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On moduli of smoothness of k-monotone functions and applications

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Abstract

Let $\mathcal{M}^k$ be the set of all k-monotone functions on $(-1, 1)$, i.e., those functions f for which the kth divided differences $[x_0, \ldots, x_k; f]$ are nonnegative for all choices of $(k + 1)$ distinct points $x_0, \ldots, x_k$ in $(-1, 1)$. We obtain estimates (which are exact in a certain sense) of kth Ditzian–Totik $\mathbb{L}_q$-moduli of smoothness of functions in $\mathcal{M}^k \cap \mathbb{L}_p(-1, 1)$, where $1 \leqslant q < p \leqslant \infty$, and discuss several applications of these estimates.

1. *Introduction*

Given an interval (closed, open or half-open) I, a function $f: I \to \mathbb{R}$ is said to be k-monotone on I if its kth divided differences $[x_0, \ldots, x_k; f] := \sum_{i=0}^{k} f(x_i)/w'(x_i)$, where $w(x) := \prod_{i=0}^{k}(x - x_i)$, are nonnegative for all selections of $k + 1$ distinct points $x_0, \ldots, x_k$ in I. We denote the class of all such functions by $\Delta^k(I)$, and note that Δ^1 and Δ^2 are convex cones of all monotone and convex functions, respectively. As usual, $\mathbb{L}_p(J)$, $1 \leqslant p \leqslant \infty$, denotes the space of all measurable functions f on J such that $\|f\|_{\mathbb{L}_p(J)} < \infty$, where $\|f\|_{\mathbb{L}_p(J)} := (\int_J |f(x)|^p\, dx)^{1/p}$ if $p < \infty$, and $\|f\|_{\mathbb{L}_\infty(J)} := \operatorname{ess\,sup}_{x \in J} |f(x)|$ (if f is continuous on $[a, b]$, then we also use the notation $\|f\|_{\mathbb{C}[a,b]} := \max_{x \in [a,b]} |f(x)|$). For simplicity, we write $\mathbb{L}_p := \mathbb{L}_p[-1, 1]$ and $\|f\|_p := \|f\|_{\mathbb{L}_p[-1,1]}$. It needs to be emphasized that the classes $\Delta^k(I)$ essentially depend on whether or not the interval I is closed. For example, convex functions in the class $\Delta^2(0, 1]$ do not have to be defined at 0 and hence have to be neither bounded nor integrable on $(0, 1]$. ($f(x) = 1/x$ is an example of one such function.) At the same time, the class $\Delta^2[0, 1]$ consists of all functions f which are convex and continuous on $(0, 1)$, defined at 0 and 1, and such that $\lim_{x \to 0^+} f(x) \leqslant f(0)$ and $\lim_{x \to 1^-} f(x) \leqslant f(1)$. Therefore, $\Delta^2[0, 1]$ consists only of bounded functions continuous everywhere put perhaps the endpoints of $[0, 1]$ and hence belonging to all $\mathbb{L}_p[0, 1]$ spaces. Clearly, $\Delta^k(\overline{I}) \subset \Delta^k(I)$ where $\overline{I}$ is the closure of I. All our results in this paper are given for the (larger) function classes $\mathcal{M}^k(I) := \Delta^k(int(I))$, where $int(S)$ denotes the interior of a set S. Note that, with this notation, $\mathcal{M}^k := \mathcal{M}^k[-1, 1] = \mathcal{M}^k(-1, 1)$, etc.

Functions from $\mathcal{M}^k(I)$ enjoy various smoothness properties. For example (see [**11**, **12**]), if $f : [a, b] \mapsto \mathbb{R}$ is k-monotone for some $k \geqslant 2$, then, for all $j \leqslant k - 2$, $f^{(j)}(x)$ exists on (a, b) and is $(k - j)$-monotone. In particular, $f^{(k-2)}(x)$ exists, is convex, and therefore satisfies a Lipschitz condition on any closed interval $[\xi, \zeta]$ contained in (a, b), is absolutely continuous on $[\xi, \zeta]$, is continuous on (a, b), and has left and right (nondecreasing) derivatives, $f_-^{(k-1)}(x)$ and $f_+^{(k-1)}(x)$ on (a,b). Moreover, the set E where $f^{(k-1)}(x)$ fails to exist is countable, and $f^{(k-1)}$ is continuous on $(a,b)\setminus E$.

For $k\in\mathbb{N}$, the kth (classical) modulus of smoothness of a function $f\in\mathbb{L}_p(J)$ is defined by

$$\omega_k(f,\delta,J)_p := \sup_{0<h\leqslant\delta} \|\Delta_h^k(f,\cdot,J)\|_{\mathbb{L}_p(J)},$$

where

$$\Delta_h^k(f,x,J) := \begin{cases} \displaystyle\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}f(x-kh/2+ih), & \text{if } x\pm kh/2\in J,\\ 0, & \text{otherwise,}\end{cases}$$

is the kth symmetric difference.

It is rather straightforward to show that, if $1\leqslant q<p\leqslant\infty$ and a function $f\in\mathbb{L}_p[a,b]$ has $s<\infty$ changes of monotonicity (note that k-monotone functions can have at most $k-1$ points of monotonicity changes), then

$$\omega_1(f,\delta,[a,b])_q \leqslant c\delta^{1/q-1/p}\,\|f\|_{\mathbb{L}_p[a,b]} \tag{1·1}$$

with a constant c depending on s. (Since $\omega_{k+1}(f,\delta,J)_p\leqslant 2^{\max\{1,1/p\}}\omega_k(f,\delta,J)_p$, $k\in\mathbb{N}$, the same inequality holds for all kth moduli.) The idea of the proof is to prove this inequality for any monotone function first (using considerations similar to what we use below to estimate Ditzian-Totik moduli of functions in $\mathcal{M}^1$), represent $[a,b]$ as a union of intervals such that f is monotone on each of them, and finally use Hölder's inequality $\|f\|_{\mathbb{L}_q(I)}\leqslant|I|^{1/q-1/p}\,\|f\|_{\mathbb{L}_p(I)}$, $q\leqslant p\leqslant\infty$, to obtain estimates near points of monotonicity changes (we omit details). Also, note that the constant c in (1·1) cannot be made independent of s, and so (1·1) may no longer be valid for an arbitrary function f from $\mathbb{L}_p[a,b]$. For example, consider $f_\beta(x):=\sin(\pi\beta x)$, $0\leqslant x\leqslant 1$, where $\beta\in\mathbb{N}$. Clearly, f_β has β changes of monotonicity, $\|f_\beta\|_{\mathbb{L}_p[0,1]}\leqslant 1$, and assuming that $\delta\leqslant 1/2$, we have

$$\begin{aligned}\left\|\Delta_h^1(f,x)\right\|_{\mathbb{L}_q[0,1]}^q &= \int_{h/2}^{1-h/2}|\sin\pi\beta(x+h/2)-\sin\pi\beta(x-h/2)|^q\,dx\\ &= 2^q\,|\sin(\pi\beta h/2)|^q\int_{h/2}^{1-h/2}|\cos(\pi\beta x)|^q\,dx\\ &\sim |\sin(\pi\beta h/2)|^q\,.\end{aligned}$$

Hence,

$$\omega_1(f,\delta,[0,1])_q \sim \sup_{0<h\leqslant\delta}|\sin(\pi\beta h/2)| \sim \min\{1,\beta\delta\}.$$

In particular, if $\delta:=1/\beta$ and $\beta\to\infty$, then $\delta^{-1/q+1/p}\omega_1(f_\beta,\delta,[0,1])_q\to\infty$, and so cannot be bounded by $\|f_\beta\|_{\mathbb{L}_p[0,1]}$.

We also remark that the power of δ in (1·1) cannot be increased. Indeed, for $p<\infty$, $f_\alpha := x^{-\alpha}$, $0<\alpha<1/p$, is monotone and $\|f_\alpha\|_{\mathbb{L}_p[0,1]}\sim c(\alpha,p)$. At the same time, $\omega_1(f_\alpha,\delta,[0,1])_q\geqslant c\delta^{1/q-\alpha}$, and so if (1·1) were valid with $\delta^{1/q-1/p+\epsilon}$, $\epsilon>0$, one could choose any $\alpha\geqslant 1/p-\epsilon/2$ in order to obtain a contradiction.

Uniform estimates of polynomial approximation in terms of classical moduli of smoothness are rather imperfect since, as is rather well known, the rate of approximation can be improved near the endpoints of an interval (see e.g. [**4**]). If approximation in the uniform norm is investigated, pointwise estimates yield constructive characterization of the smoothness classes via approximation orders achieved by algebraic polynomials. If one desires to obtain exact uniform estimates (e.g. for approximation in the $\mathbb{L}_p$ norm, $p < \infty$), then one can no longer use the classical moduli of smoothness, and the new measure of smoothness is needed. Different approaches are possible (see e.g. [**3**, chapter 13] for discussions and comparisons), but the one that has received most attention in recent years is the theory developed by Ditzian and Totik [**3**] (there is also Ivanov's τ-modulus ([**5**]) which is equivalent to the Ditzian–Totik modulus for some values of parameters which we do not discuss in this paper).

The kth Ditzian–Totik modulus of $f \in \mathbb{L}_p$ is

$$\omega_k^{\varphi}(f, t)_p := \sup_{0<h\leqslant t} \left\| \Delta_{h\varphi(\cdot)}^k (f, \cdot) \right\|_p,$$

where $\varphi(x) := \sqrt{1-x^2}$ and $\Delta_\mu^k(f, x) := \Delta_\mu^k(f, x, [-1, 1])$.

The following theorem is one of our main results.

THEOREM 1·1. *Let $k \in \mathbb{N}$, $1 \leqslant q < p \leqslant \infty$, and $f \in \mathcal{M}^k \cap \mathbb{L}_p$. Then*

$$\omega_k^{\varphi}(f, \delta)_q \leqslant c \Upsilon_\delta(k, q, p) \, \|f\|_p, \tag{1·2}$$

where

$$\Upsilon_\delta(k, q, p) := \begin{cases} \delta^{2/q-2/p}, & \text{if } k \geqslant 2, \\ \delta^{2/q-2/p}, & \text{if } k = 1 \text{ and } p < 2q, \\ (\delta\sqrt{|\ln(\delta)|})^{1/q}, & \text{if } k = 1 \text{ and } p = 2q, \\ \delta^{1/q}, & \text{if } k = 1 \text{ and } p > 2q. \end{cases}$$

Note that the estimates in Theorem 1·1 are exact in the sense that the powers of δ in (1·2) cannot be increased. In the cases $k \geqslant 2$, and $k = 1$ and $2q > p$, this follows, for example, from a simple observation (see also [**3**, page 35]) that, if $f_\epsilon(x) := (1+x)^{\epsilon - 1/p}$, $\epsilon > 0$, $\epsilon - 1/p \notin \mathbb{N}_0$, then f or $-f$ is k-monotone, $\|f_\epsilon\|_{\mathbb{L}_p[0,1]} \sim c(\epsilon, p)$, and $\omega_k^{\varphi}(f_\epsilon, \delta)_q \geqslant c\delta^{\min\{k, 2/q-2/p+2\epsilon\}}$ if $k \neq 2/q - 2/p + 2\epsilon$. In the case $k = 1$ and $2q \leqslant p$, it is sufficient to notice that $\omega_1^{\varphi}(\text{sign}(x), \delta)_q \sim \delta^{1/q}$. In fact, $\omega_k^{\varphi}(\text{sign}(x), \delta)_q \sim \delta^{1/q}$, for all $k \in \mathbb{N}$, which shows, in particular, that Theorem 1·1 is no longer true in general if the assumption that $f \in \mathcal{M}^k$ is removed. (Note that a finer analysis and other counterexamples are possible.)

Perhaps Theorem 1·1 would not be too interesting by itself, and our main motivation in considering it and writing this note is several applications which we discuss in Section 2. In particular, as we discuss in Section 2, Theorem 1·1 provides a new method of proving some known as well as several new results.

2. *Applications*

1. Recall the following well-known result which holds for all (and not only k-monotone) functions in $\mathbb{L}_q$ (see [**3**, theorem 7·2·1]): for $k \in \mathbb{N}$ and $1 \leqslant q \leqslant \infty$, there exists a constant c which depends only on k such that, for any $f \in \mathbb{L}_q$ and $n \geqslant k-1$, there is a polynomial $r_n \in \Pi_n$ such that

$$\|f - r_n\|_q \leqslant c\omega_k^{\varphi}(f, n^{-1})_q. \tag{2·1}$$

Now, let $1 \leqslant q < p \leqslant \infty$, and let $f \in \mathbb{L}_p \cap \mathcal{M}^k$, $k \in \mathbb{N}$, be such that $f^{(m)} \in \mathbb{L}_p$ with $0 \leqslant m \leqslant k-1$. Then, using Theorem 1·1 and the fact that $f^{(m)} \in \mathcal{M}^{k-m}$, we conclude that for any $n \geqslant k-1$, there exists a polynomial r_n of degree $\leqslant n$ such that

$$\|f - r_n\|_q \leqslant c\omega_k^{\varphi}(f, n^{-1})_q \leqslant cn^{-m}\omega_{k-m}^{\varphi}\left(f^{(m)}, n^{-1}\right)_q \leqslant cn^{-m}\Upsilon_{1/n}(k-m, q, p)\left\|f^{(m)}\right\|_p.$$

In other words,

$$\|f - r_n\|_q \leqslant c\left\|f^{(m)}\right\|_p \times \begin{cases} n^{-m-2/q+2/p}, & \text{if } m \leqslant k-2, \text{ or } m = k-1 \text{ and } p < 2q, \\ n^{-m-1/q}|\ln n|^{1/(2q)}, & \text{if } m = k-1 \text{ and } p = 2q, \\ n^{-m-1/q}, & m = k-1 \text{ and } p > 2q. \end{cases} \tag{2·2}$$

Remark 2·1. If $k = 1$ or $k = 2$, then the polynomial r_n in (2·2) may be chosen to be from $\mathcal{M}^k$. This immediately follows (see [**9**, **13**, **14**]) from the fact that, for $k = 1$ or $k = 2$, any $f \in \mathcal{M}^k \cap \mathbb{L}_q$, $1 \leqslant q \leqslant \infty$, and $n \geqslant k-1$, there exists a polynomial $r_n \in \Pi_n \cap \mathcal{M}^k$ such that $\|f - r_n\|_q \leqslant c\omega_k^{\varphi}(f, n^{-1})_q$, where c is an absolute constant.

Now, recalling a consequence of Hölder's inequality $\|g\|_q \leqslant \|g\|_p^{\alpha}\|g\|_1^{1-\alpha}$, $\alpha := p(q-1)/q(p-1)$, $1 \leqslant q < p \leqslant \infty$, and using Theorem 1·1 and (2·1) we have the following estimates for $f \in \mathcal{M}^k$ such that $f^{(m)} \in \mathbb{L}_p$, $0 \leqslant m \leqslant k-1$:

$$\begin{aligned} \|f - r_n\|_q &\leqslant c\omega_k^{\varphi}(f, n^{-1})_q \leqslant cn^{-m}\omega_{k-m}^{\varphi}\left(f^{(m)}, n^{-1}\right)_q \\ &\leqslant cn^{-m}\sup_{0<h\leqslant n^{-1}}\left\|\Delta_{h\varphi(\cdot)}^{k-m}\left(f^{(m)}, \cdot\right)\right\|_p^{\alpha}\left\|\Delta_{h\varphi(\cdot)}^{k-m}\left(f^{(m)}, \cdot\right)\right\|_1^{1-\alpha} \\ &\leqslant cn^{-m}\,\omega_{k-m}^{\varphi}\left(f^{(m)}, n^{-1}\right)_p^{\alpha}\,\omega_{k-m}^{\varphi}\left(f^{(m)}, n^{-1}\right)_1^{1-\alpha} \\ &\leqslant cn^{-m}\,\omega_{k-m}^{\varphi}\left(f^{(m)}, n^{-1}\right)_p^{\alpha}\,\Upsilon_{1/n}(k-m, 1, p)^{1-\alpha}\left\|f^{(m)}\right\|_p^{1-\alpha}. \end{aligned} \tag{2·3}$$

Recalling that

$$\Upsilon_{1/n}(k-m, 1, p) = \begin{cases} n^{-2+2/p}, & \text{if } 0 \leqslant m \leqslant k-2, \\ n^{-2+2/p}, & \text{if } m = k-1 \text{ and } p < 2, \\ n^{-1}\sqrt{\ln n}, & \text{if } m = k-1 \text{ and } p = 2, \\ n^{-1}, & \text{if } m = k-1 \text{ and } p > 2, \end{cases}$$

we have, for $1 < q < p \leqslant \infty$ and $0 \leqslant m \leqslant k-1$,

$$\|f - r_n\|_q = o(n^{-m}l(n)), \tag{2·4}$$

where

$$l(n) := \begin{cases} n^{-2/q+2/p}, & \text{if } 0 \leqslant m \leqslant k-2, \text{ or } m = k-1 \text{ and } p < 2, \\ (n^{-1}\sqrt{\ln n})^{2/q-1}, & \text{if } m = k-1 \text{ and } p = 2, \\ n^{-(p-q)/q(p-1)}, & \text{if } m = k-1 \text{ and } p > 2. \end{cases}$$

In particular, for $0 \leqslant m \leqslant k-1$, $p = \infty$, and $f \in \mathcal{M}^k$ such that $f^{(m)} \in \mathbb{L}_{\infty}$, we have

$$\|f - r_n\|_q = o\left(n^{-m-\min\{(k-m)/q, 2/q\}}\right), \quad 1 < q < \infty. \tag{2·5}$$

Additionally, if $m = 0$, then for $f \in \mathcal{M}^k \cap L_{\infty}$, we have

$$\|f - r_n\|_q = o\left(n^{-\min\{k/q, 2/q\}}\right), \quad 1 < q < \infty. \tag{2·6}$$

If $k = 1$ or $k = 2$, then by Remark 2·1, r_n in inequalities (2·3)–(2·6) can be chosen to be from $\mathcal{M}^k$. In the case $k = 2$, (2·6) is the main result in [**10**]. Finally, we mention that the above estimates improve [**8**, theorem 3].

2. Recently, Konovalov, Leviatan and Maiorov [**6**] investigated the orders of best approximation by polynomials and ridge functions of certain classes of k-monotone radial functions.[1] They obtained several asymptotically exact estimates. Our Theorem 1·1 yields a different (and simpler) proof of the upper estimates in [**6**]. Moreover, we are able to obtain results for monotone and convex polynomial approximation as an immediate consequence of Theorem 1·1, some known estimates, and lower estimates in [**6**]. In order to discuss this further, we need to introduce some new notation.

Let $\mathcal{M}^k\mathbb{B}_p$ denote the intersection of $\mathcal{M}^k$ with the unit ball in $\mathbb{L}_p$, i.e., $\mathcal{M}^k\mathbb{B}_p$ is the set of all k-monotone functions f on $(-1, 1)$ such that $\|f\|_p \leqslant 1$, and let Π_n be the space of all algebraic polynomials of degree $\leqslant n$. Also,

$$E(\mathcal{M}^k\mathbb{B}_p, \Pi_n)_q := \sup_{f\in\mathcal{M}^k\mathbb{B}_p} \inf_{r_n\in\Pi_n} \|f - r_n\|_q$$

denotes the rate of approximation of the set $\mathcal{M}^k\mathbb{B}_p$ by Π_n. It was shown in [**6**] that, for $1 \leqslant q \leqslant p \leqslant \infty$,

$$E(\mathcal{M}^k\mathbb{B}_p, \Pi_n)_q \asymp \begin{cases} n^{-\min\{1/q,2/q-2/p\}}, & \text{if } k = 1 \text{ and } p \neq 2q, \\ n^{-2/q+2/p}, & \text{if } k \geqslant 2, \end{cases} \tag{2·7}$$

where, for positive sequences (a_n) and (b_n), $a_n \asymp b_n$ means that $c_1 a_n \leqslant b_n \leqslant c_2 a_n$ for some positive constants c_1 and c_2 and all $n \in \mathbb{N}$.

The upper estimates in (2·7) now immediately follow from (2·2) with $m = 0$ (taking into account that, in the case $k = 1$ and $p \neq 2q$, $\Upsilon_\delta(k, q, p) = \delta^{\min\{1/q,2/q-2/p\}}$).

Now, let

$$E(\mathcal{M}^k\mathbb{B}_p, \Pi_n \cap \mathcal{M}^k)_q := \sup_{f\in\mathcal{M}^k\mathbb{B}_p} \inf_{r_n\in\Pi_n\cap\mathcal{M}^k} \|f - r_n\|_q$$

be the rate of approximation of the set $\mathcal{M}^k\mathbb{B}_p$ by k-monotone polynomials in Π_n. Clearly,

$$E(\mathcal{M}^k\mathbb{B}_p, \Pi_n)_q \leqslant E(\mathcal{M}^k\mathbb{B}_p, \Pi_n \cap \mathcal{M}^k)_q.$$

Therefore, (2·2), the observation after it, and lower estimates in (2·7) immediately imply that

$$E(\mathcal{M}^k\mathbb{B}_p, \Pi_n \cap \mathcal{M}^k)_q \asymp \begin{cases} n^{-\min\{1/q,2/q-2/p\}}, & \text{if } k = 1 \text{ and } p \neq 2q, \\ n^{-2/q+2/p}, & \text{if } k = 2. \end{cases}$$

Note that, in the case $k = 1$ and $p = 2q$, we get

$$E(\mathcal{M}^1\mathbb{B}_p, \Pi_n \cap \mathcal{M}^1)_{p/2} \leqslant cn^{-2/p}(\ln n)^{1/p}, \quad n \geqslant 2.$$

3. *Auxiliary results and proof of Theorem* 1·1

It is convenient to denote

$$\mathfrak{D}_\alpha := \{x \;:\; x \pm \alpha\varphi(x) \in [-1, 1]\} = \left\{x \;:\; |x| \leqslant \frac{1-\alpha^2}{1+\alpha^2}\right\}.$$

Then, $\Delta^k_{h\varphi(x)}(f, x) = 0$ if $x \notin \mathfrak{D}_{kh/2}$.

[1] The author is indebted to the authors of [**6**] for discussion of their new results.

LEMMA 3·1. *Let* $0 < \alpha < 1$ *and* $-\alpha \leqslant \beta \leqslant \alpha$. *Then, for any integrable function* f, *we have*

$$\int_{\mathfrak{D}_\alpha} f(x+\beta\varphi(x))\,dx = \frac{1}{1+\beta^2}\int_{-1+2\alpha(\alpha+\beta)/(1+\alpha^2)}^{1-2\alpha(\alpha-\beta)/(1+\alpha^2)} f(y)\left(1+\frac{\beta y}{\sqrt{1-y^2+\beta^2}}\right)dy.$$

Proof. The function $g(x) = x+\beta\varphi(x)$ is strictly increasing on $\mathfrak{D}_\alpha$, since the only critical point x_0 of g (in the case $\beta \neq 0$) satisfies

$$x_0^2 = \frac{1}{1+\beta^2} \geqslant \frac{1}{1+\alpha^2} > \frac{1-\alpha^2}{1+\alpha^2} > \left(\frac{1-\alpha^2}{1+\alpha^2}\right)^2,$$

and hence $x_0 \notin \mathfrak{D}_\alpha$. Now, solving the equation $y = x+\beta\varphi(x)$ we get $x = (1+\beta^2)^{-1}(y - \beta\sqrt{1-y^2+\beta^2})$, and it remains to change the variable of integration.

The following auxiliary result is interesting in its own right, and is more general than Theorem 1·1. While we only need the cases $k = 1$ and $k = 2$ in Theorem 3·2 in order to prove Theorem 1·1, the proof for arbitrary $k \in \mathbb{N}$ is not much longer. However, since it is somewhat technical we postpone it until the last section.

THEOREM 3·2. *Let* $k \in \mathbb{N}$, $f \in \mathfrak{M}^k \cap \mathbb{L}_1$ *and* $\delta \leqslant 1/k$. *If k is <u>even</u>, then*

$$\omega_k^\varphi(f,\delta)_1 \leqslant c(k)(\|f\|_{\mathbb{L}_1[-1,-1+k^2\delta^2]} + \|f\|_{\mathbb{L}_1[1-k^2\delta^2,1]} + \delta^k\,\|f\|_1).$$

If k is <u>odd</u>, then

$$\begin{aligned}\omega_k^\varphi(f,\delta)_1 \leqslant c(k)\big(&\|f\|_{\mathbb{L}_1[-1,-1+k^2\delta^2]} + \|f\|_{\mathbb{L}_1[1-k^2\delta^2,1]}\\ &+ \sup_{0<h\leqslant\delta} h^k \left\|f(y)(1-y^2)^{-k/2}\right\|_{\mathbb{L}_1[-1+k^2h^2/2,1-k^2h^2/2]}\big).\end{aligned}$$

The following corollary immediately follows by Hölder's inequality and the fact (in the case for odd k) that, for $1 \leqslant p' \leqslant \infty$ (with $1/p' + 1/p = 1$),

$$\left\|(1-y^2)^{-k/2}\right\|_{\mathbb{L}_{p'}[-1+k^2h^2/2,1-k^2h^2/2]} \leqslant c(k)\begin{cases} h^{-k+2/p'}, & \text{if } kp' > 2,\\ |\ln(h)|^{1/p'}, & \text{if } kp' = 2,\\ 1, & \text{if } kp' < 2.\end{cases}$$

In particular, if $k \geqslant 3$, then $\|(1-y^2)^{-k/2}\|_{\mathbb{L}_{p'}[-1+k^2h^2/2,1-k^2h^2/2]} \leqslant c(k)h^{-k+2/p'}$, and, if $k = 1$, then

$$\left\|(1-y^2)^{-1/2}\right\|_{\mathbb{L}_{p'}[-1+h^2/2,1-h^2/2]} \leqslant c\begin{cases} h^{-1+2/p'}, & \text{if } p' > 2,\\ \sqrt{|\ln(h)|}, & \text{if } p' = 2,\\ 1, & \text{if } p' < 2.\end{cases}$$

COROLLARY 3·3. *Let* $k \in \mathbb{N}$, $f \in \mathfrak{M}^k \cap \mathbb{L}_p$, $1 \leqslant p \leqslant \infty$. *Then*

$$\omega_k^\varphi(f,\delta)_1 \leqslant c(k)\,\|f\|_p\begin{cases} \delta^{2-2/p}, & \text{if } k \geqslant 2, \text{ or } k = 1 \text{ and } 1 \leqslant p < 2,\\ \delta\sqrt{|\ln(\delta)|}, & \text{if } k = 1 \text{ and } p = 2,\\ \delta, & k = 1 \text{ and } 2 < p \leqslant \infty.\end{cases}$$

We now generalize the estimates in Corollary 3·3 in the case $k = 1$ and $k = 2$ providing estimates of $\omega_k^\varphi(f,\delta)_q$ for all $1 \leqslant q < \infty$. We need two auxiliary lemmas.

LEMMA 3·4. *Let* $1 \leqslant q < \infty$*, and let* $f \in \mathbb{L}_q$ *be nonnegative on* $[-1, 1]$*. Then,*

$$\omega_1^\varphi(f, \delta)_q \leqslant \omega_1^\varphi(f^q, \delta)_1^{1/q}. \tag{3·1}$$

Proof. The following inequalities immediately follow from the convexity of x^q and positivity of $(1+x)^q - x^q - 1$ for $x > 0$ and $q \geqslant 1$:

$$2^{1-q}(a+b)^q \leqslant a^q + b^q \leqslant (a+b)^q, \quad a \geqslant 0, b \geqslant 0 \text{ and } q \geqslant 1. \tag{3·2}$$

Since by (3·2), $|a_1 - a_2|^q \leqslant |a_1^q - a_2^q|$, $a_1 \geqslant 0$ and $a_2 \geqslant 0$, $q \geqslant 1$, for any nonnegative function f we have

$$\left|\Delta_\mu^1(f, x)\right|^q \leqslant \left|\Delta_\mu^1(f^q, x)\right|,$$

which implies (3·1).

LEMMA 3·5. *Let* $1 \leqslant q < \infty$*, and let* $f \in \mathcal{M}^2 \cap \mathbb{L}_q$ *be nonnegative on* $[-1, 1]$*. Then,*

$$\omega_2^\varphi(f, \delta)_q \leqslant 2^{1-1/q}\omega_2^\varphi(f^q, \delta)_1^{1/q}. \tag{3·3}$$

Proof. If $a_1 \geqslant 0$, $a_2 \geqslant 0$, $a_3 \geqslant 0$, and $a_1 - 2a_2 + a_3 \geqslant 0$ and $q \geqslant 1$, then using (3·2) we have

$$(a_1 - 2a_2 + a_3)^q + (2a_2)^q \leqslant (a_1 + a_3)^q \leqslant 2^{q-1}\left(a_1^q + a_3^q\right),$$

and so

$$(a_1 - 2a_2 + a_3)^q \leqslant 2^{q-1}\left(a_1^q - 2a_2^q + a_3^q\right).$$

This implies that, for a convex and nonnegative function f,

$$\left(\Delta_\mu^2(f, x)\right)^q \leqslant 2^{q-1}\Delta_\mu^2(f^q, x),$$

and (3·3) immediately follows.

Now, taking into account that, for a nonnegative f, $\|f^q\|_{p/q}^{1/q} = \|f\|_p$, and using Lemmas 3·4 and 3·5, and Corollary 3·3 (with p/q instead of p) we get the following result.

COROLLARY 3·6. *Let* $k = 1$ *or* $k = 2$*,* $1 \leqslant q < p \leqslant \infty$*, and let* $f \in \mathcal{M}^k \cap \mathbb{L}_p$ *be nonnegative. Then*

$$\omega_k^\varphi(f, \delta)_q \leqslant c(k)\,\|f\|_p \begin{cases} \delta^{2/q - 2/p}, & \text{if } k = 2, \text{ or } k = 1 \text{ and } p < 2q, \\ (\delta\sqrt{|\ln(\delta)|})^{1/q}, & \text{if } k = 1 \text{ and } p = 2q, \\ \delta^{1/q}, & k = 1 \text{ and } p > 2q. \end{cases}$$

We are now ready to prove Theorem 1·1. First, note that, if (1·2) holds for functions f_1 and f_2 which have the same sign at all points in $[-1, 1]$ (i.e., $f_1(x)f_2(x) \geqslant 0$, $-1 \leqslant x \leqslant 1$), then it is also valid for $f_1 + f_2$. Indeed, since

$$\|f_1\|_p + \|f_2\|_p \leqslant 2\,\|\max(|f_1|, |f_2|)\|_p \leqslant 2\,\||f_1| + |f_2|\|_p = 2\,\|f_1 + f_2\|_p,$$

we have

$$\begin{aligned} \omega_k^\varphi(f_1 + f_2, \delta)_q &\leqslant \omega_k^\varphi(f_1, \delta)_q + \omega_k^\varphi(f_2, \delta)_q \leqslant c\Upsilon_\delta(k, q, p)\left(\|f_1\|_p + \|f_2\|_p\right) \\ &\leqslant c\Upsilon_\delta(k, q, p)\,\|f_1 + f_2\|_p. \end{aligned} \tag{3·4}$$

The following lemma shows that it is sufficient to prove Theorem 1·1 for functions $f \in \mathcal{M}^k$ such that $f^{(i)}(0) = 0$, $0 \leqslant i \leqslant k-1$, where $f^{(k-1)}(0) := f_-^{(k-1)}(0)$ (any number between $f_-^{(k-1)}(0)$ and $f_+^{(k-1)}(0)$ would do), since $\omega_k^\varphi(f - T_{k-1}(f), \delta)_q = \omega_k^\varphi(f, \delta)_q$.

This lemma is an immediate corollary of a stronger [**7**, theorem 1] taking into account [**2**] (see also [**1**, Theorem 4·6·3]).

LEMMA 3·7 ([**7**]). *Let $k \in \mathbb{N}$, $0 < p \leqslant \infty$, and $f \in \mathcal{M}^k \cap \mathbb{L}_p$. Denote by $T_{k-1}(f, x) := \sum_{i=0}^{k-1}(i!)^{-1} f^{(i)}(0)x^i$ the McLaurin polynomial of degree $\leqslant k-1$, where $f^{(k-1)}(0) := f_-^{(k-1)}(0)$. Then, there exists a constant $c = c(k, p)$ such that*

$$\|f - T_{k-1}(f, \cdot)\|_p \leqslant c\,\|f\|_p\,.$$

Proof of Theorem 1·1. Let $f \in \mathcal{M}^k \cap \mathbb{L}_p$ be such that $f^{(i)}(0) = 0$, $0 \leqslant i \leqslant k-2$, and $f_-^{(k-1)}(0) = 0$. Then, as is easily shown by induction (see also [**7**, lemma 7]), $f \in \mathcal{M}^j[0, 1]$ and $(-1)^{k-j} f \in \mathcal{M}^j[-1, 0]$ for all $j = 0, \ldots, k-1$. Now, let

$$f_1(x) := \begin{cases} 0, & \text{if } -1 \leqslant x \leqslant 0, \\ f(x), & \text{if } 0 < x \leqslant 1, \end{cases} \qquad \text{and} \qquad f_2(x) := \begin{cases} f(x), & \text{if } -1 \leqslant x \leqslant 0, \\ 0, & \text{if } 0 < x \leqslant 1, \end{cases}$$

Note that the functions f_1 and f_2 have the same sign on $[-1, 1]$ ($f_1(x) f_2(x) = 0$ for all x), and that $f = f_1 + f_2$.

If $k = 1$, then $f_1(x)$ and $-f_2(-x)$ are both nonnegative functions in $\mathcal{M}^1 \cap \mathbb{L}_p$, and if $k \geqslant 2$, then $f_1(x)$ and $(-1)^k f_2$ are both nonnegative functions in $\mathcal{M}^2 \cap \mathbb{L}_p$. Therefore, Corollary 3·6 and (3·4) imply that (1·2) is satisfied for $f = f_1 + f_2$ (taking into account that $\omega_k^\varphi(f, \delta)_q \leqslant c\omega_2^\varphi(f, \delta)_q$ for $k \geqslant 2$, see [**3**, theorem 4·1·3]).

4. *Proof of Theorem* 3·2

It is convenient to denote

$$\mathfrak{J}(\beta, y) := \frac{1}{1+\beta^2}\left(1 + \frac{\beta y}{\sqrt{1 - y^2 + \beta^2}}\right).$$

Taking into account that $\Delta^k_{h\varphi(x)}(f, x) \geqslant 0$, for every x, and using Lemma 3·1 with $\alpha = kh/2$ and $\beta = (i - k/2)h$, $0 \leqslant i \leqslant k$, we have

$$\begin{aligned}
\left\|\Delta^k_{h\varphi(x)}(f, x)\right\|_1 &= \int_{\mathfrak{D}_{kh/2}} \Delta^k_{h\varphi(x)}(f, x)\,dx \\
&= \sum_{i=0}^k \binom{k}{i}(-1)^{k-i} \int_{\mathfrak{D}_{kh/2}} f(x + (i - k/2)h\varphi(x))\,dx \\
&= \sum_{i=0}^k \binom{k}{i}(-1)^{k-i} \int_{-1+4kih^2/(4+k^2h^2)}^{1-4k(k-i)h^2/(4+k^2h^2)} f(y)\,\mathfrak{J}((i - k/2)h, y)\,dy \\
&= \sum_{i=0}^k \binom{k}{i}(-1)^{k-i} \left(\int_{-1+4kih^2/(4+k^2h^2)}^{-1+4k^2h^2/(4+k^2h^2)} + \int_{-1+4k^2h^2/(4+k^2h^2)}^{1-4k^2h^2/(4+k^2h^2)}\right. \\
&\qquad \left. + \int_{1-4k^2h^2/(4+k^2h^2)}^{1-4k(k-i)h^2/(4+k^2h^2)}\right) f(y)\,\mathfrak{J}((i - k/2)h, y)\,dy \\
&=: \sum_{i=0}^k \binom{k}{i}(-1)^{k-i}\,(\mathfrak{I}_1 + \mathfrak{I}_2 + \mathfrak{I}_3)\,.
\end{aligned}$$

Since $|\mathfrak{J}((i-k/2)h, y)| \leqslant 2$, $0 \leqslant i \leqslant k$, we have

$$\begin{aligned}
&\left|\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\,(\mathfrak{I}_1+\mathfrak{I}_3)\right| \\
&\quad \leqslant 2\sum_{i=0}^{k}\binom{k}{i}\left(\int_{-1+4kih^2/(4+k^2h^2)}^{-1+4k^2h^2/(4+k^2h^2)} + \int_{1-4k^2h^2/(4+k^2h^2)}^{1-4k(k-i)h^2/(4+k^2h^2)}\right)|f(y)|\,dy \\
&\quad \leqslant 2^{k+1}\left(\|f\|_{\mathbb{L}_1[-1,-1+4k^2h^2/(4+k^2h^2)]} + \|f\|_{\mathbb{L}_1[1-4k^2h^2/(4+k^2h^2),1]}\right) \\
&\quad \leqslant c(k)\left(\|f\|_{\mathbb{L}_1[-1,-1+k^2h^2]} + \|f\|_{\mathbb{L}_1[1-k^2h^2,1]}\right). \qquad (4\cdot1)
\end{aligned}$$

Now,

$$\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\,\mathfrak{I}_2 = \int_{-1+4k^2h^2/(4+k^2h^2)}^{1-4k^2h^2/(4+k^2h^2)} f(y)A_k(y,h)\,dy, \qquad (4\cdot2)$$

where

$$\begin{aligned}
A_k(y,h) &:= \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\,\mathfrak{J}((i-k/2)h, y) \\
&= \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\frac{1}{1+(i-k/2)^2h^2}\left(1+\frac{(i-k/2)hy}{\sqrt{1-y^2+(i-k/2)^2h^2}}\right). \qquad (4\cdot3)
\end{aligned}$$

Changing the order of summation (i.e., letting $j=k-i$) we get

$$A_k(y,h) = \sum_{j=0}^{k}\binom{k}{j}(-1)^{j}\frac{1}{1+(j-k/2)^2h^2}\left(1-\frac{(j-k/2)hy}{\sqrt{1-y^2+(j-k/2)^2h^2}}\right). \qquad (4\cdot4)$$

Therefore, adding (4·3) and (4·4) we have

$$2A_k(y,h) = \sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^i}{1+(i-k/2)^2h^2}\left((-1)^k+1+\frac{(i-k/2)hy\left((-1)^k-1\right)}{\sqrt{1-y^2+(i-k/2)^2h^2}}\right).$$

In particular, if k is even, then

$$A_k(y,h) = \sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^i}{1+(i-k/2)^2h^2}$$

and, if k is odd, then

$$A_k(y,h) = \sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{i+1}}{1+(i-k/2)^2h^2}\cdot\frac{(i-k/2)hy}{\sqrt{1-y^2+(i-k/2)^2h^2}}.$$

We now consider the cases for even and odd k separately.

Case I: even k. It is well known that if $g^{(m)}$ is continuous on $[a,b]$ and if $x_0, x_1, \ldots, x_m$ are any $m+1$ distinct points in $[a,b]$, then for some $\xi \in (a,b)$, $[x_0, \ldots, x_m; f] = g^{(m)}(\xi)/m!$. Since

$$\Delta_h^m(f,x) = m!\,h^m\,[x-mh/2, x-mh/2+h, \ldots, x+mh/2;\ f],$$

we conclude that, if $g^{(m)}$ is continuous on $[x-mh/2, x+mh/2]$, then for some $\xi \in (x-mh/2, x+mh/2)$,

$$\Delta_h^m(g,x) = h^m g^{(m)}(\xi). \qquad (4\cdot5)$$

We now note that it follows from the definition of the kth symmetric difference that, if $g(t) := (1+t^2)^{-1}$, then

$$A_k(y,h) = (-1)^k \Delta_h^k(g,0)$$

and (4·5) implies that for some $\xi \in (-kh/2, kh/2)$

$$|A_k(y,h)| = h^k |g^{(k)}(\xi)|.$$

Since $g \in \mathbb{C}^\infty[-1,1]$, we conclude that $|g^{(k)}(\xi)| \leqslant c(k)$ and so $|A_k(y,h)| \leqslant c(k)h^k$. It now follows from (4·2) that

$$\left|\sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i}\, \mathfrak{I}_2\right| \leqslant c(k)h^k \int_{-1+4k^2h^2/(4+k^2h^2)}^{1-4k^2h^2/(4+k^2h^2)} |f(y)|\, dy \leqslant c(k)h^k \,\|f\|_1.$$

Hence, recalling (4·1),

$$\left\|\Delta_{h\varphi(x)}^k(f,x)\right\|_1 \leqslant c(k)\left(\|f\|_{\mathbb{L}_1[-1,-1+k^2h^2]} + \|f\|_{\mathbb{L}_1[1-k^2h^2,1]} + h^k\,\|f\|_1\right)$$

and so for even k we have

$$\omega_k^{\varphi}(f,\delta)_1 \leqslant c(k)\left(\|f\|_{\mathbb{L}_1[-1,-1+k^2\delta^2]} + \|f\|_{\mathbb{L}_1[1-k^2\delta^2,1]} + \delta^k\,\|f\|_1\right).$$

Case II: odd k. Let $y \in [-1+4k^2h^2/(4+k^2h^2), 1-4k^2h^2/(4+k^2h^2)]$ be fixed and denote $\gamma := \sqrt{1-y^2}$ and

$$\tilde{g}(t) := \frac{t}{(1+t^2)\sqrt{\gamma^2+t^2}}.$$

Then,

$$A_k(y,h) = (-1)^{k-1}\, y\, \Delta_h^k(\tilde{g},0)$$

and so by (4·5)

$$|A_k(y,h)| = |y|\, h^k\, |\tilde{g}^{(k)}(\xi)|, \quad \xi \in (-kh/2, kh/2).$$

To estimate the kth derivative of $\tilde{g}(t)$ for $t \in [-kh/2, kh/2]$ we note that $\gamma \geqslant kh/2$ and so $|t| \leqslant \gamma$. Now, notice that $\tilde{g}(t) = G(t/\gamma)$ where

$$G(x) := \frac{1}{1+\gamma^2x^2} \cdot \frac{x}{\sqrt{1+x^2}}.$$

It is not difficult to show that $\|G^{(k)}\|_{\mathbb{C}[-1,1]} \leqslant c(k)$, and so

$$\left|\tilde{g}^{(k)}(t)\right| = \left|\gamma^{-k}G^{(k)}(t/\gamma)\right| \leqslant c(k)\gamma^{-k}, \quad \text{for all} \quad |t| \leqslant \gamma. \tag{4·6}$$

Hence,

$$|A_k(y,h)| \leqslant c(k)h^k(1-y^2)^{-k/2},$$

and it follows from (4·2) that

$$\begin{aligned}\left|\sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i}\, \mathfrak{I}_2\right| &\leqslant c(k)h^k \int_{-1+4k^2h^2/(4+k^2h^2)}^{1-4k^2h^2/(4+k^2h^2)} |f(y)|(1-y^2)^{-k/2}\, dy \\ &\leqslant c(k)h^k \left\|f(y)(1-y^2)^{-k/2}\right\|_{\mathbb{L}_1[-1+k^2h^2/2,1-k^2h^2/2]}.\end{aligned}$$

Therefore, recalling (4·1) we have

$$\left\| \Delta^k_{h\varphi(x)}(f,x) \right\|_1 \leqslant c(k)\Big(\|f\|_{\mathbb{L}_1[-1,-1+k^2h^2]} + \|f\|_{\mathbb{L}_1[1-k^2h^2,1]} + h^k \left\| f(y)(1-y^2)^{-k/2} \right\|_{\mathbb{L}_1[-1+k^2h^2/2,1-k^2h^2/2]} \Big)$$

and finally

$$\omega^\varphi_k(f,\delta)_1 \leqslant c(k)\Big(\|f\|_{\mathbb{L}_1[-1,-1+k^2\delta^2]} + \|f\|_{\mathbb{L}_1[1-k^2\delta^2,1]} + \sup_{0<h\leqslant\delta} h^k \left\| f(y)(1-y^2)^{-k/2} \right\|_{\mathbb{L}_1[-1+k^2h^2/2,1-k^2h^2/2]} \Big).$$

The proof of Theorem 3·2 is now complete.

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