NORTH-HOLLAND

A Note on the Convexity of the Sum of Subpermanents

Kirill A. Kopotun

Department of Mathematical Sciences
University of Alberta
Edmonton, Alberta
Canada T6G 2G1

ABSTRACT

Let Ω_n denote the set of all $n \times n$ doubly stochastic matrices, and let $\sigma_k(A)$ be the sum of all subpermanents of order k of matrix A. We prove that $\sigma_2(A)$ and $\sigma_3(A)$ are convex on Ω_n for $n \geq 2$ and $n \geq 4$, respectively, and also conjecture the following: For every $k \geq 3$ there exists $n_k \geq k+1$ such that the inequality $\sigma_k(\alpha J_n + (1-\alpha)A) \leq \alpha\sigma_k(J_n) + (1-\alpha)\sigma_k(A)$ holds for all $\alpha \in [0, 1]$ and all $A \in \Omega_n$ with $n \geq n_k$, where $J_n = (1/n)_{i,j=1}^n \in \Omega_n$. It is shown that this conjecture is true for $k \leq 4$ with $n_3 = 4$ and $n_4 = 6$.

1. INTRODUCTION

Let Ω_n denote the set of all $n \times n$ doubly stochastic matrices, I_n be the $n \times n$ identity matrix, and $J_n = (1/n)_{i,j=1}^n \in \Omega_n$, the matrix each of whose entries equals $1/n$. We also denote by $\sigma_k(A)$ the sum of all subpermanents of A of order k, $1 \leq k \leq n$. In particular, $\sigma_n(A) = \mathrm{per}(A)$, and $\sigma_1(A) = n$ if $A \in \Omega_n$.

It is well known that the permanent function is not convex on Ω_n for $n \geq 3$, and that it is convex on Ω_2 (see [1] and [18], for example). However, some weaker relations than those for convex functions have been established. For example, Brualdi and Newman [1] showed that for all $\alpha \in [0, 1]$ and $A \in \Omega_n$

$$\mathrm{per}(\alpha I_n + (1-\alpha)A) \leq \alpha + (1-\alpha)\,\mathrm{per}(A). \tag{1}$$

Wang [22] called a matrix $B \in \Omega_n$ a *star* if it satisfies the inequality

$$\mathrm{per}(\alpha B + (1-\alpha)A) \leq \alpha\,\mathrm{per}(B) + (1-\alpha)\,\mathrm{per}(A) \tag{2}$$

for all $\alpha \in [0, 1]$ and $A \in \Omega_n$, and conjectured that the only stars for $n \geq 3$ are permutation matrices. This conjecture remains unsettled.

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The following characterization of stars is due to Brualdi and Newman [1]: A matrix $B \in \Omega_n$ is a star if and only if the inequality

$$\sum_{i,j=1}^{n} b_{ij} \operatorname{per}(A_{ij}) \leq \operatorname{per}(B) + (n-1) \operatorname{per}(A) \tag{3}$$

holds for all $A \in \Omega_n$, where A_{ij} denotes the $(n-1) \times (n-1)$ matrix obtained by deleting the ith row and jth column of A. (In fact, it was shown in [1] that in this characterization the inequality (3) is necessary, and that it is sufficient with the assumption that equality in (3) occurs only if $A = B$. However, this assumption can be removed. For further discussions see Section 2.)

Brualdi and Newman [1] also showed that J_3 is not a star. Wang [22] noted that letting $A = (I_n + P_n)/2$ [where P_n is the full-cycle permutation matrix corresponding to the full cycle $(12 \cdots n)$] in (3) shows that if B is a star, then $\operatorname{per}(B) \geq 2^{1-n}$. Hence, J_n is not a star for $n \geq 3$.

Lih and Wang [11] conjectured that

$$\operatorname{per}(\alpha J_n + (1-\alpha)A) \leq \alpha \operatorname{per}(J_n) + (1-\alpha) \operatorname{per}(A) \tag{4}$$

for $\alpha \in [\frac{1}{2}, 1]$ and $A \in \Omega_n$. They proved (4) for $n = 3$, and also in the particular case $\alpha = \frac{1}{2}$ and $n = 4$ (see also [4]).

Hwang [8] conjectured that the permanent function is convex on the straight line segment joining J_n and $(J_n + A)/2$ for all $A \in \Omega_n$ and proved it for $n = 3$ (see also Remark 4 in Section 4).

It is fairly natural to inquire whether σ_k has properties similar to (1)–(4) of the permanent function. Recently, Malek [15] proved that if $A \in \Omega_n$, then $\sigma_2(\alpha J_n + (1-\alpha)A) \leq \alpha\sigma_2(J_n) + (1-\alpha)\sigma_2(A)$ for $\alpha \in [0, 1]$, and $\sigma_3(\alpha J_n + (1-\alpha)A) \leq \alpha\sigma_3(J_n) + (1-\alpha)\sigma_3(A)$ for $\alpha \in [\frac{1}{2}, 1]$. Using a method developed by Marcus and Minc [17], he also showed the validity of the inequality $\sigma_k(\alpha J_n + (1-\alpha)A) \leq \alpha\sigma_k(J_n) + (1-\alpha)\sigma_k(A)$ for normal $A \in \Omega_n$ with all eigenvalues in the sector $[-\pi/2k, \pi/2k]$ of the complex plane.

A further discussion of the properties of σ_k is the main subject of this note.

2. PROBLEMS, CONJECTURES, RESULTS

Following Wang, we introduce the following convention. Let F be a function defined on Ω_n. We call a matrix $B \in \Omega_n$ an *F-star* if it satisfies the inequality

$$F(\alpha B + (1-\alpha)A) \leq \alpha F(B) + (1-\alpha)F(A) \tag{5}$$

for all $\alpha \in [0, 1]$ and $A \in \Omega_n$. For example, a per-star is simply a star in the sense of the definition (2). Clearly, a function F is convex on Ω_n if and only if every matrix in Ω_n is an F-star. Below we consider the cases when $F = \sigma_k$, $k = 2, \dots, n$.

In view of the results quoted in Section 1 the following questions naturally arise: Is it true that for every $k = 2, \dots, n-1$ the sum of all subpermanents of order k, σ_k, is a convex function on Ω_n? If not, then what can we say about σ_k-stars?

Using the ideas of Brualdi and Newman [1], it is not difficult to show the validity of the following characterization of σ_k-stars similar to (3): A matrix $B \in \Omega_n$ is a σ_k-star if and only if

$$\sum_{i,j=1}^{n} b_{ij}\sigma_{k-1}(A_{ij}) \leq \sigma_k(B) + (k-1)\sigma_k(A) \tag{6}$$

for all $A \in \Omega_n$.

Indeed, this characterization immediately follows from the trivial observation

$$(g \in C^1[0, \varepsilon],\ g(0) = 0 \ \text{ and } \ g(\alpha) \geq 0,\ \alpha \in [0, \varepsilon]) \Rightarrow (g'(0) \geq 0)\,, \tag{7}$$

the identity

$$\frac{\partial}{\partial \alpha}\sigma_k(\alpha B + (1-\alpha)A)\bigg|_{\alpha=0} = \sum_{i,j=1}^{n} b_{ij}\sigma_{k-1}(A_{ij}) - k\sigma_k(A)\,, \tag{8}$$

and the following lemma, which is a stronger version of Lemma 1 of [1] for differentiable functions.

LEMMA 1. *Let C be a nonempty convex set of a vector space, f be a real-valued differentiable function defined over C, and x be a fixed element of C. If*

$$f(x) - f(y) - \frac{\partial}{\partial \alpha} f(\alpha x + (1-\alpha)y)\bigg|_{\alpha=0} \geq 0 \tag{9}$$

for all $y \in C$, then the inequality

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y) \tag{10}$$

is valid for all $\alpha \in [0, 1]$ and all y in C.

Proof. Suppose (10) is not satisfied for some y in C and some $\alpha_0 \in (0, 1)$, and consider the line segment between y and x. Denoting for simplicity

$\Delta_y(\alpha) := f(\alpha x + (1-\alpha)y)$ and $\omega(\alpha) := \alpha\Delta_y(1) + (1-\alpha)\Delta_y(0) - \Delta_y(\alpha)$, thus, we have $\omega(\alpha_0) < 0$ and $\omega(0) = 0$. Since $\omega(\alpha)$ is a continuous function, then there exists $\alpha_1 \in [0, \alpha_0)$ such that $\omega(\alpha_1) = 0$ and $\omega(\alpha) < 0$ for all $\alpha \in (\alpha_1, \alpha_0]$. Now, using the mean-value theorem and the fact that $\omega(\alpha)$ is differentiable, we conclude that there is $\alpha_2 \in (\alpha_1, \alpha_0)$ such that

$$\omega'(\alpha_2) = \frac{\omega(\alpha_0) - \omega(\alpha_1)}{\alpha_0 - \alpha_1} = \frac{\omega(\alpha_0)}{\alpha_0 - \alpha_1}. \tag{11}$$

Let $z := \alpha_2 x + (1-\alpha_2)y \in \mathcal{C}$; then (9) implies

$$f(x) - f(z) - \frac{\partial}{\partial\alpha}\Delta_z(\alpha)\bigg|_{\alpha=0} \geq 0. \tag{12}$$

Also, since

$$\begin{aligned}\Delta_z(\alpha) &= f(\alpha x + (1-\alpha)z) = f([\alpha + \alpha_2(1-\alpha)]x + (1-\alpha)(1-\alpha_2)y) \\ &= \Delta_y(\alpha + \alpha_2 - \alpha\alpha_2),\end{aligned}$$

then

$$\frac{\partial}{\partial\alpha}\Delta_z(\alpha)\bigg|_{\alpha=0} = \frac{\partial}{\partial\alpha}\Delta_y(\alpha + \alpha_2 - \alpha\alpha_2)\bigg|_{\alpha=0} = \Delta_y'(\alpha_2)(1-\alpha_2).$$

Therefore, using (11), (12), and the last equality, we have

$$\begin{aligned}0 &\leq f(x) - f(z) - \Delta_y'(\alpha_2)(1-\alpha_2) \\ &= \Delta_y(1) - \Delta_y(\alpha_2) - \Delta_y'(\alpha_2)(1-\alpha_2) \\ &= \Delta_y(1) - \Delta_y(\alpha_2) - (1-\alpha_2)\left(\Delta_y(1) - \Delta_y(0) - \frac{\omega(\alpha_0)}{\alpha_0 - \alpha_1}\right) \\ &= \alpha_2\Delta_y(1) + (1-\alpha_2)\Delta_y(0) - \Delta_y(\alpha_2) + \frac{1-\alpha_2}{\alpha_0 - \alpha_1}\omega(\alpha_0) \\ &= \omega(\alpha_2) + \frac{1-\alpha_2}{\alpha_0 - \alpha_1}\omega(\alpha_0) < 0.\end{aligned}$$

The contradiction obtained verifies the validity of the lemma. ■

In this paper we investigate the case $k \leq 3$ and obtain some partial results for $k > 3$ and $B = J_n$. Namely, the following theorems are proved.

THEOREM 1 ($k = 2$ and 3). *The functions $\sigma_2(A)$ and $\sigma_3(A)$ are convex on Ω_n for $n \geq 2$ and $n \geq 4$, respectively.*

THEOREM 2 ($k = 4$).

(i) *For any* $n \geq 6$ *the matrix* J_n *is a* σ_4*-star.*
(ii) *For* $n = 5$ *the following inequality is valid for* $\alpha \in [0.43, 1]$ *and all* $A \in \Omega_n$:

$$\sigma_4(\alpha J_n + (1-\alpha)A) \leq \alpha\sigma_4(J_n) + (1-\alpha)\sigma_4(A). \tag{13}$$

Recall that J_n is *not* a σ_3-star (σ_4-star) for $n = 3$ ($n = 4$), and therefore these cases are excluded from the assertions of the theorems. Theorems 1 and 2 give some support to the following conjecture (the case $k = 2$ is not considered because of its triviality).

CONJECTURE 1. *For every* $k \geq 3$ *there exists* $n_k \geq k+1$ *such that the inequality*

$$\sigma_k\left(\alpha J_n + (1-\alpha)A\right) \leq \alpha\sigma_k(J_n) + (1-\alpha)\sigma_k(A) \tag{14}$$

holds for all $\alpha \in [0, 1]$ *and all* $A \in \Omega_n$ *with* $n \geq n_k$.

In other words, the matrix J_n *is a* σ_k*-star for sufficiently large* n.

Using the characterization (6) and the identity

$$\sum_{i,j=1}^{n} \sigma_{k-1}(A_{ij}) = (n-k+1)^2\sigma_{k-1}(A), \qquad A \in \Omega_n, \tag{15}$$

we can rewrite Conjecture 1 in the following equivalent form:

CONJECTURE 1′. *For every* $k \geq 3$ *there exists* $n_k \geq k+1$ *such that for all* $A \in \Omega_n$, $n \geq n_k$, *the following inequality holds:*

$$(k-1)\sigma_k(A) + \sigma_k(J_n) \geq \frac{(n-k+1)^2}{n}\sigma_{k-1}(A). \tag{16}$$

It follows from Theorems 1 and 2 that the conjecture is true for $k = 3$ ($n_3 = 4$) and $k = 4$ ($n_4 = 6$). The question about the validity of (14) and (16) for $k = 4$, $n = 5$ and $k \geq 5$ remains open.

Conjecture 1′ is a strengthening of the following well-known Holens-Doković conjecture in the case $n \geq n_k \geq k+1$.

CONJECTURE (Holens [7] and Doković [3]). *If* $A \in \Omega_n$ *and* $2 \leq k \leq n$, *then*

$$k\sigma_k(A) \geq \frac{(n-k+1)^2}{n}\sigma_{k-1}(A) \tag{17}$$

with equality in the case $2 \leq k \leq n-1$ *only if* $A = J_n$.

Indeed, (17) immediately follows from (16) and the following Tverberg-Friedland inequality (see [5] and [20]):

$$\text{if} \quad A \in \Omega_n \text{ and } A \neq J_n, \quad \text{then} \quad \sigma_k(A) > \sigma_k(J_n), \quad 2 \leq k \leq n. \tag{18}$$

The Holens-Doković conjecture is known to be true for $k \leq 3$ [3] and $k = 4$, $n \geq 5$ [10]. It is equivalent to the assertion that the function $\sigma_k(\theta J_n + (1-\theta)A)$ is decreasing in the interval [0, 1]. This assertion is known as the *monotonicity* conjecture and was partially resolved for some special classes of matrices (see [6, 9, 12, 16, 18], and [19], for example).

It follows from the above-mentioned result of Malek [15] that Conjecture 1 (1′) is valid for normal matrices in Ω_n all whose eigenvalues lie in the sector $[-\pi/2k, \pi/2k]$ of the complex plane. In fact, the following stronger result can be easily proved (note that we do not require the condition $k \leq n-1$).

THEOREM 3. *Let $A \in \Omega_n$ be normal and $2 \leq k \leq n$. If all eigenvalues of A lie in the sector $[-\pi/2k, \pi/2k]$ of the complex plane, then the following inequality holds for all $\alpha \in [0, 1]$:*

$$\begin{aligned} &\alpha\sigma_k(J_n) + (1-\alpha)\sigma_k(A) - \sigma_k(\alpha J_n + (1-\alpha)A) \\ &\qquad \geq \frac{(k-2)!}{2n^{k-2}} \binom{n-2}{k-2}^2 \alpha(1-\alpha)\|A - J_n\|^2 . \end{aligned} \tag{19}$$

Using (7), (8), (15), and (19), one immediately gets.

COROLLARY 1. *If A satisfies the hypotheses of Theorem 3, then the following inequality holds:*

$$\begin{aligned} &(k-1)\sigma_k(A) - \frac{(n-k+1)^2}{n}\sigma_{k-1}(A) + \sigma_k(J_n) \\ &\qquad \geq \frac{(k-2)!}{2n^{k-2}} \binom{n-2}{k-2}^2 \|A - J_n\|^2 . \end{aligned} \tag{20}$$

COROLLARY 2. *If A satisfies the hypotheses of Theorem 3 and $A \neq J_n$, then*

$$(k-1)\sigma_k(A) + \sigma_k(J_n) > \frac{(n-k+1)^2}{n}\sigma_{k-1}(A) . \tag{21}$$

Finally, we remark that it is straightforward to check that the function

$$\widetilde{F}_k(A) := (k-1)\sigma_k(A) - \frac{(n-k+1)^2}{n}\sigma_{k-1}(A) + \sigma_k(J_n)$$

has a strict local minimum at J_n. Thus, it follows from Theorem 3 of [10] that if all entries of an $\widetilde{F}_k$-minimizing matrix A on Ω_n are positive, then $A = J_n$.

Section 3 contains proofs of Theorems 1–3. Some relevant remarks concerning Conjecture 1 (1′) are given in Section 4.

3. PROOFS

Throughout this section we let $A = (a_{ij})_{i,j=1}^n \in \Omega_n$ and $\sum := \sum_{i,j=1}^n$. The following formulae for σ_2, σ_3, and σ_4 (see [3] and [10]) are used:

$$\sigma_2(A) = \frac{1}{2}\sum a_{ij}^2 + \frac{n(n-2)}{2}, \tag{22}$$

$$\sigma_3(A) = \frac{2}{3}\sum a_{ij}^3 + \frac{n-4}{2}\sum a_{ij}^2 + \frac{n(n^2-6n+10)}{6}, \tag{23}$$

and

$$\begin{aligned}\sigma_4(A) = {} & \frac{3}{2}\sum a_{ij}^4 + \frac{2}{3}(n-6)\sum a_{ij}^3 \\ & + \frac{n^2-10n+28}{4}\sum a_{ij}^2 + \frac{1}{8}\left(\sum a_{ij}^2\right)^2 \\ & + \frac{1}{4}\sum_{1\le i_1<i_2\le n}\left(\sum_{j=1}^n a_{i_1j}a_{i_2j}\right)^2 + \frac{1}{4}\sum_{1\le j_1<j_2\le n}\left(\sum_{i=1}^n a_{ij_1}a_{ij_2}\right)^2 \\ & - \frac{5}{8}\sum_{i=1}^n\left(\sum_{j=1}^n a_{ij}^2\right)^2 - \frac{5}{8}\sum_{j=1}^n\left(\sum_{i=1}^n a_{ij}^2\right)^2 \\ & + \frac{n}{24}(n^3-12n^2+52n-84).\end{aligned} \tag{24}$$

The estimate in the following lemma is well known as the Jensen inequality (see Lemma 1 of [13], for example), and will be used in the proof of Theorem 2.

LEMMA A. *Let $x_1, x_2, \ldots, x_m$ be nonnegative numbers, and let $\sum_{i=1}^m x_i = p$. If $s > 1$, then*

$$\sum_{i=1}^m x_i^s \ge \frac{p^s}{m^{s-1}}. \tag{25}$$

Equality holds if and only if $x_i = p/m$, $i = 1, \ldots, m$.

Proof of Theorem 1. The assertion of Theorem 1 immediately follows from (22) and (23), the fact that the sum of convex functions is also convex, and the observation that $f(x) = x^s$ is convex on [0, 1] for any $s \ge 2$. ∎

Proof of Theorem 2. The proof of Theorem 2 is rather straightforward, but computationally involved. First, we show that the inequality (16) is valid for $k = 4$ and $n \geq 6$. Using exactly the same considerations as in the proof of the Holens-Doković conjecture for $k = 4$, $n \geq 5$ in [10] (i.e., applying the inequalities (16), (18), (19), and (17) from [10]) we obtain the following estimate for every real r:

$$\begin{aligned}\widetilde{F}_4(A) &= 3\sigma_4(A) - \frac{(n-3)^2}{n}\sigma_3(A) + \sigma_4(J_n)\\ &\geq \left(\frac{32n^2 - 273n - 36}{24n} + 9r + \frac{3}{4n-4}\right)\sum a_{ij}^3\\ &\quad + \left[\frac{n^3 - 10n^2 + 21n + 69}{4n} - \frac{3}{2n-2} - \frac{9}{2}\left(r^2 + \frac{2r}{n}\right)\right]\sum a_{ij}^2\\ &\quad + \frac{9}{2}r^2 + \frac{3n}{4n-4} - \frac{6n^4 - 60n^3 + 176n^2 + 123n - 36}{24n^2}.\end{aligned}$$

Now we choose $r = \frac{2}{5}$; then the coefficients of $\sum a_{ij}^3$ and $\sum a_{ij}^2$ are equal to

$$\frac{32n^2 - 273n - 36}{24n} + \frac{18}{5} + \frac{3}{4n-4}$$

and

$$\frac{n^3 - 10n^2 + 21n + 69}{4n} - \frac{3}{2n-2} - \frac{18}{25} - \frac{18}{5n},$$

respectively. Since they are positive for $n \geq 6$, we can use the inequality (25) for $s = 2$ and $s = 3$. Hence,

$$\begin{aligned}\widetilde{F}_4(A) &\geq \left(\frac{32n^2 - 273n - 36}{24n} + \frac{18}{5} + \frac{3}{4n-4}\right)\frac{1}{n}\\ &\quad + \left(\frac{n^3 - 10n^2 + 21n + 69}{4n} - \frac{3}{2n-2} - \frac{18}{25} - \frac{18}{5n}\right)\\ &\quad + \frac{18}{25} + \frac{3n}{4n-4} - \frac{6n^4 - 60n^3 + 176n^2 + 123n - 36}{24n^2}\\ &= 0 = \widetilde{F}_4(J_n).\end{aligned}$$

It follows from Lemma A that for any $n \geq 6$ and $A \in \Omega_n$ the equality $\widetilde{F}_4(A) = 0$ occurs if and only if $A = J_n$. The proof is complete.

For the proof of the inequality (13) for $\alpha \in [0.43, 1]$ the following lemma, which is verified by straightforward computations (see also [17], for example), will be useful.

LEMMA B. *If X is an arbitrary n-square matrix and s is a scalar, then*

$$\sigma_k(sJ_n + X) = \sum_{\nu=1}^{k} \frac{(k-\nu)!}{n^{k-\nu}} \binom{n-\nu}{k-\nu}^2 s^{k-\nu}\sigma_\nu(X) + s^k\sigma_k(J_n). \tag{26}$$

In particular,

$$\sigma_2(sJ_n + X) = \sigma_2(X) + \frac{(n-1)^2}{n} s\sigma_1(X) + s^2\sigma_2(J_n), \tag{27}$$

$$\begin{aligned}\sigma_3(sJ_n + X) &= \sigma_3(X) + \frac{(n-2)^2}{n} s\sigma_2(X)\\ &\quad + \frac{(n-1)^2(n-2)^2}{2n^2} s^2\sigma_1(X) + s^3\sigma_3(J_n),\end{aligned} \tag{28}$$

and

$$\begin{aligned}\sigma_4(sJ_n + X) &= \sigma_4(X) + \frac{(n-3)^2}{n} s\sigma_3(X) + \frac{(n-2)^2(n-3)^2}{2n^2} s^2\sigma_2(X)\\ &\quad + \frac{(n-1)^2(n-2)^2(n-3)^2}{6n^3} s^3\sigma_1(X) + s^4\sigma_4(J_n).\end{aligned} \tag{29}$$

Using (29) with $n = 5$, we have for any $A \in \Omega_5$

$$\begin{aligned}&\alpha(1-\alpha)F_\alpha(A)\\ &\quad := \alpha\sigma_4(J_5) + (1-\alpha)\sigma_4(A) - \sigma_4\left(\alpha J_5 + (1-\alpha)A\right)\\ &\quad = \alpha(1-\alpha)(\alpha^2 - 3\alpha + 3)\sigma_4(A) - \tfrac{4}{5}\alpha(1-\alpha)^3\sigma_3(A)\\ &\qquad - \tfrac{18}{25}\alpha^2(1-\alpha)^2\sigma_2(A) - \tfrac{96}{25}\alpha^3(1-\alpha) + \tfrac{24}{25}\alpha(1-\alpha^3).\end{aligned}$$

Since for $\alpha = 1$ the inequality (13) becomes an equality, it is sufficient to consider $\alpha < 1$. Using the Holens-Đoković inequality for $k = 4$, $n = 5$, we get the following estimate for $F_\alpha(A)$:

$$F_\alpha(A) \geq \frac{-3\alpha^2 + 5\alpha - 1}{5}\sigma_3(A) - \tfrac{18}{25}\alpha(1-\alpha)\sigma_2(A) + \tfrac{24}{25}(1 + \alpha - 3\alpha^2).$$

Now, using (22) and (23), one has

$$\begin{aligned}F_\alpha(A) \geq{} & \frac{2(-3\alpha^2 + 5\alpha - 1)}{15}\sum a_{ij}^3\\ & + \frac{3\alpha^2 + 7\alpha - 5}{50}\sum a_{ij}^2 + \frac{3\alpha^2 - 41\alpha + 19}{150}.\end{aligned}$$

Since the coefficient of $\sum a_{ij}^3$ is nonnegative for $\alpha \in [0.43,\ 1]$, we are able to use the following inequality (see (13) of [10]), which is valid for $(a_{ij})_{i,j=1}^n \in \Omega_n$:

$$\left(\sum_{j=1}^n a_{ij}^2\right)^2 \le \sum_{j=1}^n a_{ij}^3, \qquad i = 1, \ldots, n. \tag{30}$$

Hence,

$$F_\alpha(A) \ge \sum_{i=1}^5 \left(\frac{2(-3\alpha^2 + 5\alpha - 1)}{15} \left(\sum_{j=1}^5 a_{ij}^2\right)^2 + \frac{3\alpha^2 + 7\alpha - 5}{50} \sum_{j=1}^5 a_{ij}^2 \right) + \frac{3\alpha^2 - 41\alpha + 19}{150}.$$

If $\alpha \in [0.43,\ 1]$, then the function

$$f(x) = \frac{2(-3\alpha^2 + 5\alpha - 1)}{15} x^2 + \frac{3\alpha^2 + 7\alpha - 5}{50} x$$

is increasing on $[1/5, +\infty)$, and therefore $f(x) \ge f(\frac{1}{5})$ for all $x \ge \frac{1}{5}$. Together with the estimate $\sum_{j=1}^5 a_{ij}^2 \ge \frac{1}{5}$, $i = 1, \ldots, 5$, which follows from Lemma A, this implies

$$F_\alpha(A) \ge \frac{2}{75}(-3\alpha^2 + 5\alpha - 1) + \frac{3\alpha^2 + 7\alpha - 5}{50} + \frac{3\alpha^2 - 41\alpha + 19}{150} = 0$$

with equality if and only if $A = J_5$. The proof of Theorem 2 is now complete. ■

Proof of Theorem 3. The proof is based on Lemma B and the following result of Marcus and Minc [17].

LEMMA C [17].

(i) *If S is a real n-square matrix each of whose row and column sums is* 0, *then $\sigma_2(S) = \|S\|^2/2 \ge 0$ with equality if and only if $S = \mathbf{0}$.*
(ii) *If $A \in \Omega_n$ is normal and such that all eigenvalues of A lie in the sector $[-\pi/2k,\ \pi/2k]$ of the complex plane, then $\sigma_1(A - J_n) = 0$ and $\sigma_\nu(a - J_n) \ge 0$, $\nu = 2, \ldots, k$. In the case $\nu = 2$ equality can occur if and only if $A = J_n$.*

Lemma C together with (26) yields the inequalities

$$\begin{aligned}
&\alpha\sigma_k(J_n) + (1-\alpha)\sigma_k(A) - \sigma_k\left(\alpha J_n + (1-\alpha)A\right) \\
&\quad = \alpha\sigma_k(J_n) + (1-\alpha)\sigma_k(J_n + (A - J_n)) - \sigma_k\left(J_n + (1-\alpha)(A-J_n)\right) \\
&\quad = \frac{(k-2)!}{n^{k-2}}\binom{n-2}{k-2}^2 \alpha(1-\alpha)\sigma_2(A - J_n) \\
&\qquad + \sum_{\nu=3}^{k} \frac{(k-\nu)!}{n^{k-\nu}}\binom{n-\nu}{k-\nu}^2 (1-\alpha)[1-(1-\alpha)^{\nu-1}]\sigma_\nu(A-J_n) \\
&\quad \geq \frac{(k-2)!}{2n^{k-2}}\binom{n-2}{k-2}^2 \alpha(1-\alpha)\|A - J_n\|^2,
\end{aligned}$$

which complete the proof of Theorem 3. ∎

4. REMARKS

1.

The following conjecture of Wang is known to be true for $n = 3$ (Wang [21]) and $n = 4$ (Chang [2]).

CONJECTURE (Wang [21]). *The inequality*

$$\operatorname{per}\left(\frac{nJ_n + A}{n+1}\right) \leq \operatorname{per}(A)$$

holds for all $A \in \Omega_n$.

We propose the following generalization.

CONJECTURE 2. *For all* $A \in \Omega_n$ *and* $k = 2, \ldots, n$ *the inequality*

$$\sigma_k\left(\frac{nJ_n + A}{n+1}\right) \leq \sigma_k(A)$$

holds.

Conjecture 2 is clearly weaker than the Holens-Doković conjecture and is true for $k \leq 4$. This follows from Theorem 1 for $k = 2$ and for $k = 3$, $n \geq 4$, from Theorem 2 for $k = 4$, $n \geq 5$, from Wang [21] for $k = n = 3$, and from Chang [2] for $k = n = 4$. Also, Theorem 3 (see also [15]) implies that Conjecture 2 is valid for normal $A \in \Omega_n$ whose eigenvalues all lie in the sector $[-\pi/2k, \pi/2k]$ of the complex plane.

2.

Using (22) and (23) as in the proof of Theorem 1, one can show that the function $\sigma_3(A) - s(n)\sigma_2(A)$ is convex on Ω_n, $n \geq 3$, if $s(n) \leq n - 4$. In particular, $\sigma_3(A) - [(n-2)^2/3n]\sigma_2(A)$ is convex on Ω_n for $n \geq 5$.

3.

Even though the permanent function is not convex on Ω_n, $n \geq 3$, there is hope that it is convex on some subset(s) of Ω_n. In fact, this is the case for $\Omega_3^0 \subset \Omega_3$, where Ω_n^0 denotes the set of all matrices in Ω_n with zero main diagonal. Indeed, if $A \in \Omega_3^0$, then $a_{11} = a_{22} = a_{33} = 0$, $a_{12} = a_{23} = a_{31} = x$, and $a_{13} = a_{21} = a_{32} = 1 - x$, $0 \leq x \leq 1$, and therefore $\operatorname{per}(A) = \frac{2}{3}\sum a_{ij}^3 - \frac{1}{2}\sum a_{ij}^2 + \frac{1}{2} = 3x^2 - 3x + 1$. Since $f(x) = 3x^2 - 3x + 1$ is a convex function, convexity of $\operatorname{per}(A)$ on Ω_3^0 follows.

4.

We propose a different (short) proof of the following lemma, which is the main auxiliary result in [8].

LEMMA (Lemma 3 of [8]). *For any* $A \in \Omega_3$, $f_A''(1/2) = \frac{2}{3}\sigma_2(A - J_3) + 3\operatorname{per}(A - J_3) \geq 0$, *with equality if and only if either* $A = J_3$ *or* A *is a permutation of* $(3J_3 - I_3)/2$.

Proof. Using (27) and (28) with $s = -1$, $n = 3$, we write

$$\begin{aligned} f_A''(\tfrac{1}{2}) &= \tfrac{2}{3}\left[\sigma_2(A) - \tfrac{4}{3}\sigma_1(A) + \sigma_2(J_3)\right] \\ &\quad + 3\left[\sigma_3(A) - \tfrac{1}{3}\sigma_2(A) + \tfrac{2}{9}\sigma_1(A) - \sigma_3(J_3)\right] \\ &= 3\sigma_3(A) - \tfrac{1}{3}\sigma_2(A) \geq 0\,. \end{aligned}$$

The last inequality is the Holens-Doković conjecture for $k = n = 3$, which was proved by Doković [3]. It was also shown in [3] that equality is attained if and only if $A = J_3$ or A is a permutation of $(3J_3 - I_3)/2$. ■

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