frac{\delta\rho_n(z)}{\varepsilon\varphi(x)} \le \frac{7\delta\rho_n(x)}{\varepsilon\varphi(x)} \le \frac{7\delta}{\varepsilon n}\left(1 + \frac{\sqrt{2}}{nt}\right) \le \frac{7\delta t}{\varepsilon}\left(2 + 4\sqrt{2}\right) \le 54\delta t/\varepsilon.$$

Therefore, for almost all $x \in I \cup J$, denoting $h^* := h/\varphi(x)$ we have

$$\begin{aligned} w(x)\Delta_h^r(f, x, I \cup J) &= w(x)\Delta_{h^*\varphi(x)}^r(f, x, I \cup J) \\ &\le \sup_{0<h\le 54\delta t/\varepsilon} \left\| w(\cdot)\Delta_{h\varphi(\cdot)}^r(f, \cdot, I \cup J) \right\|, \end{aligned}$$

and the proof of (4.4) is complete. $\square$

Corollary 4.10. *Let $\mathcal{Z} \in \mathbb{Z}_M$, $w \in \mathcal{W}^*(\mathcal{Z})$, $r \in \mathbb{N}$, $B > 0$, and let $0 < t < c_0$, where c_0 is such that $\max_{1\le j\le M} \rho(c_0, z_j) \le \delta(\mathcal{Z})/(2B)$ (for example, $c_0 := \min\{1, \delta(\mathcal{Z})/(4B)\}$ will do). Then,*

$$\omega_\varphi^r(f, 1, B(1 + 1/(2r)), t)_w \le c\omega_\varphi^r(f, 1, B, t)_w,$$

where the constant c depends only on r, B and the weight w.

Taking into account that $(1 + 1/(2r))^m \geq 2$ for $m = \lceil 1/\log_2(1 + 1/(2r))\rceil$, we immediately get the following result.

Corollary 4.11. *Let $\mathcal{Z} \in \mathbb{Z}_M$, $w \in \mathcal{W}^*(\mathcal{Z})$, $r \in \mathbb{N}$, $B > 0$, and let $0 < t < c_0$, where c_0 is such that $\max_{1\leq j\leq M} \rho(c_0, z_j) \leq \delta(\mathcal{Z})/(2B)$ (for example, $c_0 := \min\{1, \delta(\mathcal{Z})/(4B)\}$ will do). Then,*

$$\omega_\varphi^r(f, 1, B, t)_w \leq c\omega_\varphi^r(f, 1, B/2, t)_w,$$

where the constant c depends only on r, B and the weight w.

Proof of Corollary 4.10. For each $1 \leq j \leq M$, let $\varepsilon_j := B\rho(t, z_j)$ and note that $\varepsilon_j < \delta(\mathcal{Z})/2$. It follows from Lemma 4.7 and the remark after it that, for any $q_j \in \Pi_r$, $\delta_j := \varepsilon_j/(2r)$ and τ_j such that $\rho(\tau_j, z_j) = B\rho(t, z_j) = \varepsilon_j$, we have

$$\|w(f - q_j)\|_{J_j^r} \leq c\Omega_\varphi^r(f, 27\tau_j/r)_{I_j^r\cup J_j^r,w} + c\,\|w(f - q_j)\|_{I_j^r},$$

where $I_j^r := [z_j + \varepsilon_j/2, z_j + \varepsilon_j]$ and $J_j^r := [z_j + \varepsilon_j, z_j + \varepsilon_j + \delta_j]$, and

$$\|w(f - q_j)\|_{J_j^l} \leq c\Omega_\varphi^r(f, 27\tau_j/r)_{I_j^l\cup J_j^l,w} + c\,\|w(f - q_j)\|_{I_j^l},$$

where $I_j^l := [z_j - \varepsilon_j, z_j - \varepsilon_j/2]$ and $J_j^l := [z_j - \varepsilon_j - \delta_j, z_j - \varepsilon_j]$. Note that, if $z_j = 1$ or -1, we do not consider I_j^r, J_j^r or I_j^l, J_j^l, respectively.

We now note that $[z_j - \varepsilon_j, z_j + \varepsilon_j] \cap [-1, 1] = \mathcal{Z}_{B,t}^j$ and $[z_j - \varepsilon_j - \delta_j, z_j + \varepsilon_j + \delta_j] \cap [-1, 1] = \mathcal{Z}_{\widetilde{B},t}^j$, where $\widetilde{B} := B(1 + 1/(2r))$. Letting $q_j \in \Pi_r$ be such that

$$\|w(f - q_j)\|_{\mathcal{Z}_{B,t}^j} \leq cE_r(f, \mathcal{Z}_{B,t}^j)_w,$$

we have

$$\begin{aligned}
E_r(f, \mathcal{Z}_{\widetilde{B},t}^j)_w &\leq \|w(f - q_j)\|_{\mathcal{Z}_{\widetilde{B},t}^j} \\
&\leq \|w(f - q_j)\|_{J_j^l} + \|w(f - q_j)\|_{\mathcal{Z}_{B,t}^j} + \|w(f - q_j)\|_{J_j^r} \\
&\leq c\,\|w(f - q_j)\|_{\mathcal{Z}_{B,t}^j} + c\Omega_\varphi^r(f, 27\tau_j/r)_{I_j^l\cup J_j^l,w} + c\Omega_\varphi^r(f, 27\tau_j/r)_{I_j^r\cup J_j^r,w} \\
&\leq cE_r(f, \mathcal{Z}_{B,t}^j)_w + c\Omega_\varphi^r(f, 27\tau_j/r)_{I_j^l\cup J_j^l,w} + c\Omega_\varphi^r(f, 27\tau_j/r)_{I_j^r\cup J_j^r,w}.
\end{aligned}$$

Now, note that

$$I_j^r \cup J_j^r \subset \mathfrak{I}_{B/2,t} \quad \text{and} \quad I_j^l \cup J_j^l \subset \mathfrak{I}_{B/2,t}.$$

Hence, taking into account that $\tau_j \leq \max\{B, \sqrt{B}\}t$ we get

$$\begin{aligned}
E_r(f, \mathcal{Z}_{\widetilde{B},t}^j)_w &\leq cE_r(f, \mathcal{Z}_{B,t}^j)_w + c\Omega_\varphi^r(f, 27\tau_j/r)_{\mathfrak{I}_{B/2,t},w} \\
&\leq cE_r(f, \mathcal{Z}_{B,t}^j)_w + c\Omega_\varphi^r\left(f, 27\max\{B, \sqrt{B}\}t/r\right)_{\mathfrak{I}_{B/2,t},w}.
\end{aligned}$$

Now, with $c_* := 27\max\{B, \sqrt{B}\}/r$, Proposition 4.1(vi) and Corollary 4.6 imply

$$\Omega_\varphi^r(f, c_*t)_{\mathfrak{I}_{B/2,t},w} \leq \Omega_\varphi^r\left(f, B/(2\max\{c_*, c_*^2\}), c_*t\right)_w \leq c\Omega_\varphi^r(f, 1, t)_w.$$

Therefore,

$$\begin{aligned}
\omega_\varphi^r(f,1,\widetilde{B},t)_w &= \Omega_\varphi^r(f,1,t)_w + \sum_{j=1}^{M} E_r(f,\mathcal{Z}^j_{\widetilde{B},t})_w \\
&\le c\Omega_\varphi^r(f,1,t)_w + c\sum_{j=1}^{M} E_r(f,\mathcal{Z}^j_{B,t})_w \\
&\le c\omega_\varphi^r(f,1,B,t)_w,
\end{aligned}$$

and the proof is complete. □

5. Auxiliary results

Theorem 5.1. *(See [7, (6.10)].) Let W be a 2π-periodic function which is an A^* weight on $[0,2\pi]$. Then there is a constant $C>0$ such that if T_n is a trigonometric polynomial of degree at most n and E is a measurable subset of $[0,2\pi]$ of measure at most Λ/n, $1\le\Lambda\le n$, then*

$$\|T_nW\|_{[0,2\pi]} \le C^{\Lambda}\|T_nW\|_{[0,2\pi]\setminus E}.$$

The following result is essentially proved in [7]. However, since it was not stated there explicitly we sketch its very short proof below.

Corollary 5.2. *Let $w\in A^*_{L^*}$. If $E\subset[-1,1]$ is such that $\int_E(1-x^2)^{-1/2}dx\le\lambda/n$ with $\lambda\le n/2$, then for each $P_n\in\Pi_n$, we have*

$$\|P_nw\|_{[-1,1]} \le c\|P_nw\|_{[-1,1]\setminus E},$$

where the constant c depends only on λ and L^*.

Proof. Let $W(t):=w(\cos t)$, $T_n(t):=P_n(\cos t)$ and $\widetilde{E}:=\{0\le t\le 2\pi \mid \cos t\in E\}$. Note that W is a 2π-periodic function which is an A^* weight on $[0,2\pi]$ (see [7, p. 68]), and

$$\operatorname{meas}(\widetilde{E}) = \int_{\widetilde{E}} dt = 2\int_{\widetilde{E}\cap[0,\pi]} dt = 2\int_E (1-x^2)^{-1/2}dx \le 2\lambda/n.$$

Hence,

$$\|P_nw\| = \|T_nW\|_{[0,2\pi]} \le c\|T_nW\|_{[0,2\pi]\setminus\widetilde{E}} = c\|P_nw\|_{[-1,1]\setminus E}. \qquad \square$$

Lemma 5.3. *(See [7, (7.27)].) Let w be an A^* weight on $[-1,1]$. Then, for all $n\in\mathbb{N}$ and $P_n\in\Pi_n$,*

$$\|P_nw\| \sim \|P_nw_n\|$$

with the equivalence constants independent of P_n and n.

It is convenient to denote $\varphi_n(x):=\varphi(x)+1/n$, $n\in\mathbb{N}$, and note that $w:=\varphi$ is an A^* weight and $w_n\sim\varphi_n$ on $[-1,1]$.

One of the applications of Corollary 5.2 is the following quite useful result.

Theorem 5.4. *Let w be an A^* weight, $n \in \mathbb{N}$, $0 \le \mu \le n$. Then, for any $P_n \in \Pi_n$,*

$$\|w\varphi^\mu P_n\| \sim \|w_n \varphi^\mu P_n\| \tag{5.1}$$

and

$$\|w\lambda_n^\mu P_n\| \sim \|w_n \lambda_n^\mu P_n\|\,, \tag{5.2}$$

where $\lambda_n(x) := \max\left\{\sqrt{1-x^2}, 1/n\right\}$, and the equivalence constants are independent of μ, n and P_n.

Proof. We start with the equivalence (5.1). Let $m := 2\lfloor \mu/2 \rfloor$. Then m is an even integer such that $\mu - 2 < m \le \mu$ (note that $m = 0$ if $\mu < 2$), and $Q_{n+m} := \varphi^m P_n \in \Pi_{n+m} \subset \Pi_{2n}$.

Since w is an A^* weight, then $w\varphi^\gamma$, $\gamma > 0$, is also an A^* weight (see Remark 2.3) and

$$(w\varphi^\gamma)_n \sim w_n \varphi_n^\gamma,$$

where the equivalence constants depend on $\lceil \gamma \rceil$ and the doubling constant of w.

Hence, denoting $E_n := [-1+n^{-2}, 1-n^{-2}]$, $\eta := \mu - m$, noting that $0 \le \eta < 2$ (and so $\lceil \eta \rceil$ is either 0, 1 or 2 allowing us to replace constants that depend on $\lceil \eta \rceil$ by those independent of η), and using Lemmas 5.3 and 2.4, Corollary 5.2, and the observation that $w_n(x) \sim w_k(x)$ if $n \sim k$, we have

$$\begin{aligned}\|\varphi^\mu w P_n\| &= \|\varphi^\eta w Q_{n+m}\| \sim \|(w\varphi^\eta)_n Q_{n+m}\| \sim \|(w\varphi^\eta)_n Q_{n+m}\|_{E_n} \\ &\sim \|w_n \varphi_n^\eta Q_{n+m}\|_{E_n} \sim \|w_n \varphi^\eta Q_{n+m}\|_{E_n}\,.\end{aligned}$$

Since $w_n\varphi^\eta$ is an A^* weight (see Remark 2.3), we can continue as follows:

$$\|w_n \varphi^\eta Q_{n+m}\|_{E_n} \sim \|w_n \varphi^\eta Q_{n+m}\| = \|w_n \varphi^\mu P_n\|\,.$$

Note that none of the constants in the equivalences above depend on μ. This completes the proof of (5.1).

Now, let $\mathcal{E}_n := \left\{x \;\middle|\; \sqrt{1-x^2} \le 1/n\right\}$ and note that $\lambda_n(x) = 1/n$ if $x \in \mathcal{E}_n$, and $\lambda_n(x) = \varphi(x)$ if $x \in [-1,1] \setminus \mathcal{E}_n$. Using (5.1) we have

$$\begin{aligned}\|w\lambda_n^\mu P_n\| &\le \|w\lambda_n^\mu P_n\|_{\mathcal{E}_n} + \|w\lambda_n^\mu P_n\|_{[-1,1]\setminus\mathcal{E}_n} \\ &= n^{-\mu}\,\|wP_n\|_{\mathcal{E}_n} + \|w\varphi^\mu P_n\|_{[-1,1]\setminus\mathcal{E}_n} \\ &\le n^{-\mu}\,\|wP_n\| + \|w\varphi^\mu P_n\| \\ &\le c_0\left(n^{-\mu}\,\|w_n P_n\| + \|w_n\varphi^\mu P_n\|\right) \\ &\le 2c_0\,\|w_n\lambda_n^\mu P_n\|\,.\end{aligned}$$

In the other direction, the sequence of inequalities is exactly the same (switching w and w_n). This verifies (5.2). $\square$

If we allow constants to depend on μ, then we have the following result.

Corollary 5.5. *Let w be an A^* weight, $n \in \mathbb{N}$ and $\mu \ge 0$. Then, for any $P_n \in \Pi_n$,*

$$\|w\varphi_n^\mu P_n\| \sim \|w\varphi^\mu P_n\| \sim \|w_n\varphi^\mu P_n\| \sim \|w_n\varphi_n^\mu P_n\|\,,$$

where all equivalence constants are independent of n and P_n.

Proof. Since $\lambda_n(x) \le \varphi_n(x) \le 2\lambda_n(x)$ and $\varphi(x) \le \varphi_n(x)$, we immediately get from Theorem 5.4

$$\|w\varphi^\mu P_n\| \sim \|w_n\varphi^\mu P_n\| \le \|w_n\varphi_n^\mu P_n\| \sim \|w\varphi_n^\mu P_n\| .$$

At the same time,

$$\|w\varphi^\mu P_n\| \sim \|(w\varphi^\mu)_n P_n\| \sim \|w_n\varphi_n^\mu P_n\| ,$$

and the proof is complete. □

Theorem 5.6 *(Markov–Bernstein type theorem). Let w be an A^* weight and $r \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$ and $P_n \in \Pi_n$,*

$$n^{-r}\left\|w\varphi^r P_n^{(r)}\right\| \sim n^{-r}\left\|w_n\varphi^r P_n^{(r)}\right\| \sim \left\|w_n\rho_n^r P_n^{(r)}\right\| \sim \left\|w\rho_n^r P_n^{(r)}\right\| \le c\,\|wP_n\| \sim \|w_nP_n\| ,$$

where the constant c and all equivalence constants are independent of n and P_n.

Proof. The statement of the lemma is an immediate consequence of Corollary 5.5 and either of the estimates

$$\left\|w_n\rho_n^r P_n^{(r)}\right\| \le c\,\|w_nP_n\|$$

(see [3, Lemma 6.1], for example), or

$$\left\|w\varphi^r P_n^{(r)}\right\| \le cn^r\,\|wP_n\|$$

(see [7, (7.29)] or [8, (2.5)]), where the constant c depends only on r and the A^* constant of w. □

Lemma 5.7. *Let w be an A^* weight, $A > 0$ and $\mathcal{Z} \in \mathbb{Z}_M$. Then for any $n, r \in \mathbb{N}$, $1 \le j \le M$, and any polynomials $Q_n \in \Pi_n$ and $q_r \in \Pi_r$ satisfying $Q_n^{(\nu)}(z_j) = q_r^{(\nu)}(z_j)$, $0 \le \nu \le r-1$, the following inequality holds*

$$\|w(Q_n - q_r)\|_{\mathcal{Z}^j_{A,1/n}} \le cn^{-r}\left\|w\varphi^r Q_n^{(r)}\right\| ,$$

where the constant c depends only on r, A and the weight w.

Proof. Denote $I := \mathcal{Z}^j_{A,1/n}$, $z := z_j$, and note that $(Q_n - q_r)^{(\nu)}(z) = 0$, $0 \le \nu \le r-1$. Using Taylor's theorem with the integral remainder we have

$$Q_n(x) - q_r(x) = \frac{1}{(r-1)!}\int_z^x (x-u)^{r-1}Q_n^{(r)}(u)du,$$

which implies

$$\begin{aligned}\|w(Q_n - q_r)\|_I &\le \sup_{x\in I} w(x)\left|\int_z^x (x-u)^{r-1}Q_n^{(r)}(u)du\right| \le \left\|Q_n^{(r)}\right\|_I \sup_{x\in I} w(x)|x-z|^r \\ &\le (A\rho_n(z))^r\left\|Q_n^{(r)}\right\|_I \sup_{x\in I} w(x) \le c\left\|\rho_n^r Q_n^{(r)}\right\|_I \frac{1}{|I|}w(I),\end{aligned}$$

where, in the last inequality, we used the fact that w is an A^* weight and $\rho_n(x) \sim \rho_n(z)$, $x \in I$. Now, since w is doubling, $w(I)/|I| \le cw[z - \rho_n(z), z + \rho_n(z)]/|I| \le cw_n(z) \le cw_n(x)$, $x \in I$, and so

$$\|w(Q_n - q_r)\|_I \le c \left\|w_n \rho_n^r Q_n^{(r)}\right\|_I \le cn^{-r} \left\|w\varphi^r Q_n^{(r)}\right\|,$$

where the last estimate follows from Theorem 5.6. $\square$

Lemma 5.8. *Let $\mathcal{Z} \in \mathbb{Z}_M$, $w \in \mathcal{W}^*(\mathcal{Z})$, $c_* > 0$, $n, r \in \mathbb{N}$, $A > 0$ and $0 < t \le c_*/n$. Then, for any $P_n \in \Pi_n$, we have*

$$\Omega_\varphi^r(P_n, A, t)_w \le ct^r \left\|w\varphi^r P_n^{(r)}\right\|,$$

where c depends only on r, c_* and the weight w.

Remark 5.9. Using the same method as the one used to prove [4, Lemma 8.2] one can show that a stronger result than Lemma 5.8 is valid. Namely, if f is such that $f^{(r-1)} \in \mathrm{AC}_{\mathrm{loc}}((-1,1) \setminus \mathcal{Z})$ and $\left\|w\varphi^r f^{(r)}\right\| < \infty$, then one can show that

$$\Omega_\varphi^r(f, A, t)_w \le ct^r \left\|w\varphi^r f^{(r)}\right\|, \quad t > 0.$$

However, Lemma 5.8 whose proof is simpler and shorter is sufficient for our purposes.

Proof of Lemma 5.8. It follows from [3, Lemma 7.2] and Corollary 5.5 that, for any $c_* > 0$ and $0 < t \le c_*/n$,

$$\omega_\varphi^r(P_n, t)_{w_n} \le ct^r \left\|w_n \varphi^r P_n^{(r)}\right\| \le ct^r \left\|w\varphi^r P_n^{(r)}\right\|,$$

where the constants c depend on r, c_* and the weight w. Therefore, since any A^* weight w satisfies $w(x) \le cw_n(x)$, for any $x \in [-1,1]$ and $n \in \mathbb{N}$ (see (2.4)), we have

$$\begin{aligned}\Omega_\varphi^r(P_n, A, t)_w &= \sup_{0<h\le t} \left\|w(\cdot)\Delta_{h\varphi(\cdot)}^r(P_n, \cdot, \mathcal{I}_{A,h})\right\| \le c \sup_{0<h\le t} \left\|w_n(\cdot)\Delta_{h\varphi(\cdot)}^r(P_n, \cdot, [-1,1])\right\| \\ &\le c\omega_\varphi^r(P_n, t)_{w_n} \le ct^r \left\|w\varphi^r P_n^{(r)}\right\|. \quad \square\end{aligned}$$

6. Direct theorem

Theorem 6.1. *Let $w \in \mathcal{W}^*(\mathcal{Z})$, $r, \nu_0 \in \mathbb{N}$, $\nu_0 \ge r$, $\vartheta > 0$, $f \in \mathbb{L}_\infty^w$ and $B > 0$. Then, there exists $N \in \mathbb{N}$ depending on r, ϑ and the weight w, such that for every $n \ge N$, there is a polynomial $P_n \in \Pi_n$ satisfying*

$$\|w(f - P_n)\| \le c\omega_\varphi^r(f, 1, B, \vartheta/n)_w \tag{6.1}$$

and

$$\left\|w\varphi^\nu P_n^{(\nu)}\right\| \le cn^\nu \omega_\varphi^r(f, 1, B, \vartheta/n)_w, \quad r \le \nu \le \nu_0, \tag{6.2}$$

where constants c depend only on r, ν_0, B, ϑ and the weight w.

We use an idea from [8, Section 3.2] and deduce Theorem 6.1 from the following result that was proved in [3].

Theorem 6.2. *(See [3, Theorem 5.3 (p = ∞)].) Let w be a doubling weight, $r, \nu_0 \in \mathbb{N}$, $\nu_0 \ge r$, and $f \in \mathbb{L}_\infty[-1,1]$. Then, for every $n \ge r$ and $0 < \vartheta \le 1$, there exists a polynomial $P_n \in \Pi_n$ such that*

$$\|w_n(f - P_n)\| \le c\omega_\varphi^r(f, \vartheta/n)_{w_n}$$

and

$$\left\|w_n \rho_n^\nu P_n^{(\nu)}\right\| \le c\omega_\varphi^r(f, \vartheta/n)_{w_n}, \quad r \le \nu \le \nu_0,$$

where constants c depend only on r, ν_0, ϑ and the doubling constant of w.

Proof of Theorem 6.1. Since $\omega_\varphi^r(f, 1, B, t)_w$ is a nondecreasing function of t, without loss of generality we can assume that with $\vartheta \le 1/(2r)$. Suppose that $N \in \mathbb{N}$ is such that $N \ge \max\{r, 100/(\vartheta\delta(\mathcal{Z}))\}$, $n \ge N$, and let $(x_i)_{i=0}^n$ be the Chebyshev partition of $[-1,1]$, *i.e.*, $x_i = \cos(i\pi/n)$, $0 \le i \le n$ (for convenience, we also denote $x_i := -1$, $i \ge n+1$, and $x_i := 1$, $i \le -1$). As usual, we let $I_i := [x_i, x_{i-1}]$ for $1 \le i \le n$. Note that each (nonempty) interval $[z_j, z_{j+1}]$, $0 \le j \le M$, contains at least 10 intervals I_i.

For each $1 \le j \le M$, denote

$$\nu_j := \min\left\{i \;\middle|\; 1 \le i \le n \text{ and } z_j \in I_i\right\} \quad \text{and} \quad J_j := [x_{\nu_j+1}, x_{\nu_j-2}].$$

Note that min in the definition of ν_j is needed if z_j belongs to more than one (closed) interval I_i (in which case ν_j is chosen so that z_j is the left endpoint of I_{ν_j}). Let $q_j \in \Pi_r$ be a polynomial of near best weighted approximation of f on J_j, *i.e.*, $\|w(f - q_j)\|_{J_j} \le cE_r(f, J_j)_w$, $1 \le j \le M$, and define

$$F(x) := \begin{cases} q_j(x), & \text{if } x \in J_i,\ 1 \le j \le M, \\ f(x), & \text{otherwise.} \end{cases}$$

Since (see [4, p. 27], for example) $|I_i|/3 \le |I_{i+1}| \le 3|I_i|$, $1 \le i \le n-1$, and $\rho_n(x) \le |I_i| \le 5\rho_n(x)$ for all $x \in I_i$ and $1 \le i \le n$, we conclude that

$$\begin{aligned} \max\{|x_{\nu_j+1} - z_j|, |x_{\nu_j-2} - z_j|\} &\le \max\{|I_{\nu_j+1}| + |I_{\nu_j}|, |I_{\nu_j}| + |I_{\nu_j-1}|\} \\ &\le 4|I_{\nu_j}| \le 20\rho_n(z_j) \le (20/\vartheta^2)\rho(\vartheta/n, z_j), \end{aligned}$$

and so

$$J_j \subset \mathcal{Z}^j_{20/\vartheta^2, \vartheta/n}, \quad 1 \le j \le M.$$

Therefore,

$$\begin{aligned} \|w(F - f)\| &= \max_{1\le j\le M} \|w(q_j - f)\|_{J_i} \le c \max_{1\le j\le M} E_r(f, J_j)_w \\ &\le c\sum_{j=1}^M E_r(f, \mathcal{Z}^j_{20/\vartheta^2, \vartheta/n})_w \le c\omega_\varphi^r(f, 1, 20/\vartheta^2, \vartheta/n)_w. \end{aligned} \tag{6.3}$$

We now estimate $\omega_\varphi^r(F, \vartheta/n)_{w_n}$ in terms of the modulus of f. Let $0 < h \le \vartheta/n$ and x such that $[x - rh\varphi(x)/2, x + rh\varphi(x)/2] \subset [-1,1]$ be fixed, and consider the following three cases.

Case 1: $x \in \mathcal{R}_1 := \left\{x \;\middle|\; [x - rh\varphi(x)/2, x + rh\varphi(x)/2] \subset J_j, \text{ for some } 1 \le j \le M\right\}$.

Then, for some $1 \le j \le M$, $\Delta^r_{h\varphi(x)}(F, x, [-1,1]) = \Delta^r_{h\varphi(x)}(q_j, x, [-1,1]) = 0$, and so

$$\left\| w_n(\cdot)\Delta^r_{h\varphi(\cdot)}(F, \cdot, [-1,1]) \right\|_{\mathcal{R}_1} = 0.$$

Case 2: $x \in \mathcal{R}_2 := \left\{ x \;\middle|\; [x - rh\varphi(x)/2, x + rh\varphi(x)/2] \cap \cup_{j=1}^M J_i = \emptyset \right\}$.

Then, taking into account that

$$\begin{aligned} J_j &\supset [z_j - |I_{\nu_j+1}|, z_j + |I_{\nu_j-1}|] \supset [z_j - |I_{\nu_j}|/3, z_j + |I_{\nu_j}|/3] \\ &\supset [z_j - \rho_n(z_j)/3, z_j + \rho_n(z_j)/3] = \mathcal{Z}^j_{1/3,1/n}, \end{aligned} \tag{6.4}$$

we conclude that $x \in \mathcal{I}_{1/3,1/n}$, and so $w_n(x) \sim w(x)$ by Lemma 3.4(v). Also, (6.4) implies that

$$[-1,1] \setminus \cup_{j=1}^M J_j \subset [-1,1] \setminus \cup_{j=1}^M \mathcal{Z}^j_{1/3,1/n} \subset \mathcal{I}_{1/3,1/n} \subset \mathcal{I}_{1/3,h},$$

and so $[x - rh\varphi(x)/2, x + rh\varphi(x)/2] \subset \mathcal{I}_{1/3,h}$. Therefore, $\Delta^r_{h\varphi(x)}(F, x, [-1,1]) = \Delta^r_{h\varphi(x)}(f, x, \mathcal{I}_{1/3,h})$, and

$$\left\| w_n(\cdot)\Delta^r_{h\varphi(\cdot)}(F, \cdot, [-1,1]) \right\|_{\mathcal{R}_2} \le c \left\| w(\cdot)\Delta^r_{h\varphi(\cdot)}(f, \cdot, \mathcal{I}_{1/3,h}) \right\|_{\mathcal{R}_2}.$$

Case 3: $x \in \mathcal{R}_3^j$, for some $1 \le j \le M$, where $\mathcal{R}_3^j$ is the set of all x such that $[x - rh\varphi(x)/2, x + rh\varphi(x)/2]$ has nonempty intersections with J_j and $([-1,1] \setminus J_j)^{cl}$, *i.e.*,

$$\mathcal{R}_3^j := \left\{ x \;\middle|\; x_{\nu_j+1} \;\text{ or }\; x_{\nu_j-2} \in [x - rh\varphi(x)/2, x + rh\varphi(x)/2] \right\}.$$

Note that, because of the restrictions on N, $[x - rh\varphi(x)/2, x + rh\varphi(x)/2]$ cannot have nonempty intersection with more than one interval J_i, and, in fact, $\mathcal{R}_3^j$ is "far" from all intervals J_i with $i \ne j$.

Without loss of generality, we can assume that $x_{\nu_j+1} \in [x - rh\varphi(x)/2, x + rh\varphi(x)/2]$, since the other case follows by symmetry. Taking into account that $x - rh\varphi(x)/2$ and $x + rh\varphi(x)/2$ are both increasing functions in x, we have

$$\operatorname{dist}\{z_j, [x - rh\varphi(x)/2, x + rh\varphi(x)/2]\} = z_j - x - rh\varphi(x)/2 \ge z_j - \widetilde{x} - rh\varphi(\widetilde{x})/2,$$

where $\widetilde{x}$ is such that $\widetilde{x} - rh\varphi(\widetilde{x})/2 = x_{\nu_j+1}$. Note that $\widetilde{x} < x_{\nu_j}$ since

$$x_{\nu_j} - rh\varphi(x_{\nu_j})/2 > x_{\nu_j} - r\vartheta\rho_n(x_{\nu_j})/2 \ge x_{\nu_j} - r\vartheta|I_{\nu_j+1}|/2 > x_{\nu_j+1},$$

and so $\widetilde{x} \in I_{\nu_j+1}$. Therefore,

$$\begin{aligned} &\operatorname{dist}\{z_j, [x - rh\varphi(x)/2, x + rh\varphi(x)/2]\} \\ &\quad \ge z_j - x_{\nu_j+1} - rh\varphi(\widetilde{x}) \ge |I_{\nu_j+1}| - r\vartheta\rho_n(\widetilde{x}) \\ &\quad \ge (1 - r\vartheta)|I_{\nu_j+1}| \ge (1 - r\vartheta)\rho_n(z_j)/3 = \rho_n(z_j)/6. \end{aligned}$$

Also,

$$\max\left\{ |y - z_j| \;\middle|\; y \in [x - rh\varphi(x)/2, x + rh\varphi(x)/2] \right\} = z_j - x + rh\varphi(x)/2 \le z_j - \widehat{x} + rh\varphi(\widehat{x})/2,$$

where $\widehat{x}$ is such that $\widehat{x} + rh\varphi(\widehat{x})/2 = x_{\nu_j+1}$. Now, $\widehat{x} > x_{\nu_j+2}$ since

$$x_{\nu_j+2} + rh\varphi(x_{\nu_j+2})/2 < x_{\nu_j+2} + r\vartheta\rho_n(x_{\nu_j+2})/2 < x_{\nu_j+2} + r\vartheta|I_{\nu_j+2}|/2 < x_{\nu_j+1},$$

and so $\widehat{x} \in I_{\nu_j+2}$. Therefore,

$$\begin{aligned}&\max\left\{|y-z_j| \;\middle|\; y\in[x-rh\varphi(x)/2, x+rh\varphi(x)/2]\right\}\\&\le z_j - x_{\nu_j+1} + rh\varphi(\widehat{x}) \le x_{\nu_j-1} - x_{\nu_j+1} + r\vartheta\rho_n(\widehat{x})\\&\le |I_{\nu_j+1}| + |I_{\nu_j}| + r\vartheta|I_{\nu_j+2}| \le (20+45r\vartheta)\rho_n(z_j) \le 50\rho_n(z_j).\end{aligned}$$

Hence,

$$[x-rh\varphi(x)/2, x+rh\varphi(x)/2] \subset \mathcal{J}_{1/6,1/n}\cap \mathcal{Z}^j_{50,1/n} \subset \mathcal{J}_{1/6,h}\cap \mathcal{Z}^j_{50,1/n}.$$

Lemma 3.4(v) implies that $w_n(x) \sim w(x)$. Also, for any $y \in [x-rh\varphi(x)/2, x+rh\varphi(x)/2]$, $|x-y| \le rh\varphi(x)/2 \le r\vartheta\rho_n(x)/2 = \rho_n(x)/4$, and so Lemma 3.4(iv) yields $w(y) \sim w(x)$.

This implies

$$\begin{aligned}&\left\| w_n(\cdot)\Delta^r_{h\varphi(\cdot)}(F,\cdot,[-1,1])\right\|_{\mathcal{R}^j_3}\\&\le \left\| w_n(\cdot)\Delta^r_{h\varphi(\cdot)}(f,\cdot,[-1,1])\right\|_{\mathcal{R}^j_3} + \left\| w_n(\cdot)\Delta^r_{h\varphi(\cdot)}(F-f,\cdot,[-1,1])\right\|_{\mathcal{R}^j_3}\\&\le \left\| w_n(\cdot)\Delta^r_{h\varphi(\cdot)}(f,\cdot,\mathcal{J}_{1/6,h})\right\|_{\mathcal{R}^j_3} + \left\| w_n(\cdot)\sum_{i=0}^r\binom{r}{i}\left|(F-f)(\cdot - rh/2 + ih\varphi(\cdot))\right|\right\|_{\mathcal{R}^j_3}\\&\le c\left\| w(\cdot)\Delta^r_{h\varphi(\cdot)}(f,\cdot,\mathcal{J}_{1/6,h})\right\|_{\mathcal{R}^j_3} + c\left\|w(q_j-f)\right\|_{J_j}\\&\le c\left\| w(\cdot)\Delta^r_{h\varphi(\cdot)}(f,\cdot,\mathcal{J}_{1/6,h})\right\|_{\mathcal{R}^j_3} + cE_r(f,\mathcal{Z}^j_{20/\vartheta^2,\vartheta/n})_w.\end{aligned}$$

Combining the above cases we conclude that

$$\begin{aligned}\omega^r_\varphi(F,\vartheta/n)_{w_n} &\le c\sup_{0<h\le\vartheta/n}\left\| w(\cdot)\Delta^r_{h\varphi(\cdot)}(f,\cdot,\mathcal{J}_{1/6,h})\right\| + c\sum_{j=1}^M E_r(f,\mathcal{Z}^j_{20/\vartheta^2,\vartheta/n})_w\\&\le c\omega^r_\varphi(f,1/6,20/\vartheta^2,\vartheta/n)_w \le c\omega^r_\varphi(f,1,20/\vartheta^2,\vartheta/n)_w.\end{aligned}$$

We now recall that Theorem 5.6 implies that $\left\|w\varphi^\nu P_n^{(\nu)}\right\| \le cn^\nu\left\|w_n\rho_n^\nu P_n^{(\nu)}\right\|$, and so applying Theorem 6.2 for the function F as well as the fact that $w(x) \le cw_n(x)$ (see (2.4)) we conclude that (6.1) and (6.2) are proved with $\omega^r_\varphi(f,1,20/\vartheta^2,\vartheta/n)_w$ instead of $\omega^r_\varphi(f,1,B,\vartheta/n)_w$ on the right-hand side.

Now, if $B \ge 20/\vartheta^2$, then $\omega^r_\varphi(f,1,20/\vartheta^2,\vartheta/n)_w \le \omega^r_\varphi(f,1,B,\vartheta/n)_w$. If $B < 20/\vartheta^2$, then, since $\vartheta/n < \delta(\mathcal{Z})/(80/\vartheta^2) < 1$, Corollary 4.11 implies that $\omega^r_\varphi(f,1,20/\vartheta^2,\vartheta/n)_w \le c\omega^r_\varphi(f,1,2^{-m}\cdot 20/\vartheta^2,\vartheta/n)_w \le c\omega^r_\varphi(f,1,B,1/n)_w$, where $m := \lceil \log_2(20/(B\vartheta^2)\rceil \in \mathbb{N}$, and the constant c depends only on r, B, ϑ and the weight w.

The proof is now complete. $\square$

7. Inverse theorem

Theorem 7.1. *Suppose that $\mathcal{Z} \in \mathbb{Z}_M$, $w \in \mathcal{W}^*(\mathcal{Z})$, $f \in \mathbb{L}^w_\infty$, $A, B > 0$ and $n, r \in \mathbb{N}$. Then*

$$\omega^r_\varphi(f,A,B,n^{-1})_w \le cn^{-r}\sum_{k=1}^n k^{r-1}E_k(f,[-1,1])_w,$$

where the constant c depends only on r, A, B, and the weight w.

Proof. Let $P_n^* \in \Pi_n$ denote a polynomial of (near) best approximation to f with weight w, *i.e.*,

$$c\,\|w(f - P_n^*)\| \le \inf_{P_n\in\Pi_n} \|w(f-P_n)\| = E_n(f,[-1,1])_w.$$

We let $N \in \mathbb{N}$ be such that $2^N \le n < 2^{N+1}$. To estimate $\Omega_\varphi^r(f,A,n^{-1})_w$, using Lemma 4.3 we have

$$\begin{aligned}
\Omega_\varphi^r(f,A,n^{-1})_w &\le \Omega_\varphi^r(f,A,2^{-N})_w \\
&\le \Omega_\varphi^r(f-P_{2^N}^*,A,2^{-N})_w + \Omega_\varphi^r(P_{2^N}^*,A,2^{-N})_w \\
&\le c\,\|w(f-P_{2^N}^*)\| + \Omega_\varphi^r(P_{2^N}^*,A,2^{-N})_w \\
&\le cE_{2^N}(f,[-1,1])_w + \Omega_\varphi^r(P_{2^N}^*,A,2^{-N})_w.
\end{aligned}$$

Now, using

$$P_{2^N}^* = P_1^* + \sum_{i=0}^{N-1}(P_{2^{i+1}}^* - P_{2^i}^*) \tag{7.1}$$

as well as Lemma 5.8 we have

$$\Omega_\varphi^r(P_{2^N}^*,A,2^{-N})_w \le \sum_{i=0}^{N-1} \Omega_\varphi^r\left(P_{2^{i+1}}^* - P_{2^i}^*, A, 2^{-N}\right)_w \le c2^{-Nr}\sum_{i=0}^{N-1}\left\|w\varphi^r\left(P_{2^{i+1}}^* - P_{2^i}^*\right)^{(r)}\right\|.$$

Now, for each $1 \le j \le M$, taking into account that $\mathcal{Z}_{B,t_1}^j \subset \mathcal{Z}_{B,t_2}^j$ if $t_1 \le t_2$, we have

$$\begin{aligned}
E_r(f,\mathcal{Z}_{B,1/n}^j)_w \le E_r(f,\mathcal{Z}_{B,2^{-N}}^j)_w &\le \|w(f-P_{2^N}^*)\|_{\mathcal{Z}_{B,2^{-N}}^j} + E_r(P_{2^N}^*,\mathcal{Z}_{B,2^{-N}}^j)_w \\
&\le cE_{2^N}(f,[-1,1])_w + \|w(P_{2^N}^* - q_r(P_{2^N}^*))\|_{\mathcal{Z}_{B,2^{-N}}^j},
\end{aligned}$$

where $q_r(g)$ denotes the Taylor polynomial of degree $< r$ at z_j for g. Using (7.1) again, noting that

$$q_r(P_{2^N}^*) = P_1^* + \sum_{i=0}^{N-1} q_r(P_{2^{i+1}}^* - P_{2^i}^*), \tag{7.2}$$

and taking Lemma 5.7 into account we have

$$\begin{aligned}
\|w(P_{2^N}^* - q_r(P_{2^N}^*))\|_{\mathcal{Z}_{B,2^{-N}}^j} &\le \sum_{i=0}^{N-1}\|w\left((P_{2^{i+1}}^* - P_{2^i}^*) - q_r(P_{2^{i+1}}^* - P_{2^i}^*)\right)\|_{\mathcal{Z}_{B,2^{-N}}^j} \\
&\le c\sum_{i=0}^{N-1} 2^{-Nr}\left\|w\varphi^r(P_{2^{i+1}}^* - P_{2^i}^*)^{(r)}\right\|.
\end{aligned}$$

Hence,

$$\omega_\varphi^r(f,A,B,n^{-1})_w \le cE_{2^N}(f,[-1,1])_w + c2^{-Nr}\sum_{i=0}^{N-1}\left\|w\varphi^r\left(P_{2^{i+1}}^* - P_{2^i}^*\right)^{(r)}\right\|.$$

Now, using Theorem 5.6 we have

$$\begin{aligned}\omega_\varphi^r(f,A,B,n^{-1})_w &\le cE_{2^N}(f,[-1,1])_w + c2^{-Nr}\sum_{i=0}^{N-1} 2^{ir}\left\|w(P^*_{2^{i+1}}-P^*_{2^i})\right\| \\ &\le c2^{-Nr}\sum_{i=0}^{N} 2^{ir}E_{2^i}(f,[-1,1])_w \\ &\le cn^{-r}\left(E_1(f,[-1,1])_w + \sum_{i=1}^{N}\sum_{k=2^{i-1}+1}^{2^i} k^{r-1}E_k(f,[-1,1])_w\right) \\ &\le cn^{-r}\sum_{k=1}^{n} k^{r-1}E_k(f,[-1,1])_w,\end{aligned}$$

with all constants c depending only on r, A, B, and the weight w. $\square$

8. Realization functionals

For $w\in\mathcal{W}^*(\mathcal{Z})$, $r\in\mathbb{N}$, and $f\in\mathbb{L}_\infty^w$, we define the following "realization functional" as follows

$$R_{r,\varphi}(f,t,\Pi_n)_w := \inf_{P_n\in\Pi_n}\left(\|w(f-P_n)\| + t^r\left\|w\varphi^r P_n^{(r)}\right\|\right),$$

and note that $R_{r,\varphi}(f,t_1,\Pi_n)_w \sim R_{r,\varphi}(f,t_2,\Pi_n)_w$ if $t_1\sim t_2$.

Theorem 8.1. *Let $\mathcal{Z}\in\mathbb{Z}_M$, $w\in\mathcal{W}^*(\mathcal{Z})$, $f\in\mathbb{L}_\infty^w$, $A,B>0$, $r\in\mathbb{N}$, and let $\vartheta_2\ge\vartheta_1>0$. Then, there exists a constant $N\in\mathbb{N}$ depending only on r, ϑ_1, and the weight w, such that, for $n\ge N$ and $\vartheta_1/n\le t\le\vartheta_2/n$,*

$$R_{r,\varphi}(f,1/n,\Pi_n)_w \sim \omega_\varphi^r(f,A,B,t)_w,$$

where the equivalence constants depend only on r, A, B, ϑ_1, ϑ_2 and the weight w.

Proof. In view of Corollary 4.6 it is sufficient to prove this lemma for $A=1$. Theorem 6.1 implies that, for every $n\ge N$ (with N depending only on r, ϑ_1 and the weight w), there exists a polynomial $P_n\in\Pi_n$ such that

$$R_{r,\varphi}(f,1/n,\Pi_n)_w \le c\omega_\varphi^r(f,1,B,\vartheta_1/n)_w \le c\omega_\varphi^r(f,1,B,t)_w. \tag{8.1}$$

Now, let P_n be an arbitrary polynomial from Π_n, $n\in\mathbb{N}$. Lemmas 4.3 and 5.8 imply that

$$\begin{aligned}\Omega_\varphi^r(f,1,t)_w &\le c\Omega_\varphi^r(f-P_n,1,t)_w + c\Omega_\varphi^r(P_n,1,t)_w \\ &\le c\|w(f-P_n)\| + cn^{-r}\left\|w\varphi^r P_n^{(r)}\right\|,\end{aligned} \tag{8.2}$$

where constants c depend only on r, ϑ_2 and the weight w. Also, taking into account that $\mathcal{Z}^j_{B,t}\subset\mathcal{Z}^j_{B,\vartheta_2/n}\subset\mathcal{Z}^j_{B\vartheta_2\max\{\vartheta_2,1\},1/n}$ and using Lemma 5.7, we have

$$\begin{aligned}\sum_{j=1}^{M} E_r(f,\mathcal{Z}^j_{B,t})_w &\le c\|w(f-P_n)\| + \sum_{j=1}^{M}\inf_{q\in\Pi_r}\|w(P_n-q)\|_{\mathcal{Z}^j_{B\vartheta_2\max\{\vartheta_2,1\},1/n}} \\ &\le c\|w(f-P_n)\| + cn^{-r}\left\|w\varphi^r P_n^{(r)}\right\|.\end{aligned} \tag{8.3}$$

Therefore, for any $n \in \mathbb{N}$, $\vartheta_2 > 0$ and $0 < t \le \vartheta_2/n$,

$$\omega_\varphi^r(f, 1, B, t)_w \le cR_{r,\varphi}(f, 1/n, \Pi_n)_w, \tag{8.4}$$

which completes the proof of the theorem. $\square$

Theorem 8.1 implies, in particular, that $\omega_\varphi^r(f, A_1, B_1, t_1)_w \sim \omega_\varphi^r(f, A_2, B_2, t_2)_w$ if $t_1 \sim t_2$ with equivalence constants independent of f.

Finally, we remark that the moduli $\omega_\varphi^r(f, A, B, t)_w$ are not equivalent to the following weighted K-functional

$$K_{r,\varphi}(f, t)_w := \inf_{g^{(r-1)} \in \mathrm{AC}_{\mathrm{loc}}} \left(\|w(f - g)\| + t^r \left\| w\varphi^r g^{(r)} \right\| \right).$$

This follows from counterexamples constructed in [6], where additional discussions and negative results can be found.

Appendix A

The following lemma shows that $E_r(f, \mathcal{Z}_{B,t}^j)_w$ in the definition of the complete modulus (3.5) can be replaced with $\|w(f - q_j)\|_{\mathcal{Z}_{B,t}^j}$, where q_j is a polynomial of (near) best weighted approximation to f on any subinterval of $\mathcal{Z}_{B,t}^j$ of length $\ge c\rho(t, z_j)$.

Lemma A.1. *Suppose that $\mathcal{Z} \in \mathbb{Z}_M$, $w \in \mathcal{W}^*(\mathcal{Z})$, $f \in \mathbb{L}_\infty^w$, and suppose that intervals I and J are such that $I \subset J \subset [-1, 1]$ and $|J| \le c_0|I|$. Then, for any $r \in \mathbb{N}$, if $q \in \Pi_r$ is a polynomial of near best approximation to f on I with weight w,* i.e.,

$$\|w(f - q)\|_I \le c_1 E_r(f, I)_w,$$

then q is also a polynomial of near best approximation to f on J. In other words,

$$\|w(f - q)\|_J \le cE_r(f, J)_w,$$

where the constant c depends only on r, c_0, c_1 and the weight w.

Proof. The proof is similar to that of [4, Lemma A.1]. First, we assume that $|I| \le \delta(\mathcal{Z})/2$, and so I may contain at most one z_j from $\mathcal{Z}$. Now, we denote by a the midpoint of I and let $n \in \mathbb{N}$ be such that $\rho_{n+1}(a) < |I|/1000 \le \rho_n(a)$. Then, $\rho_n(a) \sim |I|$ and, as was shown in the proof of [4, Lemma A.1], I contains at least 5 adjacent intervals $I_{\nu+i}$, $i = 2, 1, 0, -1, -2$. Moreover, one of those intervals, I_μ, is such that $|I_\mu| \sim |I|$ and $I_\mu \subset \mathcal{I}_{c,1/n}$ with some absolute constant c, and Lemma 3.4(iv) implies that $w(x) \sim w(y)$, for $x, y \in I_\mu$, with equivalence constants depending only on w.

Suppose now that $\widetilde{q}$ is a polynomial of near best weighted approximation of f on J, *i.e.*, $\|w(f - \widetilde{q})\|_J \le cE_r(f, J)_w$. Then, taking into account that $|I_\mu| \sim |I| \sim |J|$ and using the fact that w is doubling, we have

$$\begin{aligned} \|w(\widetilde{q} - q)\|_J &\le L^*|J|^{-1} \|\widetilde{q} - q\|_J\, w(J) \le c|I_\mu|^{-1} \|\widetilde{q} - q\|_{I_\mu} w(I_\mu) \\ &\le cw(x_\mu) \|\widetilde{q} - q\|_{I_\mu} \le c \|w(\widetilde{q} - q)\|_{I_\mu}. \end{aligned}$$

Therefore,

$$\begin{aligned}\|w(f-q)\|_J &\le c\,\|w(f-\widetilde{q})\|_J + c\,\|w(\widetilde{q}-q)\|_J \\ &\le c\,\|w(f-\widetilde{q})\|_J + c\,\|w(\widetilde{q}-q)\|_I \\ &\le c\,\|w(f-\widetilde{q})\|_J + c\,\|w(\widetilde{q}-f)\|_I + c\,\|w(f-q)\|_I \\ &\le c\,\|w(f-\widetilde{q})\|_J + c\,\|w(f-q)\|_I \\ &\le cE_r(f,J)_w + cE_r(f,I)_w \\ &\le cE_r(f,J)_w,\end{aligned}$$

and the proof is complete if $|I| \le \delta(\mathcal{Z})/2$.

If $|I| > \delta(\mathcal{Z})/2$, then $|I| \sim |J| \sim 1$, and we take $n \in \mathbb{N}$ to be such that I contains at least $4M+4$ intervals I_i. Then I contains 4 adjacent intervals I_i not containing any points from $\mathcal{Z}$, and we can use the same argument as above. $\square$

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