UNIFORM ESTIMATES OF THE COCONVEX APPROXIMATION OF FUNCTIONS BY POLYNOMIALS

K. A. Kopotun

1. In the paper we consider the question on the coconvex approximation by polynomials of functions with deteriorating smoothness at the endpoints of a segment. We denote by $\mathring{W}^r$ the class of continuous functions f on [−1, 1] that have the absolutely continuous (r − 1)-th derivative locally in (−1, 1), and

$$|f^{(r)}(x)(1-x^2)^{r/2}|\leqslant 1 \tag{0.1}$$

for almost all $x \in [-1, 1]$.

For $r \geq 3$ the following theorem will be proved:

THEOREM 1. Let $r \in \mathbf{N}$, $r \neq 4$, and $I: = [-1, 1]$. If a function $f = f(x)$ is convex on I, and $f \in \mathring{W}^r$, then for any natural number $n \geq r - 1$ there exists an algebraic polynomial $P_n = P_n(x)$ of degree $\leq n$ that is convex on I, and such that,

$$|f(x)-P_n(x)|\leqslant Cn^{-r},\ C=C(r)=\text{const},\ x\in I. \tag{0.2}$$

The corresponding theorem for the approximation without restrictions was proved by Ditzian and Totik [1, pp. 40-41, 79-83] (see also Dzyadyk [2, Chap. IX]). A similar theorem for the comonotone approximation in the case r = 1, 2 follows from the paper by Leviatan [3], and in the case $r \geq 3$ it was proved by Dzyubenko, Listopad, and Shevchuk [4] by using the method from [5]. A modification of the method is used in the present paper too. Theorem 1 for r = 1, 2 also is a consequence of the paper by Leviatan [3]. It follows from Theorem 2 that Theorem 1 does not hold for all r, contrary to the corresponding theorems for the approximation without restrictions, and the comonotone approximation. Namely, the theorem is not true for r = 4.

THEOREM 2. $\forall n\in\mathbf{N}\ \forall C\in\mathbf{R},\ \exists f\in\mathring{W}^4,\ f''(x)\geqslant 0,\ x\in I:\ \forall P_n,\ P_n''(x)\geqslant 0,\ x\in I\ \exists x_0\in I:\ |f(x_0)-P_n(x_0)|>C.$

We use the notation from [5]:

Let L(x, g, [a, b]) be the Lagrange polynomial of degree $\leq r - 3$ that interpolates the function g at the points $a + i(b - a)/(r - 3)$, $i = \overline{0, r - 3}$, $r > 3$;

$$\Delta_n(y):=n^{-2}+\sqrt{1-y^2}n^{-1},\ y\in I;\ \Delta:=\Delta_n(x),\ x\in I;$$

$$x_j:=\cos(j\pi/n),\ j=\overline{0,n};$$
$$\bar{x}_j:=\cos(j\pi/n-\pi/2n),\ j=\overline{1,n};$$
$$x_j^0:=\cos(j\pi/n-\pi/4n),\ j<n/2;$$
$$x_j^0:=\cos(j\pi/n-3\pi/4n),\ j\geqslant n/2;$$
$$I_j:=[x_j,x_{j-1}],\ h_j:=x_{j-1}-x_j,\ j=\overline{1,n};$$
$$t_{j,n}:=(x-x_j^0)^{-2}\cos^2 2n\arccos x+(x-\bar{x}_j)^{-2}\sin^2 2n\arccos x$$

is an algebraic polynomial of degree $\leq 4n - 2$;

$$T_j(x):=\int_{-1}^{x}t_{j,n}^{3r}(y)dy\left(\int_{-1}^{1}t_{j,n}^{3r}(y)dy\right)^{-1},$$

$$\mathcal{T}_j(x):=\int_{-1}^{x}(y-x_j)(x_{j-1}-y)t_{j,n}^{3r+1}(y)dy$$
$$\cdot\left(\int_{-1}^{1}(y-x_j)(x_{j-1}-y)t_{j,n}^{3r+1}(y)dy\right)^{-1}$$

T. G. Shevchenko Kiev State University. Translated from Matematicheskie Zametki, Vol. 51, No. 3, pp. 35-46, March, 1992. Original article submitted June 24, 1991.

are polynomials of degree $\le 6r(2n-1)+1$ and $\le 2(3r+1)(2n+1)$, respectively;

$$J_{n,r}(t)=(\sin nt/2/\sin t/2)^{28r}\left(\int_{-\pi}^{\pi}(\sin nt/2/\sin t/2)^{28r}\,dt\right)^{-1}$$

is a kernel of the Jackson type;

$$\mathcal{D}_n(y,x)=\frac{1}{(28r-1)!}\frac{\partial^{28r}}{\partial x^{28r}}(x-y)^{28r-1}\int_{\arccos x-\alpha}^{\arccos x+\alpha}J_{n,r}(t)\,dt$$

is a polynomial kernel of the Dzyadyk type, in which $\alpha = \arccos y$, $x, y \in I$, and C, C_i are positive numbers that depend on r only.

Also we use the following inequalities

$$\Delta_n^2(y)\leqslant 4\Delta(|x-y|+\Delta),\ x\in I,\ y\in I;$$
$$2(|x-y|+\Delta)>|x-y|+\Delta_n(y)>(|x-y|+\Delta)/2;$$
$$h_{j\pm1}<3h_j;\ \Delta<h_j<5\Delta \text{ for } x\in I_j.$$

In Proposition 2 and Lemmas 3-7 of the next section, we assume that $r \ge 5$.

2. Some Lemmas and Definitions. Similarly to the proof of Lemma 6 from [6, p. 17-19], it is easy to check the following inequalities

$$1-x_{j-1}<\int_{-1}^{1}\tilde T_j(x)\,dx<1-x_j;$$
$$1-x_{j-1}<\int_{-1}^{1}T_j(x)\,dx<1-x_j,\quad j=\overline{1,n}.$$

From this it follows that there exist numbers $\alpha = \alpha(j) \in (0, 1)$ and $\beta = \beta(j) \in (0, 1)$, such that, for the polynomials

$$\tilde\sigma_j(x):=\int_{-1}^{x}(\alpha\tilde T_j(y)+(1-\alpha)\tilde T_{j+1}(y))\,dy,$$
$$\sigma_j(x):=\int_{-1}^{x}(\beta T_j(y)+(1-\beta)T_{j+1}(y))\,dy,\ j=\overline{1,n-1},$$

we have the equalities

$$\tilde\sigma_j(1)=\sigma_j(1)=1-x_j \tag{1.1}$$

(a similar consideration was applied in the proof of the theorem from [7]).

We denote $\tau_j := h_j(|x-x_j|+h_j)^{-1}$, for short. We put $\chi_j(x) := 0$ if $x \le x_j$, $\chi_j(x) := 1$ if $x > x_j$, and write $(x-x_j)_+ := \int_{-1}^{x}\chi_j(t)\,dt$.

Proposition 1. The following estimates hold:

$$0<-\tilde\sigma_j''(x)\leqslant C_1h_j^{-1}\tau_j^{6r},\ x\in I_j\cup I_{j+1}, \tag{1.2}$$
$$|\tilde\sigma_j''(x)|\leqslant C_1h_j^{-1}\tau_j^{6r},\ x\in I, \tag{1.3}$$
$$|(x-x_j)_+-\tilde\sigma_j(x)|\leqslant C_1h_j\tau_j^{6r-2},\ x\in I, \tag{1.4}$$
$$C_2h_j^{-1}\tau_j^{6r}\leqslant\sigma_j''(x)\leqslant C_1h_j^{-1}\tau_j^{6r},\ x\in I, \tag{1.5}$$
$$|(x-x_j)_+-\sigma_j(x)|\leqslant C_1h_j\tau_j^{6r-2},\ x\in I. \tag{1.6}$$

The proof of Proposition 1 is similar to the proof of Lemma 6 from [6], where we take into account the equalities (1.1) and the inequalities $h_{j+1}^{-1}\tau_{j+1}^{6r} < 3^{12r-1}h_j^{-1}\tau_j^{6r}$, $x \in I$.

LEMMA 1. Suppose that a set E consists of some segments I_{j_i}. The polynomial

$$\bar Q_n(x,E):=n^{-r}\sum_{i\in\{i\}}h_{j_i}^{-1}(\sigma_{j_i}(x)-\tilde\sigma_{j_i}(x))$$

of degree $\le 2(3r+1)(2n+1)$, where $\{i\} := \{i \mid I_{j_i} \in E,\ I_{j_i+1} \in E\}$, satisfies the inequalities

$$|\bar{Q}_n(x, E)| \leqslant C_3 n^{-r}, \quad x \in I,$$
$$\bar{Q}_n''(x, E) \geqslant -C_4 \Delta^{-2} n^{-r}, \quad x \in E,$$
$$\bar{Q}_n''(x, E) \geqslant C_5 \Delta^{-2} n^{-r} (\Delta / (\operatorname{dist}(x, \tilde{\tilde{E}}) + \Delta))^{12r-2}, \quad x \in I \setminus E,$$

where $\tilde{\tilde{E}} := E \setminus \{I_{j_i} | I_{j_i \pm 1} \bar{\in} E\}$.

Proof. The following estimate is a consequence of the inequalities (1.4) and (1.6):

$$|\bar{Q}_n(x, E)| \leqslant n^{-r} \sum_i 2C_1 \tau_{j_i}^{6r-2},$$

and from this

$$|\bar{Q}_n(x, E)| \leqslant 2C_1 n^{-r} \sum_{j=1}^{n} \tau_j^{6r-2} \leqslant$$
$$\leqslant C_1 n^{-r} 2^{12r} 5^{9r} \Delta^{3r-1,5} \int_{-1}^{1} (|x-t| + \Delta)^{-3r+0,5} dt \leqslant C_3 n^{-r}, \quad x \in I.$$

From the inequalities (1.2), (1.3), and (1.5), we get

$$\bar{Q}_n''(x, E) \geqslant -n^{-r} \sum_i \sigma_{j_i}''(x) h_{j_i}^{-1} \geqslant -n^{-r} h_{j^*}^{-1} (|\sigma_{j^*}''(x)| + 3|\sigma_{j^*-1}''(x)|),$$

where the index j^* is chosen in such a way that $x \in I_{j^*}$, i.e., $\bar{Q}_n''(x, E) \geq -C_4 \Delta^{-2} n^{-r}$ for $x \in E$. Finally, from (1.2) and (1.5), we get

$$\bar{Q}_n''(x, E) \geqslant C_2 n^{-r} \sum_i h_{j_i}^{-2} \tau_{j_i}^{6r} \geqslant C_2 n^{-r} h_{j^*}^{-2} \tau_{j^*}^{6} \geqslant$$
$$\geqslant C_5 \Delta^{-2} n^{-r} (\Delta / (\operatorname{dist}(x, \tilde{\tilde{E}}) + \Delta))^{12r-2}, \quad x \in I \setminus E,$$

where j^* is chosen in such a way that I_{j^*} is the interval from $\tilde{\tilde{E}}$ that is closest to x, i.e., $\operatorname{dist}(x, \tilde{\tilde{E}}) = \operatorname{dist}(x, I_{j^*})$. The lemma is proved.

LEMMA 2. Let $0 \leq g''(x) \leq n^{-r} \Delta^{-2}$, $x \in I$. Then the polynomial

$$R_n(x, g) := \sum_{j=1}^{n-1} [x_{j+1}, x_j, x_{j-1}; g] (x_{j-1} - x_{j+1}) \sigma_j(x) +$$
$$+ g(x_{n-1}) + [x_n, x_{n-1}; g](x - x_{n-1})$$

of degree $\leq 6r(2n - 1) + 2$ is convex on I, and moreover,

$$|g(x) - R_n(x, g)| \leqslant C_6 n^{-r}, \quad x \in I. \tag{1.7}$$

Proof. Since the function g is convex, therefore, $[x_{j+1}, x_j, x_{j-1}; g] \geq 0$, and, by using (1.5), the polynomial $R_n(x, g)$ is convex. We shall prove the inequality (1.7). By the Lagrange formula, we have

$$|[x_{j+1}, x_j, x_{j-1}; g]| = \frac{1}{2} |g''(\theta)| \leqslant 113 n^{-r} h_j^{-2},$$
$$\theta \in [x_{j+1}, x_{j-1}];$$
$$|[x_i, x, x_{i-1}; g]| \leqslant 13 n^{-r} h_i^{-2}, \quad x \in I_i.$$

From this and using (1.6), for $x \in (x_i, x_{i-1}]$ we get

$$|g(x) - R_n(x, g)| = |[x_i, x, x_{i-1}; g](x - x_i)(x - x_{i-1}) +$$
$$+ \sum_{j=1}^{n-1} [x_{j+1}, x_j, x_{j-1}; g](x_{j-1} - x_{j+1})((x - x_j)_+ - \sigma_j(x))| \leqslant$$
$$\leqslant 13 n^{-r} + \sum_{j=1}^{n-1} 113 n^{-r} h_j^{-2} 4 h_j C_1 h_j \tau_j^{6r-2} \leqslant C_6 n^{-r}, \quad x \in I.$$

The lemma is proved.

For a function $g = g(x)$ that has the second derivative on $[-1, 1]$, we write

$$\mathcal{L}(x, q) := g(-1) + g'(-1)(x+1) + \int_{-1}^{x}\int_{-1}^{t} L(y, g'', I)\,dy\,dt,$$
$$L(x, y) := g(x) + g'(x)(y-x) + \int_{x}^{y}\int_{x}^{z} L(t, g'', [x, x+\Delta])\,dt\,dz.$$

Proposition 2. If $g \in \mathring{W}^r$, then the following inequalities hold:

1) $|g(x) - \mathcal{L}(x, g)| \le C_7$, $x \in I$;

2) $|g(y) - L(x, y)| \le C_8 n^{-r}(|x - y| + \Delta)^{2r}\Delta^{-2r}$, $[x, x + \Delta] \subset I$, $y \in I$.

Proof. Let $y_i := -1 + 2i/(r - 3)$, $i = \overline{0, r - 3}$. Then

$$|g(x) - \mathcal{L}(x, g)| \le$$
$$\times \int_{-1}^{x}\int_{-1}^{t} |(y-y_0)\ldots(y-y_{r-3})| \int_0^1\int_0^{t_1}\ldots\int_0^{t_{r-3}}(1+$$
$$+(y_0 + (y_1 - y_0)t_1 + \ldots + (y - y_{r-3})t_{r-2}))^{-r/2}\,dt_{r-2}\ldots dt_1\,dy\,dt +$$
$$+\int_{-1}^{x}\int_{-1}^{t} |(y-y_0)\ldots| \int_0^1\ldots\int_0^{t_{r-3}}(1-$$
$$-(y_0 + \ldots + (y - y_{r-3})t_{r-2}))^{-r/2}\,dt_{r-2}\ldots dt =: \mathcal{J}_1 + \mathcal{J}_2.$$

We prove, for instance, the boundedness of the first integral. Put $g_r(t) = (1 + t)^{r/2-2}$ if r is odd, and $g_r(t) = (1 + t)^{r/2-2}\ln(1 + t)$ if r is even, which gives $g_r^{(r-2)}(t) = C_9(1 + t)^{-r/2}$. We have

$$\mathcal{J}_1 \le C_9^{-1}\int_{-1}^{x}\int_{-1}^{t} |(y-y_0)\ldots(y-y_{r-3})| \int_0^1\ldots$$
$$\ldots\int_0^{t_{r-3}} g_r^{(r-2)}(y_0 + \ldots + (y - y_{r-3})t_{r-2})\,dt_{r-2}\ldots dt =$$
$$= C_9^{-1}\int_{-1}^{x}\int_{-1}^{t} |g_r(y) - L(y, g_r, I)|\,dy\,dt \le C_{10},$$

because the estimate $|g_r(y) - L(y, g_r, I)| \le C_9^*$, $y \in I$, follows from [2, pp. 159-161].

2) We fix $x \in I_i$ and set $\bar{y}_i := x + \Delta i/(r - 3)$, $i = \overline{0, r - 3}$. Then

$$|g(y) - L(x, y)| \le$$
$$\le \int_x^y\int_x^z |(t-\bar{y}_0)\ldots(t-\bar{y}_{r-3})| \int_0^1\int_0^{t_1}\ldots$$
$$\ldots\int_0^{t_{r-3}}(1-(\bar{y}_0 + (\bar{y}_1 - \bar{y}_0)t_1 + \ldots + (t - \bar{y}_{r-3})t_{r-2})^2)^{-r/2}\,dt_{r-2}\ldots$$
$$\ldots dt\,dz =: G(y).$$

We consider three cases.

a) If $-1 + n^{-2} \le x$, $x + \Delta \le 1 - n^{-2}$, $-1 + n^{-2} \le y \le 1 - n^{-2}$, then $2\sqrt{1 - x^2} > n\Delta$, $16\sqrt{1 - y^2} > n\Delta^2/(|x - y| + \Delta)$, and

$$G(y) \le C\left(\frac{|x-y|+\Delta}{n\Delta^2}\right)^r (|x-y|+\Delta)^{r-2}|x-y|^2 \le$$
$$\le C_8 n^{-r}\left(\frac{|x-y|+\Delta}{\Delta}\right)^{2r}$$

b) Let $xy > 0$. We consider the case $x \in [-1, 0)$ and $y \in [-1, 0)$ (the case $x \in [0, 1]$ can be proved in a similar way). We write $x^* = \min\{x, y\}$, $y^* = \max\{x, y\}$. Using a), we consider the case $x^* < -1 + n^{-2}$ only. Similarly to 1), we have the following estimate:

$$G(y) \le C\int_x^y\int_x^z |\bar{g}_r(t) - L(t, \bar{g}_r, [x, x+\Delta])|\,dt\,dz.$$

where $\bar{g}_r(t) = (1+t)^{r/2-2}$ if r is odd, and $\bar{g}_r(t) = (1+t)^{r/2-2}\ln\left(\frac{1+t}{1+y^*}\right)$. if r is even.

We notice that $|\bar{g}_r(t)| \le (1+y^*)^{r/2-2}$, $t \in [-1, y^*]$. Then

$$G(y) \leqslant C\int_x^y\int_x^z (1+y^*)^{r/2-2}\left(1+C\left(\frac{|x-y|+\Delta}{\Delta}\right)^{r-3}\right)dt\,dz \leqslant$$
$$\leqslant C(|x-y|+\Delta)^{r/2}\left(\frac{|x-y|+\Delta}{\Delta}\right)^{r-3} \leqslant C_8 n^{-r}\left(\frac{|x-y|+\Delta}{\Delta}\right)^{2r}$$

c) For the remaining cases we have $C \le n^{-r}(|x-y|+\Delta)^{2r}\Delta^{-2r}$, and the estimate $G(y) \le C$ can be proved similarly to 1). The proposition is proved.

LEMMA 3. Suppose that a function $\Phi \ni \mathring{W}^r$ and a set $F \supset I$ are given. If $\Phi''(x) = 0$ for $x \in F$, then the polynomial

$$D_n(x,\Phi) := \int_{-1}^1 (\Phi(y) - \mathscr{L}(y,\Phi))\mathscr{D}_n(y,x)dy + \mathscr{L}(x,\Phi)$$

of degree $\le 14r(n-1)$ approximates the function Φ and its derivatives, and

$$|\Phi^{(p)}(x) - D_n^{(p)}(x,\Phi)| \leqslant C_{11}\Delta^{-p}n^{-r}\left(\frac{\Delta}{\Delta + \operatorname{dist}(x, I\backslash F)}\right)^{12r-2}, \tag{1.8}$$
$$x \in I,\ p = 0 \vee 1 \vee 2.$$

Proof. Put $g(x) := \Phi(x) - \mathscr{L}(x, \Phi)$. It follows from Proposition 2 that $|g(x)| \le C_7$, $x \in I$. We assume that $x + \Delta \in I$, where x is fixed, for convenience. Similarly as in Lemma 3 from [6], we reduce the proof of (1.8) to an estimate of the integral

$$\mathcal{J} = \int_{-1}^1 (L(x,y) - g(y))\frac{\partial^p}{\partial x^p}\mathscr{D}_n(y,x)dy.$$

Using Proposition 2 and Proposition 1 from [5], we obtain

$$|\mathcal{J}| \leqslant \int_{-1}^1 |L(x,y) - g(y)|C_{12}\Delta^{14r-2-p}(|x-y|+\Delta)^{-14r+1}dy \leqslant$$
$$\leqslant C_8C_{12}n^{-r}\int_{-1}^1 \Delta^{12r-2-p}(|x-y|+\Delta)^{-12r+1}dy \leqslant C_{11}n^{-r}\Delta^{-p},$$

which means that the inequality (1.8) is proved in the case $x \bar{\in} F$.

In the case $[x, x+\Delta]$ the polynomial $L(x, y)$ coincides with $g(y)$ for $y \in F$. Therefore, if $x \in F$, then assuming $[x, x+\Delta] \subset F$, for convenience, we have

$$|\mathcal{J}| \leqslant \int_{I\backslash F} C_8C_{12}n^{-r}\Delta^{12r-2-p}(|x-y|+\Delta)^{-12r+1}dy \leqslant$$
$$\leqslant 2C_8C_{12}n^{-r}\Delta^{12r-2-p}\int_{\operatorname{dist}(x, I\backslash F)}^{+\infty}(t+\Delta)^{-12r+1}dt \leqslant C_{11}\frac{n^{-r}\Delta^{12r-2-p}}{(\Delta + \operatorname{dist})^{12r-2}}.$$

The lemma is proved.

LEMMA 4. Suppose that the following is given: a function $g \in \mathring{W}^r$, and a set $\mathscr{Y}_j$ that consists of $2r-5$ adjacent intervals I_j, i.e.,

$$\mathscr{Y}_j = I_j \cup I_{j+1} \cup \ldots \cup I_{j+2(r-3)}.$$

If for any $i = \overline{0, 2r-6}$ there exists a point $\tilde{x}_i \in I_{j+i}$ such that $|g''(\tilde{x}_i)| \le n^{-r}\Delta_n^{-2}(\tilde{x}_i)$, then $|g''(x)| \le C_{13}n^{-r}\Delta^{-2}$ for all $x \in \mathscr{Y}_j$.

Proof. Let $\ell(x, g'', \tilde{x}_{2p})$ be the Lagrange polynomial of degree $\le r-3$ that approximates g'' at $\tilde{x}_{2p}$, $p = \overline{0, r-3}$. We represent the derivative g'' in the following form

$$g''(x)=[g''(x)-L(x, g'', \mathcal{Y}_j)]-l(x, g''-L, \tilde{x}_{2p})+l(x, g'', \tilde{x}_{2p}).$$

We estimate the last term

$$|l(x, g'', \tilde{x}_{2p})|=\left|\sum_{p=0}^{r-3} g''(\tilde{x}_{2p})\prod_{\substack{0\leq i\leq r-3,\\ i\neq p}}\frac{x-\tilde{x}_{2i}}{\tilde{x}_{2p}-\tilde{x}_{2i}}\right|\leqslant$$
$$\leqslant 3^{2r}\sum_{p=0}^{r-3} n^{-r}\Delta_n^{-2}(\tilde{x}_{2p})\leqslant C_{14}n^{-r}\Delta^{-2}.$$

The relevant estimate for the second term $\ell(x, g'' - L, \tilde{x}_{2p})$ follows from the above and from the estimate $|g''(x) - L(x, g'', \mathcal{Y}_j)| \leq C_{15}n^{-r}\Delta^{-2}$, which follows from the proof of Proposition 2. The lemma is proved.

Let a function $f = f(x)$ be convex on I and $f \in \mathring{W}^r$.

Definition 1. An interval I_j we will call the interval of type I if for all $x \in I_j$, $f''(x) \leq C_{13}(C_4 + C_5)n^{-r}\Delta^{-2}$; the interval of type II if it is not the interval of type I and for all $x \in I_j$, $f''(x) \geq (C_4 + C_5)\cdot n^{-r}\Delta^{-2}$. The remaining intervals I_j we call the intervals of type III. We denote by E_1, E_2, and E_3 the sum of the intervals of type I, type II, and type III, respectively.

LEMMA 5. The number of adjacent intervals of the type III is $\leq(2r - 6)$, i.e., each set $\mathcal{Y}_j$, $j = \overline{0, n - 2r + 6}$, consists of at least one interval I_j that is not of the type III.

Lemma 5 follows from Lemma 4.

We represent the set $E_1 \cup E_3 \cup \{I_j \in E_2 | I_{j\pm1} \bar{\in} E_2\}$ as a finite union of disjoint intervals. We denote by G_1 the set of all intervals such that they contain at least $(4r - 8)$ intervals I_j.

Therefore, $G_1 = [x_{j_1}, x_{j_0}] \cup [x_{j_1}, x_{j_2}] \cap\ldots$, where $0 \leq j_\nu < j_{\nu+1} \leq n$. We write $\bar{j}_\nu := j_\nu + (1 + (-1)^\nu)/2$. If $|x_{j_\nu}| = 1$, then we put $S_\nu(x) := 1$; if $|x_{j_\nu}| \neq 1$, then we put

$$S_\nu(x):=\int_x^{x_{j_\nu}}(y-\bar{x}_{j_\nu})^{r-2}(x_{i_\nu}-y)^{r-2}dy\cdot$$
$$\cdot\left(\int_{\bar{x}_{j_\nu}}^{x_{j_\nu}}(y-\bar{x}_{j_\nu})^{r-2}(x_{i_\nu}-y)^{r-2}dy\right)^{-1}.$$

Definition 2. Define $g_1(x) := 0$ for $x \bar{\in} G_1$, $g_1(x) := f''(x)S_\nu(x)$ for $x \in [x_{\bar{j}_\nu}, x_{j_\nu}]$, and $g_1(x) := f''(x)$ in the remaining cases. Define $g_2(x) := f''(x) - g_1(x)$. We set

$$f_1(x):=f(-1)+f'(-1)(x+1)+\int_{-1}^x\int_{-1}^t g_1(y)dy\,dt;$$
$$f_2(x):=\int_{-1}^x\int_{-1}^t g_2(y)\,dy\,dt.$$

Obviously, $f(x) = f_1(x) + f_2(x)$.

LEMMA 6. The functions g_1 and g_2 are nonnegative, and the following inequalities hold

$$|g_1(x)|\leqslant C_{16}n^{-r}\Delta^{-2},\quad x\in I;$$
$$|g_2^{(r-2)}(x)(1-x^2)^{r/2}|\leqslant C_{17},\quad x\in I. \tag{1.9}$$

Proof. Obviously, the functions g_1 and g_2 are nonnegative. The first of the inequalities (1.9) follows from $|S_\nu(x)| \leq 1$ and the inequality $|f''(x)| \leq Cn^{-r}\Delta^{-2}$, $x \in G_1$, which is proved similarly to Lemma 4, keeping in mind Lemma 5.

Now we prove the inequality

$$|g_1^{(r-2)}(x)(1-x^2)^{r/2}|\leqslant C_{18},\quad x\in I. \tag{1.10}$$

Fix a point $x \in I$. If $g_1(x) = 0$ or $g_1(x) = f''(x)$, then the inequality (1.10) is obvious. Therefore, it is enough to prove it for $x \in [x_{\bar{j}_\nu}, x_{j_\nu}]$, $|x_{j_\nu}| \neq 1$. Since the interval

$[x_{j\nu}^-, x_{j\nu}]$ does not contain ± 1, therefore, we have the inequality $|f^{(r)}(x)| \le 2^r n^{-r}\Delta^{-r}$. From this, using the estimate $|f''(x)| \le Cn^{-r}\Delta^{-2}$, $x \in G_1$, and with the help of inequalities of the Kolmogorov type, we obtain $|f^{(j+2)}(x)| \le C_{19}n^{-r}\Delta^{-(j+2)}$, $j = \overline{0, r-2}$. Using the inequality $|S_\nu^{(k)}(x)| \le C_{20}\Delta^{-k}$, $k = \overline{0, r-2}$, we have

$$|g_1^{(r-2)}(x)| = \Big|\sum_{j=0}^{r-2}\binom{r-2}{j} f^{(j+r)}(x)\, S_\nu^{(r-2-j)}(x)\Big| \leqslant$$
$$\leqslant C_{19}C_{20}n^{-r}\Delta^{-r}\sum_{j=0}^{r-2}\binom{r-2}{j} \leqslant C_{18}(1-x^2)^{-r/2}.$$

The second of the inequalities (1.9) follows from (0.1) and (1.10) with $C_{17} = C_{18} + 1$. The lemma is proved.

We write $G_2 := \{x \mid \mathrm{dist}(x, \bar{E}_2) \le 3^{4r}\Delta\}$, where $\bar{E}_2 := \{I_j \mid I \in E_2,\ I_j \bar{\in} G_1\}$. It follows from Lemma 5 and Definition 2 that $g_2(x) = 0$ for $x \in I\backslash G_2$.

We notice that for $n_1 \ge n$ the following inequality holds

$$\Delta_{n_1}(x)\,(\mathrm{dist}(x, G_2)+\Delta_{n_1}(x))^{-1} \leqslant C_{21}\Delta(\mathrm{dist}(x, \bar{E}_2)+\Delta)^{-1}.$$

From Lemmas 3 and 6, and Definitions 1 and 2, we get

LEMMA 7. For any natural number $n_1 \ge n$, the polynomial $D_{n_1}(x, f_2)$ of degree $<14rn_1$ has the properties

$$|f_2(x) - D_{n_1}(x, f_2)| \leqslant C_{22}n^{-r}, \quad x \in I,$$
$$D''_{n_1}(x, f_2) \geqslant -C_{23}n_1^{-r}\Delta_{n_1}^{-2}(x)\left(\frac{\Delta}{\mathrm{dist}(x, \bar{E}_2)+\Delta}\right)^{12r-2}, \quad x \in I\backslash\bar{E}_2,$$
$$D''_{n_1}(x, f_2) \leqslant (C_4+C_5)n^{-r}\Delta^{-r} - C_{23}n_1^{-r}\Delta_{n_1}^{-2}(x), \quad x \in \bar{E}_2,$$

where $C_{22} = C_{11}C_{17}$, $C_{23} = C_{11}C_{17}C_{21}^{12r-2}$.

3. Proof of Theorem 1 in the Case $r \ge 5$. Let $n_1 \in \mathbb{N}$, $n_1 \ge n$. We write $P_{n_1}(x) := \bar{Q}_n(x, \bar{E}_2) + D_{n_1}(x, f_2) + R_n(x, f_1)$, and notice that the polynomial is of degree $<14rn_1$. The following estimates result from Lemmas 1, 2, 6, and 7:

$$|f(x) - P_{n_1}(x)| \leqslant (C_{22}+C_6C_{16}+C_3)n^{-r} = C_{24}n^{-r}, \quad x \in I,$$
$$P''_{n_1}(x) \geqslant C_5n^{-r}\Delta^{-2} - C_{23}\Delta_{n_1}^{-2}(x)\,n_1^{-r}, \quad x \in \bar{E}_2,$$
$$P''_{n_1}(x) \geqslant (3^{-50r^2}C_5n^{-r}\Delta^{-2} - C_{23}n_1^{-r}\Delta_{n_1}^{-2}(x))\left(\frac{\Delta}{\mathrm{dist}(x, \bar{E}_2)+\Delta}\right)^{12r-2},$$
$x \in I\backslash\bar{E}_2$.

It remains to choose n_1 such that the following inequality is satisfied:

$$C_5n^{-r}\Delta^{-2} > 3^{50r^2}C_{23}n_1^{-r}\Delta_{n_1}^{-2}(x).$$

For this, it is enough to take $n_1 = [[3^{51r^2}C_{23}C_5^{-1}]^{1/(r-4)} + 1]n$.

Thus, for $n > [(3^{51r^2}C_{23}C_5^{-1})^{1/(r-4)} + 1]r = C_{24}$ the theorem is proved. The case $r - 1 \le n \le C_{24}$ follows from the case $n = r - 1$, and it is enough to take $P_n(x) := \mathscr{L}(x, f) + (2C_9^*/C_9)x^2$.

4. Some Lemmas and the Proof of Theorem 1 for $r = 3$. The peculiarity of the case $r = 3$ lies in the fact that the second derivative of a function $f \in \hat{W}^3$, f'', does not exist, in general, at the endpoints of the interval I. Using this information, it is not difficult to prove the following analogs of Proposition 2 and Lemmas 2-7 from Sec. 2 in the case $r = 3$.

LEMMA 2^1. Let $g \in \hat{W}^3$, $g''(x) \ge 0$ for $x \in I$, $g''(x) \le n^{-3}\Delta^{-2}$ for $x \in I\backslash(I_1 \cup I_n)$. Then the polynomial $R_n(x, g)$ is convex on I, and moreover, $|g(x) - R_n(x, g)| \le \tilde{C}_6n^{-3}$.

We set

$$\tilde{\mathscr{L}}(x,g):=g(-1)+g'(-1)(x+1)+g''(0)(x+1)^2/2,$$
$$\tilde{L}(x,y):=g(x)+g'(x)(y-x)+g''(x+\Delta/2)(y-x)^2/2.$$

We notice that for $g \in \mathring{W}^3$, Proposition 2 is satisfied if the functions $\mathscr{L}(x, g)$ and $L(x, y)$ are substituted by $\tilde{\mathscr{L}}(x, g)$ and $\tilde{L}(x, y)$.

LEMMA 3[1]. Suppose that a function $\Phi \in \mathring{W}^3$ and a set $F \subset I$ are given. If $\Phi''(x) = 0$ for $x \in F$, then the polynomial

$$\tilde{D}_n(x,\Phi)=\int_{-1}^{1}(\Phi(y)-\tilde{\mathscr{L}}(y,\Phi))\tilde{\mathscr{D}}_n(y,x)dy+\tilde{\mathscr{L}}(x,\Phi)$$

approximates the function Φ and its derivative in such a way that

$$|\Phi(x)-\tilde{D}_n(x,\Phi)|\leqslant\tilde{C}_{11}n^{-3},\quad x\in I,$$
$$|\Phi''(x)-\tilde{D}_n''(x,\Phi)|\leqslant\tilde{C}_{11}n^{-3}\Delta^{-2}\left(\frac{\Delta}{\operatorname{dist}(x,I\backslash F)+\Delta}\right)^{34},$$
$$x\in I\backslash(I_1\cup I_n),$$
$$\tilde{D}_n''(x,\Phi)\geqslant-C_{25}n,\quad x\in I.$$

LEMMA 4[1]. Let a function $g \in \mathring{W}^3$ and an interval I_j be given ($I_j \cap (I_1 \cup I_n) = \varnothing$). If there exists a point $\tilde{x} \in I_j$ at which $|g''(\tilde{x})| \le n^{-3}\Delta_n^{-2}(\tilde{x})$, then $|g''(x)| \le \tilde{C}_{13}n^{-3}\Delta^{-2}$ for all $x \in I_j$.

Definitions 1 and 2 remain unchanged, only we replace C_i by $\tilde{C}_i$.

LEMMA 5[1]. The third type intervals can be I_1 and I_n only.

LEMMA 6[1]. The functions g_1 and g_2 are nonnegative, and the following estimates hold:

$$|g_1(x)|\leqslant\tilde{C}_{16}n^{-3}\Delta^{-2},\quad x\in I\backslash(I_1\cup I_n);$$
$$|g_2'(x)(1-x^2)^{3/2}|\leqslant\tilde{C}_{17},\quad x\in I.$$

LEMMA 7[1]. For each $n_1 \ge n$, the polynomial $\tilde{D}_{n_1}(x, f_2)$ has the properties

$$|f_2(x)-\tilde{D}_{n_1}(x,f_2)|\leqslant\tilde{C}_{22}n^{-3},\quad x\in I,$$
$$\tilde{D}_{n_1}''(x,f_2)\geqslant-C_{23}n_1^{-3}\Delta_{n_1}^{-2}(x)\left(\frac{\Delta}{\operatorname{dist}(x,\bar{E}_2)+\Delta}\right)^{34},$$
$$x\in I\backslash(\bar{E}_2\cup I_1\cup I_n),$$
$$\tilde{D}_{n_1}''(x,f_2)\geqslant(\tilde{C}_4+\tilde{C}_5)n^{-3}\Delta^{-2}-C_{23}n_1^{-3}\Delta_{n_1}^{-2}(x),\quad x\in\bar{E}_2\backslash(I_1\cup I_n),$$
$$\tilde{D}_{n_1}''(x,f_2)\geqslant-C_{25}n_1,\quad x\in I_1\cup I_n.$$

Similarly to Sec. 3, it is not difficult to show that there exists n_1 (for instance, $n_1 = [10^{225}\tilde{C}_{23}C_5^{-1}]n$), such that, the polynomial $\tilde{P}_{n_1}(x) := \bar{Q}_n(x, \bar{E}_2) + \tilde{D}_{n_1}(x, f_2) + R_n(x, f_1)$ has the properties

$$|f(x)-\tilde{P}_{n_1}(x)|\leqslant\tilde{C}_{24}n^{-3},\quad x\in I;$$
$$\tilde{P}_{n_1}''(x)\geqslant0,\quad x\in I\backslash(I_1\cup I_n);$$
$$\tilde{P}_{n_1}''(x)\geqslant-(\tilde{C}_4+C_{25})n_1,\quad x\in I_1\cup I_n.$$

LEMMA 8. For the algebraic polynomial

$$Q_n(x):=\int_{-1}^{x}\int_{-1}^{y}\left(\sin\frac{n}{2}\arccos t/\sin\frac{1}{2}\arccos t\right)^{10}n^{-10}\,dt\,dy$$

of degree $\le 5n$, the following inequalities hold

$$\bar{Q}_n''(x) \geqslant 0, \quad x\in I; \quad 0 \leqslant Q_n(x) \leqslant 2\cdot 10^4 n^{-4}, \quad x\in I;$$
$$Q_n''(x) > \left(\frac{2}{\pi}\right)^{10} > 2^{-10}, \quad x\in I_1.$$

Finally, we obtain that the polynomial

$$\tilde{\tilde{P}}_{n_1}(x) := \tilde{P}_{n_1}(x) + (\bar{C}_4 + C_{25})\, 2^{10}[10^{225}\bar{C}_{23}\bar{C}_5^{-1}]\, n(Q_n(x) + Q_n(-x))$$

is convex on I, and satisfies the inequality (0.2).

Thus, for $r = 3$ and $n > C_{26}$, Theorem 1 is proved. For the remaining n, the proof follows from the case $n = 2$, in which it is sufficient to take $\tilde{P}_2(x) := \tilde{\mathscr{L}}(x, f)$.

5. Proof of Theorem 2. We assume to the contrary that Theorem 2 is not true. Then

$$\exists n\in\mathbf{N}\ \exists C_0\in\mathbf{R},\ C_0>1:\ \forall f\in \mathring{W}^4,\ f''>0\ \exists P_n,\ P_n''\geqslant 0;$$
$$|f(x)-P_n(x)|<C_0,\quad x\in I.$$

It is well known that the inequality $|\Sigma_{i=0}{}^n a_i x^i| \le 1$, $x \in I$, implies the estimate

$$|a_i|<M_1,\quad M_1=M(n)=\text{const},\quad i=\overline{0,n}. \tag{5.1}$$

Take a function f_b such that its derivative $f_b''(x) = -bx + b - \ln b - \ln(1 - x)$, where $b = 2\exp 8nM_1C_0$. It is obvious that $f_b/4 \in \mathring{W}^4$ and $f_b''(x) > 0$, $x \in I$. Take a polynomial $P_n = P_n(x)$ such that $P_n''(x) = \Sigma_{i=0}{}^{n-2} a_i x^i$. Then

$$|f(x)-P_n(x)| = \Big|\sum_{k=2}^{n-2} a_k\left(\frac{x^{k+2}-(-1)^{k+2}}{(k+1)(k+2)} - (-1)^{k+1}(x+1)\right) +$$
$$+\left(\frac{x^3+1}{6}-x-1\right)(a_1+b)+\left(\frac{x^2-1}{2}+x+1\right)(a_0-b+\ln b)+$$
$$+M_2x+M_3+\int_{-1}^{x}\int_{-1}^{y}\ln(1-z)\,dz\,dy\Big|<C_0,\quad x\in I.$$

From (5.1) and the inequality $|\int_{-1}^{x}\int_{-1}^{y}\ln(1-z)dzdy|<7$, $x \in I$, we have

$$|a_i|<8M_1C_0,\ i=\overline{2,n-2};$$
$$|a_1+b|<8M_1C_0;\quad |a_0-b+\ln b|<8M_1C_0.$$

From this we get

$$P_n''(1)=\Sigma_{i=0}^{n-2}a_i<(n-1)M_1C_0\cdot 8-\ln b<0.$$

We have obtained a contradiction, which proves Theorem 2.

The method of proof of Theorem 2 allows us to generalize it to the classes Δ^q, $q > 2$, of functions $f \in C(I)$, such that, $\Delta_h{}^q(f, x) \ge 0$, $x \in I$, where $\Delta_h{}^q(f, x)$ is the q-th difference of the function f with the difference h. (We note that Δ^1 is the set of nondecreasing functions on I, and Δ^2 is the set of convex functions on I.) To do this it is sufficient to consider the function f_b, whose the q-th derivative is of the form

$$f_b^{(q)}(x)=-bx+b-\ln b-\ln(1-x),\quad x\in I.$$

and the inequalities $r - q \ge 2$ and $r/2 \ge r - q$ are satisfied, i.e., an analog of Theorem 1 for Δ^q, $q \ge 2$, does not hold when $r = \overline{q+2, 2q}$ (see Fig. 1). The domains I and II are still not investigated.

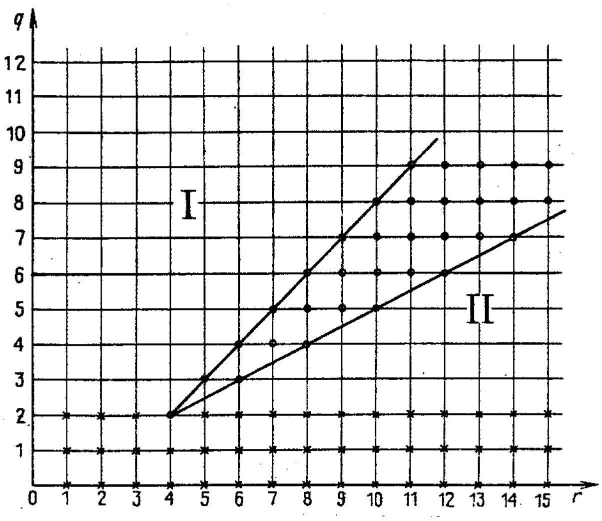

Fig. 1

The author expresses gratitude to I. A. Shevchuk for statement of the subject and valuable discussions.

LITERATURE CITED

1. Z. Ditzian and V. Totik, Moduli of Smoothness, Springer Series in Computational Mathematics, Vol. 9, Springer-Verlag, New York–Berlin (1987).
2. V. K. Dzyadyk, Introduction to the Theory of Uniform Approximation of Functions by Polynomials [in Russian], Nauka, Moscow (1977).
3. D. Leviatan, "Pointwise estimates for convex polynomial approximation," Proc. Am. Math. Soc., 98, No. 3 (1986).
4. G. A. Dzyubenko, V. V. Listopad, and I. A. Shevchuk, "Approximation of functions with deteriorating smoothness at the endpoints of an interval," in: Abstracts of Lectures of the Republic Scientific Conference. Extremal Problems of the Approximation Theory and Their Applications [in Russian], (May 29-31, 1990), Kiev (1990), p. 50.
5. I. A. Shevchuk, "On coapproximation of monotone functions," Dokl. Akad. Nauk SSSR, 308, No. 3, 537-541 (1989).
6. I. A. Shevchuk, "Comonotone approximation and polynomial Dzyadyk kernels," Preprint, Akad. Nauk UkrSSR, Inst. Mat., October 1989, Kiev (1989).
7. S. P. Maniya and I. A. Shevchuk, "The coconvex approximation of functions of the class W^r, $r > 2$," in: Abstracts of Lectures of the Republic Scientific Conference. Extremal Problems of the Approximation Theory and Their Applications [in Russian], (May 29-31, 1990), Kiev (1990), p. 87.