Full length article

Polynomial approximation with doubling weights having finitely many zeros and singularities

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Abstract

We prove matching direct and inverse theorems for (algebraic) polynomial approximation with doubling weights w having finitely many zeros and singularities (i.e., points where w becomes infinite) on an interval and not too "rapidly changing" away from these zeros and singularities. This class of doubling weights is rather wide and, in particular, includes the classical Jacobi weights, generalized Jacobi weights and generalized Ditzian–Totik weights. We approximate in the weighted $\mathbb{L}_p$ (quasi) norm $\|f\|_{p,w}$ with $0 < p < \infty$, where $\|f\|_{p,w} := \left(\int_{-1}^{1} |f(u)|^p w(u) du \right)^{1/p}$. Equivalence type results involving related realization functionals are also discussed.
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1. Introduction

The main goal of this paper is to prove matching direct and inverse theorems for polynomial approximation with doubling weights w having finitely many zeros and singularities (i.e., points

where w becomes infinite) on an interval and not too "rapidly changing". In order to discuss this further, we need to recall some notation and definitions. As usual, $\mathbb{L}_p(I)$, $0 < p < \infty$, is the set of measurable on I functions f equipped with the (quasi)norm $\|f\|_{\mathbb{L}_p(I)} := \left(\int_I |f(u)|^p du\right)^{1/p}$. We say that a function w is a doubling weight on $[-1, 1]$ if $w \in \mathbb{L}_1[-1, 1]$ is nonnegative, not identically equal to zero, and there exists a positive constant L (the so-called doubling constant of w) such that $w(2I) \le Lw(I)$, for any interval $I \subset [-1, 1]$. Here, $w(J) := \int_{J \cap [-1,1]} w(u)du$, and $2I$ denotes the interval of length $2|I|$ ($|I|$ is the length of I) with the same center as I. Doubling weights, their properties and various approximation results are discussed in a series of papers [20,21,18,19] by G. Mastroianni and V. Totik. In particular, it turns out that one can obtain many analogs of theorems for unweighted approximation by considering weights w_n which are certain averages of w depending on the degree of approximating polynomials. Recall (see e.g. [21]) that $w_n(x) := \rho_n(x)^{-1} \int_{x-\rho_n(x)}^{x+\rho_n(x)} w(u)du$, where $\rho_n(x) = n^{-1}(1 - x^2)^{1/2} + n^{-2}$. We refer the reader to [21,18,16] for further discussions of results involving w_n. At the same time, it is clear that averaging removes singularities (and "lifts" zeros) of weights, and so a natural question is whether or not one can obtain matching direct and inverse theorems for general doubling weights. This seems to be a very hard question since a general doubling weight can exhibit some rather "wild" behavior that makes it hard if not impossible to work with (while proving positive approximation results). For example, doubling weights can vanish on sets of positive measures as well as they can be "rapidly changing". Even relatively well-behaved weights (such as generalized Jacobi weights) can cause difficulties because of the presence of internal zeros/singularities. For example, see [21] for discussions of difficulties in forming weighted moduli of smoothness for generalized Jacobi weights, and [19,4] for examples showing that the original Jackson–Favard estimates are no longer valid for some specific doubling weights.

Still, if a doubling weight w has only finitely many zeros and singularities inside $[-1, 1]$ and is not too rapidly changing once one moves away from these points (i.e., if it behaves like w_n there), the matching direct and inverse results are possible (this is the main result of this paper). Earlier, this type of results was established in [21, Theorem 1.4] in the uniform norm weighted by generalized Jacobi weights with finitely many zeros in $[-1, 1]$, and in [4, Theorem 3.1] in the $\mathbb{L}_p$ norm (with $1 \le p \le \infty$) weighted by a specific weight having one zero at the origin and zeros or singularities at ± 1 (see also [2,5] for related results). However, we are not aware of any results of this type for $0 < p < 1$. Perhaps, the reason for this is that the usual method seems to be to first establish the equivalence of the moduli and some related K-functionals, and then proceed with the proofs. This method cannot work for $0 < p < 1$ since it is rather well known (see [9]) that K-functionals are often zeros if $0 < p < 1$.

Our approach is different and is actually somewhat similar to the one used in our earlier paper [16] where matching direct and (weak) inverse theorems were established for the weights w_n and all $0 < p \le \infty$. Namely, we derive the equivalence of the moduli and related "realization" functionals as a corollary of our estimates, and our proofs of direct/inverse theorems does not rely on this equivalence.

The class of doubling weights $\mathcal{W}(\mathcal{Z})$ that we introduce in Section 2 is rather wide and, in particular, includes the classical Jacobi weights, generalized Jacobi weights and generalized Ditzian–Totik weights. We approximate in the weighted $\mathbb{L}_p$ (quasi) norm with $0 < p < \infty$. For $p < \infty$, the weighted (quasi)norm is defined as $\|f\|_{\mathbb{L}_p(I),w} := \left(\int_I |f(u)|^p w(u)du\right)^{1/p}$ and $\|f\|_{p,w} := \|f\|_{\mathbb{L}_p[-1,1],w}$. We also denote by $\mathbb{L}_p^w$ the set of all functions on $[-1, 1]$ such that $\|f\|_{p,w} < \infty$.

Many of the results presented in this paper are also valid if $p = \infty$. However, one can only approximate essentially bounded functions by polynomials if the weights are essentially bounded. This puts a rather significant restriction on the weights, and the weights having the so-called property A^* (see [20,21], for example) are usually considered if $p = \infty$ instead of a wider class of doubling weights. This is the main reason why we only discuss the case $0 < p < \infty$ in this paper, and analogous results for $p = \infty$ and A^* weights will appear elsewhere.

The paper is organized as follows. In Section 2, we define a class of doubling weights $\mathcal{W}(\mathcal{Z})$ with finitely many zeros and singularities inside $[-1, 1]$ and give several equivalent conditions guaranteeing that w is in this class. Main part weighted moduli, (complete) weighted moduli as well as averaged moduli of smoothness are introduced in Section 3. A relation between the degrees of local approximation by piecewise polynomials and the main part moduli is established in Section 4. Lemma 4.2 in this section is the main result that allows us to estimate the degree of approximation away from zeros and singularities of the weight w. A Jackson type theorem with doubling weights from the class $\mathcal{W}(\mathcal{Z})$ is proved in Section 5. This is the main direct result in this paper. In Section 6, we discuss several Remez type and Markov–Bernstein type results that are needed in the proof of the inverse theorems. In Section 7, we prove two crucial lemmas (dealing with cases $1 \leq p < \infty$ and $0 < p < 1$ separately) on local approximation of polynomials of degree $< n$ by Taylor polynomials of degree $< r$, where r is related to the order of the moduli of smoothness, and n is the degree of approximating polynomials in Jackson-type estimates. The inverse results heavily depend on these lemmas. Some preliminary results needed in the proofs of inverse theorems are given in Section 8. The inverse theorems in cases $1 \leq p < \infty$ and $0 < p < 1$ are proved, respectively, in Sections 9 and 10. In Section 11, we obtain realization type results by proving the equivalence of the averaged and "regular" weighted moduli and appropriate realization functionals. An auxiliary result that is well known in the unweighted case about a polynomial of near best approximation (in weighted $\mathbb{L}_p$ with w from the class $\mathcal{W}(\mathcal{Z})$) being a near best approximant on a slightly larger interval is proved in Appendix A. This result is needed in the proof of the direct theorem and it could be used to provide an alternative proof of relations between different moduli with different parameters A (see Section 3). Finally, for readers' convenience, we repeat the main definitions and list the main notation (or provide references to appropriate statements/formulas) in Appendix B.

2. Doubling weights with finitely many zeros and singularities

Let w be a doubling weight on $[-1, 1]$ such that $w(z) = 0$ or $w(z) = \infty$ only at finitely many points z. Moreover, we assume that $w(x)$ "does not rapidly change" when x is "far" from these points z. These assumptions certainly limit the set of the weights that we consider since there are doubling weights that vanish on sets of positive measures and, at the same time, there are "rapidly changing" positive doubling weights. However, many important weights, such as generalized Jacobi weights or the so-called generalized Ditzian–Totik weights, satisfy this property (see Example 2.7 for their definitions).

We now make everything precise in the following definition noting that, throughout this paper, if $y < x$, then $[x, y] := [y, x]$ (and not $\emptyset$ as it is sometimes defined). We also denote $\varphi(x) := (1 - x^2)^{1/2}$, $\rho(h, x) := h\varphi(x) + h^2$ and note that $\rho_n(x) := \rho(1/n, x)$.

Definition 2.1. Let $M \in \mathbb{N}$ and $\mathcal{Z} := (z_j)_{j=1}^M$, $-1 \leq z_1 < \cdots < z_{M-1} < z_M \leq 1$. We say that a doubling weight w belongs to the class $\mathcal{W}(\mathcal{Z})$ (and write $w \in \mathcal{W}(\mathcal{Z})$) if, for any $\varepsilon > 0$ and $x, y \in [-1, 1]$ such that $|x - y| \leq \rho(\varepsilon, x)$ and $\mathrm{dist}\big([x, y], z_j\big) \geq \rho(\varepsilon, z_j)$ for all $1 \leq j \leq M$,

the following inequalities are satisfied

$$c_* w(y) \leq w(x) \leq c_*^{-1} w(y), \tag{2.1}$$

where the constant c_* depends only on w, and does not depend on x, y and ε.

Remark 2.2. The condition that $\mathrm{dist}\left([x, y], z_j\right) \geq \rho(\varepsilon, z_j)$, for all $1 \leq j \leq M$, guarantees that $[x, y] \cap \mathcal{Z} = \emptyset$.

Note that the set $\mathcal{Z}$ is where w can have zeros or singularities, but we do not actually require that it happens at all points in $\mathcal{Z}$. In other words, we do not exclude the possibility that w is "well behaved" at some/all points in $\mathcal{Z}$. We also note that the set $\mathcal{Z}$ is considered fixed throughout this paper, and so we refer to it in various theorems without redefining it (unless a statement/example is given for a specific $\mathcal{Z}$ in which case it will be explicitly stated). Also, note that the moduli of smoothness that we define below in Section 3 depend on $\mathcal{Z}$ and so, in particular, all constants in our estimates involving moduli will depend on M, but we are not explicitly stating this in every statement.

It is convenient to denote

$$\mathcal{Z}_{A,h}^j := \left\{ x \in [-1, 1] \,\big|\, |x - z_j| \leq A\rho(h, z_j) \right\}, \quad 1 \leq j \leq M,$$
$$\mathcal{Z}_{A,h} := \cup_{j=1}^M \mathcal{Z}_{A,h}^j,$$

and

$$\mathcal{I}_{A,h} := \left([-1, 1] \setminus \mathcal{Z}_{A,h}\right)^{cl} = \left\{ x \in [-1, 1] \,\big|\, |x - z_j| \geq A\rho(h, z_j), \text{ for all } 1 \leq j \leq M \right\}.$$

Also,

$$\mathcal{D} := \mathcal{D}(w) := \mathrm{pmin}\left\{ |z_j - z_{j-1}| \,\big|\, 1 \leq j \leq M + 1 \right\},$$

where $z_0 := -1$, $z_{M+1} := 1$ and $\mathrm{pmin}(S)$ is the smallest *positive* number from the set S of nonnegative reals.

Note that the condition $\mathrm{dist}\left([x, y], z_j\right) \geq \rho(\varepsilon, z_j)$, $1 \leq j \leq M$, in Definition 2.1 is equivalent to $[x, y] \subset \mathcal{I}_{1,\varepsilon}$.

Throughout this paper, $(x_i)_{i=0}^n$ is the Chebyshev partition of $[-1, 1]$, i.e., $x_i = \cos(i\pi/n)$, $0 \leq i \leq n$. For $1 \leq i \leq n$, we also denote $I_i := [x_i, x_{i-1}]$,

$$\psi_i := \psi_i(x) := \frac{|I_i|}{|x - x_i| + |I_i|} \quad \text{and} \quad \chi_i(x) := \chi_{[x_i, 1]}(x) = \begin{cases} 1, & \text{if } x_i \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

We need the following facts about the Chebyshev partition and the weights w_n (the facts without references are verified by straightforward, if cumbersome, computations).

- $\rho_n(x) \leq |I_i| \leq 5\rho_n(x)$ for all $x \in I_i$ and $1 \leq i \leq n$.
- $|I_i|/3 \leq |I_{i+1}| \leq 3|I_i|$ for all $1 \leq i \leq n - 1$.
- If $\alpha \geq 2$, then $\sum_{i=1}^n \psi_i(x)^\alpha \leq c$ for all $-1 \leq x \leq 1$, and $\int_{-1}^1 \psi_i(x)^\alpha dx \leq c|I_i|$ for all $1 \leq i \leq n$.
- For all $x, y \in [-1, 1]$, $\rho_n(y)^2 \leq 4\rho_n(x)(|x - y| + \rho_n(x))$.
- For any $c_0 > 0$ and $x \in [-1, 1]$, the interval $[x - c_0\rho_n(x), x + c_0\rho_n(x)]$ has nonempty intersection with at most m intervals I_i, $1 \leq i \leq n$, where m is some natural number that depends only on c_0. This follows from Proposition 4.1 whose proof we postpone until Section 4.
- For any doubling weight w, if $n \sim m$, then $w_n(x) \sim w_m(x)$, for all $x \in [-1, 1]$.

- For any doubling weight w and $n \in \mathbb{N}$, $w_n(x) \sim w_n(y)$ if $|x-y| \le c_* \rho_n(x)$, with equivalence constants depending only on c_* and the doubling constant of w (see [21, (2.3)]).
- For any doubling weight w, $n \in \mathbb{N}$, $1 \le i \le n$, $x \in [-1, 1]$ and $y \in I_i$, $w_n(x) \le c\psi_i(x)^{-s} w_n(y)$ and $w_n(y) \le c\psi_i(x)^{-s} w_n(x)$, where constants c and $s \ge 0$ depend only on the doubling constant of w (see [16, Lemma 2.5]).

We also mention that defining I_i's to be closed causes some ambiguity at the boundaries of these intervals since any two adjacent intervals in this partition have a nonempty intersection. Hence, when we make statements of type "let $x \in [-1, 1]$ and let μ be such that $x \in I_\mu$", this is ambiguous if $x = x_j$ for some $1 \le j \le n - 1$, since there are actually two intervals containing x (namely, I_j and I_{j+1}). To remedy this problem, we use the convention that, if x belongs to two adjacent (closed) intervals, we always choose the right interval as the one containing x.

We are now ready to discuss several conditions that are equivalent to the statement that a doubling weight is in the class $\mathcal{W}(\mathcal{Z})$.

Lemma 2.3. *Let w be a doubling weight. The following conditions are equivalent.*

(i) *$w \in \mathcal{W}(\mathcal{Z})$.*

(ii) *For any $n \in \mathbb{N}$ and x, y such that $[x, y] \subset \mathcal{I}_{1,1/n}$ and $|x - y| \le \rho_n(x)$, inequalities (2.1) are satisfied with the constant c_* depending only on w.*

(iii) *For some $N \in \mathbb{N}$ that depends only on w, and any $n \ge N$ and x, y such that $[x, y] \subset \mathcal{I}_{1,1/n}$ and $|x - y| \le \rho_n(x)$, inequalities (2.1) are satisfied with the constant c_* depending only on w.*

(iv) *For any $n \in \mathbb{N}$, $A, B > 0$, and x, y such that $[x, y] \subset \mathcal{I}_{A,1/n}$ and $|x - y| \le B\rho_n(x)$, inequalities (2.1) are satisfied with the constant c_* depending only on w, A and B.*

(v) *For any $n \in \mathbb{N}$ and $A > 0$,*

$$w(x) \sim w_n(x), \quad x \in \mathcal{I}_{A,1/n},$$

where the equivalence constants depend only on w and A, and are independent of x and n.

Proof. Clearly, (i) $\Rightarrow$ (ii) (one just needs to pick $\varepsilon = 1/n$), and (ii) $\Rightarrow$ (iii). Also, (iv) $\Rightarrow$ (i). Indeed, note that the statement of Definition 2.1 becomes vacuous if $\varepsilon > \sqrt{2}$ (since $\rho(\varepsilon, z_j) > 2$). Hence, assuming that $\varepsilon \le \sqrt{2}$ we pick $n = \lfloor 2/\varepsilon \rfloor \in \mathbb{N}$, $A = 1$ and $B = 4$. Then $1 < n\varepsilon \le 2$, $A\rho_n(z_j) \le \rho(\varepsilon, z_j)$ and $B\rho_n(x) \ge \rho(\varepsilon, x)$, and so if $[x, y] \subset \mathcal{I}_{1,\varepsilon}$ and $|x - y| \le \rho(\varepsilon, x)$, then $[x, y] \subset \mathcal{I}_{A,1/n}$ and $|x - y| \le B\rho_n(x)$.

Now, we will show that (iii) $\Rightarrow$ (iv). Let $n \in \mathbb{N}$ and $A, B > 0$ be given, and suppose that x, y are such that $[x, y] \subset \mathcal{I}_{A,1/n}$ and $|x - y| \le B\rho_n(x)$. Pick $m := \max\{\lceil n/\min\{A, 1\}\rceil, N\}$ and note that $A\rho_n(z_j) \ge \rho_m(z_j)$, and so $\mathcal{I}_{A,1/n} \subset \mathcal{I}_{1,1/m}$. Also, it is not difficult to check that $m/n \le \max\{N, 2/\min\{A, 1\}\} =: c^*$ and hence $\rho_n(x) \le (c^*)^2 \rho_m(x)$ which implies that $|x - y| \le B(c^*)^2 \rho_m(x) =: M\rho_m(x)$.

Hence, in order to complete the proof it is sufficient to show that, for any $x, y \in [-1, 1]$ such that $|x - y| \le M\rho_m(x)$ there are K points y_i, $1 \le i \le K$, with $K \in \mathbb{N}$ depending only on M, such that

$$[x, y] \subset \cup_{i=1}^{K-1}[y_i, y_{i+1}] \quad \text{and} \quad |y_i - y_{i+1}| \le \rho_m(y_i), \quad 1 \le i \le K - 1.$$

We will use Proposition 4.1. Let $(x_i)_{i=0}^m$ be the Chebyshev partition of $[-1, 1]$ into m intervals $I_i = [x_i, x_{i-1}]$. Suppose that $x \in I_\mu$, $1 \le \mu \le m$, denote

$$I^* := \{1 \le i \le m \mid I_i \cap [x, y] \ne \emptyset\} \quad \text{and} \quad I_* := \{1 \le i \le m \mid I_i \subset [x, y]\},$$

and let $J := \cup_{i \in I^*} I_i$.

If $I_* = \emptyset$, then $[x, y] \subset I_\mu \cup I_{\mu \pm 1}$, and J consists of at most 2 intervals I_i. If $I_* \neq \emptyset$, then recalling that $|I_{i \pm 1}| \leq 3|I_i|$ and $\rho_m(x) \leq |I_i| \leq 5\rho_m(x)$, for any $x \in I_i$, we conclude

$$|J| \leq 7 \left| \cup_{i \in I_*} I_i \right| \leq 7|x - y| \leq 7M\rho_m(x) \leq 7M|I_\nu|.$$

Proposition 4.1 implies that J consists of at most k intervals I_i, where k depends only on M. We now define $y_j^i := x_j + i|I_j|/5$, $0 \leq i \leq 5$, for all $j \in I^*$, and denote $Y := (y_i)_{i=1}^K :=$ $\cup_{j \in I^*} \{y_j^0, y_j^1, \ldots, y_j^5\}$, where $y_i < y_{i+1}$, $1 \leq i \leq K - 1$. Then, K is not bigger than $5k + 1$ and depends only on M, $[x, y] \subset J = \cup_{i=1}^{K-1} [y_i, y_{i+1}]$, and, for each $1 \leq i \leq K - 1$, $|y_i - y_{i+1}| \leq |I_{j(i)}|/5 \leq \rho_m(y_i)$.

So far, we have verified the equivalence of (i)–(iv).

We will show now that (iv) $\Rightarrow$ (v).

Let $A > 0$ and suppose that $n \in \mathbb{N}$ is such that $n > 4(A + 1)/\mathcal{D}$. This guarantees that

$$2\rho_n(x) < \mathcal{D} - A \left[\rho_n(z_j) + \rho_n(z_{j-1}) \right] \leq z_j - A\rho_n(z_j) - \left(z_{j-1} + A\rho_n(z_{j-1}) \right),$$
$$1 \leq j \leq M + 1, \ z_{j-1} \neq z_j,$$

and so $[x - \rho_n(x), x + \rho_n(x)]$ has a nonempty intersection with at most one interval from $\mathcal{Z}_{A,1/n}$. Moreover, if $[x - \rho_n(x), x + \rho_n(x)]$ does intersect an interval from $\mathcal{Z}_{A,1/n}^j$ with $z_j \neq \pm 1$, then it does not contain ± 1. Hence, if $x \in \mathcal{I}_{A,1/n}$, then either $[x, x + \rho_n(x)] \subset \mathcal{I}_{A,1/n}$ or $[x - \rho_n(x), x] \subset \mathcal{I}_{A,1/n}$, and without loss of generality, suppose that it is the former (in the latter case $[x - \rho_n(x), x] \subset \mathcal{I}_{A,1/n}$, the quantity $-\rho_n(x)$ replaces $\rho_n(x)$ and the order of integration is reversed). Then, taking into account that

$$w\left([x - \mu, x + \mu]\right) \leq w\left([x - \mu, x + 2\mu]\right)$$
$$\leq L^2 w\left([x + \mu/8, x + 7\mu/8]\right) \leq L^2 w\left([x, x + \mu]\right),$$

we have

$$w_n(x) = \frac{1}{\rho_n(x)} \int_{x - \rho_n(x)}^{x + \rho_n(x)} w(u)du \leq \frac{L^2}{\rho_n(x)} \int_x^{x + \rho_n(x)} w(u)du \leq L^2 c_*^{-1} w(x)$$

and

$$w_n(x) \geq \frac{1}{\rho_n(x)} \int_x^{x + \rho_n(x)} w(u)du \geq c_* w(x),$$

where c_* depends only on w and A.

Hence, (v) is proved for all $n \in \mathbb{N}$ such that $n > 4(A+1)/\mathcal{D}$. If $1 \leq n \leq N := \lceil 4(A+1)/\mathcal{D} \rceil$, then we use the fact that (v) is valid for $n = N + 1$, $\mathcal{I}_{A,1/n} \subset \mathcal{I}_{A,1/(N+1)}$ and that $w_n(x) \sim w_{N+1}(x)$ with equivalence constants depending only on w and N, to conclude that

$$w(x) \sim w_{N+1}(x) \sim w_n(x), \quad x \in \mathcal{I}_{A,1/n}.$$

To prove (v) $\Rightarrow$ (iv), we note that it follows from the doubling condition that, if $x, y \in [-1, 1]$ and $|x - y| \leq B\rho_n(x)$, then $w_n(x) \sim w_n(y)$ with equivalence constants depending only on B and the doubling constant of w. Hence, if (v) is valid and x, y are such that $[x, y] \subset \mathcal{I}_{A,1/n}$ and $|x - y| \leq B\rho_n(x)$, then

$$w(y) \sim w_n(y) \sim w_n(x) \sim w(x)$$

with equivalence constants depending on A, B and the weight w. This verifies (iv). □

Remark 2.4. We note that if a doubling weight w is in the class $\mathcal{W}(\mathcal{Z})$ then, in particular, it is bounded away from zero and ∞ when x is "far" from $\mathcal{Z}$. In other words,

$$\forall \varepsilon > 0 \; \exists \delta_\varepsilon > 0 : \delta_\varepsilon < w(x) < \delta_\varepsilon^{-1}, \quad \text{for all } x \text{ such that } \operatorname{dist}(x, \mathcal{Z}) \geq \varepsilon.$$

This follows from Lemma 2.3(iv) if we pick $n = \lceil 2/\varepsilon \rceil$, $A = 1$ and $B = 2n^2$.

We will now show that if a doubling weight w is monotone near points from $\mathcal{Z}$ and is bounded away from zero and infinity on the rest of the interval $[-1, 1]$ then it is in the class $\mathcal{W}(\mathcal{Z})$.

We use the usual notation $f_+(a) := \lim_{x \to a^+} f(x)$ and $f_-(a) := \lim_{x \to a^-} f(x)$.

Lemma 2.5. *Let w be a doubling weight, and suppose that there exists $0 < \alpha < \mathcal{D}/4$ such that w is monotone on $(z_j - \alpha, z_j) \cap [-1, 1]$ and on $(z_j, z_j + \alpha) \cap [-1, 1]$ for all $1 \leq j \leq M$, and suppose that $\mu_* > 0$ and $\mu^* < \infty$, where*

$$\mu_* := \min \left\{ \inf_{x \in S_\alpha} w(x), \; \min_{1 \leq j \leq M} \left\{ w_-(z_j + \alpha), w_+(z_j - \alpha) \right\} \right\}$$

and

$$\mu^* := \max \left\{ \sup_{x \in S_\alpha} w(x), \; \max_{1 \leq j \leq M} \left\{ w_-(z_j + \alpha), w_+(z_j - \alpha) \right\} \right\}$$

where $S_\alpha := \left\{ x \in [-1, 1] \mid \operatorname{dist}(x, \mathcal{Z}) \geq \alpha \right\}$.
Then w belongs to the class $\mathcal{W}(\mathcal{Z})$.

We use the convention that if a quantity is not defined then it is not present in the set whose minimum or maximum is taken. Thus, for example, if $z_1 = -1$, then $w_-(-1 - \alpha)$ is excluded from the definition of μ_* and μ^* in the statement of the lemma since this quantity is not defined.

Proof. For each $1 \leq i \leq M$, there exists $\varepsilon_i > 0$ such that

$$\mu_*/2 \leq w(x) \leq 2\mu^*,$$
$$\text{for all } x \in ([z_i + \alpha - \varepsilon_i, z_i + \alpha] \cup [z_i - \alpha, z_i - \alpha + \varepsilon_i]) \cap [-1, 1].$$

We let $\varepsilon := \min \left\{ \alpha/2, \min_{1 \leq j \leq M} \varepsilon_j \right\}$ and $N := \lceil 4/\varepsilon \rceil$. Note that N depends only on the weight w, and that the inequality $\rho_n(x) \leq \varepsilon/2$ is satisfied for all $x \in [-1, 1]$ and all $n \geq N$. Recalling that

$$S_{\alpha - \varepsilon} = \left\{ x \in [-1, 1] \mid \operatorname{dist}(x, \mathcal{Z}) \geq \alpha - \varepsilon \right\}$$

we also note that $\mu_*/2 \leq w(x) \leq 2\mu^*$, for all $x \in S_{\alpha - \varepsilon}$.

Now, let $n \geq N$ and let x, y be such that $[x, y] \subset \mathcal{I}_{1,1/n}$ and $|x - y| \leq \rho_n(x)$. We will show that Lemma 2.3(iii) is valid which implies that w is in the class $\mathcal{W}(\mathcal{Z})$. We have the following cases to consider (for convenience, suppose that $x < y$):

(a) $[x, y] \subset S_{\alpha - \varepsilon}$,
(b) $[x, y] \cap ([-1, 1] \setminus S_{\alpha - \varepsilon}) \neq \emptyset$.

Case (a): $\mu_*/2 \leq w(x), w(y) \leq 2\mu^*$, and so (2.1) is satisfied with $c_* = \mu_*/(4\mu^*)$.
Case (b): Let $I_x := [x - \rho_n(x)/6, x]$ and $I_y := [y, y + \rho_n(x)/6]$ and note that $I_{x,y} := [x - \rho_n(x)/6, y + \rho_n(x)/6]$ is such that $I_{x,y} \cap \{z_i \pm \alpha\}_{i=1}^M = \emptyset$ since

$$\operatorname{dist}(I_{x,y}, \{z_i \pm \alpha\}_{i=1}^M) \geq \operatorname{dist}([x, y], \{z_i \pm \alpha\}_{i=1}^M) - \rho_n(x)/6$$
$$\geq \varepsilon/2 - \rho_n(x)/6 \geq \varepsilon/2 - \varepsilon/12 > 0.$$

Additionally, $I_{x,y} \cap \mathcal{Z} = \emptyset$. Indeed, recalling that $\rho_n(v)^2 \leq 4\rho_n(u)(|v-u| + \rho_n(u))$, for all $u, v \in [-1, 1]$, letting $u = z_j$, $1 \leq j \leq M$, and $v = x$ and noting that $|x - z_j| \geq \rho_n(z_j)$ because $x \in \mathcal{I}_{1,1/n}$, we have

$$\rho_n(x)^2 \leq 4\rho_n(z_j)(|x - z_j| + \rho_n(z_j)) \leq 8|x - z_j|^2,$$

and so $|x - z_j| > \rho_n(x)/3$ for all $1 \leq j \leq M$.

Also, taking into account that $y \in \mathcal{I}_{1,1/n}$ which implies $|y - z_j| \geq \rho_n(z_j)$ we have

$$\begin{aligned}
\rho_n(x)^2 &\leq 4|y - z_j|(|x - z_j| + |y - z_j|) \leq 4|y - z_j|(|x - y| + 2|y - z_j|) \\
&\leq 4|y - z_j|(\rho_n(x) + 2|y - z_j|),
\end{aligned}$$

which implies that $\rho_n(x) < 6|y - z_j|$ for all $1 \leq j \leq M$.

Therefore, $I_{x,y} \cap \{z_i, z_i \pm \alpha\}_{i=1}^M = \emptyset$, and so w is monotone on $I_{x,y}$.

It follows from the properties of doubling weights (see [20, Lemma 2.1], for example) that $c_0 w(I_x) \leq w(I_y) \leq c_0^{-1} w(I_x)$ (since $|I_x| = |I_y| \sim |I_{x,y}|$) with the constant c_0 depending only on w.

Now, if w is nondecreasing on $I_{x,y}$, then

$$w(x) \leq w(y) \leq 6w(I_y)/\rho_n(x) \leq 6c_0^{-1}w(I_x)/\rho_n(x) \leq c_0^{-1}w(x),$$

and if w is nonincreasing on $I_{x,y}$, then

$$w(y) \leq w(x) \leq 6w(I_x)/\rho_n(x) \leq 6c_0^{-1}w(I_y)/\rho_n(x) \leq c_0^{-1}w(y).$$

This verifies Lemma 2.3(iii), and the proof is now complete. $\quad\square$

Corollary 2.6. *Let w be a doubling weight, and suppose that w is piecewise monotone with finitely many monotonicity intervals, i.e., let $T := (t_i)_{i=0}^K$, $K \in \mathbb{N}$, be such that $-1 = t_0 < t_1 < \cdots < t_{K-1} < t_K = 1$ and w is monotone on each interval (t_i, t_{i+1}), $0 \leq i \leq K - 1$. Moreover, assume that $\mu^* < \infty$ and $\mu_* > 0$, where*

$$\mu^* := \max\{w(t_i), w_\pm(t_i) \mid 0 \leq i \leq K, \ t_i \notin \mathcal{Z}\} \quad and$$
$$\mu_* := \min\{w(t_i), w_\pm(t_i) \mid 0 \leq i \leq K, \ t_i \notin \mathcal{Z}\}$$

(with the convention that $\max\{\emptyset\} = \min\{\emptyset\} := 1$, $w_-(-1) := w(-1)$ and $w_+(1) := w(1)$). Then w belongs to the class $\mathcal{W}(\mathcal{Z})$.

Taking into account characterization of (piecewise) monotone doubling weights (see e.g. [3]) and Lemma 2.5, it is now relatively easy to check that many well known weights are not only doubling but are also in $\mathcal{W}(\mathcal{Z})$ for some $\mathcal{Z}$.

Example 2.7. The following are examples of doubling weights from $\mathcal{W}(\mathcal{Z})$ with $\mathcal{Z} = (z_i)_{i=1}^M$, $-1 \leq z_1 < \cdots < z_{M-1} < z_M \leq 1$.

- Classical Jacobi weights:

$$w(x) = (1+x)^\alpha (1-x)^\beta, \quad \alpha, \beta > -1, \ \text{with } M = 2, z_1 = -1 \text{ and } z_2 = 1.$$

- Generalized Jacobi weights:

$$w(x) = \prod_{j=1}^M |x - z_j|^{\gamma_j}, \quad \gamma_j > -1.$$

- Generalized DT weights (see e.g. in [2, p. 134]):

$$w(x) = \prod_{j=1}^{M} |x - z_j|^{\gamma_j} \left(\ln \frac{e}{|x - z_j|} \right)^{\Gamma_j}, \quad \gamma_j > -1, \ \Gamma_j \in \mathbb{R}.$$

(Note that if these weights are defined with $\gamma_j = -1$, $\Gamma_j < -1$, for some j's, then they will be in $\mathbb{L}_1$ but will not be doubling. For example, $w(x) = |x|^{-1}(1 - \ln|x|)^{-2}$ is not doubling since, for example, for sufficiently small $t > 0$, $w([0, t]) \sim (1 - \ln t)^{-1}$ and $w([t, 2t]) \sim (1 - \ln t)^{-1}(1 - \ln(2t))^{-1}$ and so $w([0, t])/w([t, 2t]) \to \infty$ as $t \to 0^+$, which cannot happen for doubling weights.)

Remark 2.8. Of course, there are doubling weights which are not in any $\mathcal{W}(\mathcal{Z})$ classes. Doubling weights that vanish on sets of positive measures (see [24, Chapter I, Section 8.8] for an example) are an illustration of this. Also, there are doubling weights which are not A_∞ weights (see e.g. [20, Section 5] for the definition) and which do not vanish anywhere (see [14], [24, Chapter I, Section 8.8] or [20, Section 2]), and one can use the same construction for any $\mathcal{Z}$ to build a doubling weight w which will not be in $\mathcal{W}(\mathcal{Z})$.

Remark 2.9. It is clear that if w is a doubling weight from $\mathcal{W}(\mathcal{Z})$ and w_0 is such that $0 < c_1 \leq w_0(x) \leq c_2 < \infty$, $x \in [-1, 1]$ (in particular, if w_0 is a strictly positive continuous function on $[-1, 1]$), then $w_0 w$ is a doubling weight from $\mathcal{W}(\mathcal{Z})$. This immediately implies that the so-called generalized smooth Jacobi weights (see e.g. [1,22]) are doubling weights from $\mathcal{W}(\mathcal{Z})$.

Remark 2.10. Recall (see [24, Chapter V]) that, for $1 \leq p < \infty$, a weight w is said to be in A_p if

$$\left(\frac{1}{|I|} \int_I w(x)dx \right) \cdot \left(\frac{1}{|I|} \int_I w(x)^{-p'/p}dx \right)^{p/p'} \leq A < \infty, \quad 1/p + 1/p' = 1,$$

for all intervals $I \subset [-1, 1]$. It was shown in [24, p. 196] that all A_p weights are doubling, and we remark that there are doubling weights from $\mathcal{W}(\mathcal{Z})$ that do not belong to A_p (with a fixed p). For example, $w(x) = |x|^\gamma$ belongs to A_p, $p > 1$, if and only if $-1 < \gamma < p - 1$, and it is a doubling weight from $\mathcal{W}(\mathcal{Z})$, for any $\gamma > -1$ and any $\mathcal{Z}$ containing 0.

3. Moduli of smoothness

As usual, for $r \in \mathbb{N}$, let

$$\Delta_h^r(f, x, S) := \begin{cases} \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} f(x - rh/2 + ih), & \text{if } [x - rh/2, x + rh/2] \subset S, \\ 0, & \text{otherwise,} \end{cases}$$

be the rth symmetric difference. Note that S can be a union of (disjoint) intervals. Also, let $\Delta_h^r(f, x) := \Delta_h^r(f, x, [-1, 1])$.

Main part weighted modulus of smoothness is defined as

$$\Omega_\varphi^r(f, A, t)_{p,w} := \sup_{0 < h \leq t} \left\| \Delta_{h\varphi(\cdot)}^r(f, \cdot, \mathcal{I}_{A,h}) \right\|_{\mathbb{L}_p(\mathcal{I}_{A,h}), w}.$$

Note that, for small A and h, $\mathcal{I}_{A,h}$ consists of $M - 1$, M or $M + 1$ intervals depending on whether or not w has a zero/singularity at ± 1.

It is clear that moduli Ω_φ^r are not sufficient to characterize smoothness of functions (the main part weighted modulus is obviously zero for any piecewise constant function f with jump points at $\mathcal{Z}$), and we define the (complete) *weighted modulus of smoothness* as

$$\omega_\varphi^r(f, A, t)_{p,w} := \Omega_\varphi^r(f, A, t)_{p,w} + \sum_{j=1}^{M} E_r(f)_{\mathbb{L}_p(\mathcal{Z}_{2A,t}^j),w}, \tag{3.1}$$

where

$$E_r(f)_{\mathbb{L}_p(I),w} := \inf_{q \in \Pi_r} \| f - q \|_{\mathbb{L}_p(I),w}$$

and Π_r denotes the set of all algebraic polynomials of degree $< r$ (see e.g. [11, (1.11)], [4, (2.4)], [21, (1.19)] and [12, Chapter 11] for similar definitions). Note that these moduli can be defined as $\omega_\varphi^r(f, A, B, t)_{p,w}$ with $2A$ in the sets $\mathcal{Z}_{2A,t}^j$ replaced by B. It is possible to show that $\omega_\varphi^r(f, A, B, t)_{p,w}$ are equivalent for different A and B provided $B > A$ and t is small (if $0 < p < 1$), and we did not investigate the question of equivalence of these moduli in the case $B \leq A$. It will be shown in Section 11 that moduli (3.1) (as well as the averaged moduli (3.2) defined below) are equivalent for all positive A and all $t > 0$ (if $1 \leq p < \infty$) or $0 < t < t_0$, for some $t_0 > 0$ (if $0 < p < 1$). Note, however, that we cannot use this equivalence in the proof of the direct theorem (which would have simplified it considerably) since we derive it as a corollary of several results, the direct theorem being one of them.

We define the *averaged main part weighted modulus* and the (complete) *averaged weighted modulus of smoothness*, respectively, as

$$\widetilde{\Omega}_\varphi^r(f, A, t)_{p,w} := \left(\frac{1}{t} \int_0^t \int_{\mathcal{I}_{A,h}} w(x) |\Delta_{h\varphi(x)}^r(f, x, \mathcal{I}_{A,h})|^p dx dh \right)^{1/p}$$

$$= \left(\frac{1}{t} \int_0^t \left\| \Delta_{h\varphi(\cdot)}^r(f, \cdot, \mathcal{I}_{A,h}) \right\|_{\mathbb{L}_p(\mathcal{I}_{A,h}),w}^p dh \right)^{1/p}$$

and

$$\widetilde{\omega}_\varphi^r(f, A, t)_{p,w} := \widetilde{\Omega}_\varphi^r(f, A, t)_{p,w} + \sum_{j=1}^{M} E_r(f)_{\mathbb{L}_p(\mathcal{Z}_{2A,t}^j),w}. \tag{3.2}$$

The following properties of these moduli immediately follow from the definition:

(i) $\widetilde{\Omega}_\varphi^r(f, A, t)_{p,w} \leq \Omega_\varphi^r(f, A, t)_{p,w}$ and $\widetilde{\omega}_\varphi^r(f, A, t)_{p,w} \leq \omega_\varphi^r(f, A, t)_{p,w}$,
(ii) $\Omega_\varphi^r(f, A, t_2)_{p,w} \leq \Omega_\varphi^r(f, A, t_1)_{p,w}$ and $\omega_\varphi^r(f, A, t_2)_{p,w} \leq \omega_\varphi^r(f, A, t_1)_{p,w}$ if $t_1 \geq t_2$,
(iii) $\widetilde{\Omega}_\varphi^r(f, A, t_2)_{p,w} \leq (t_1/t_2)^{1/p} \widetilde{\Omega}_\varphi^r(f, A, t_1)_{p,w}$ and $\widetilde{\omega}_\varphi^r(f, A, t_2)_{p,w} \leq (t_1/t_2)^{1/p} \widetilde{\omega}_\varphi^r(f, A, t_1)_{p,w}$ if $t_1 \geq t_2$,
(iv) $\Omega_\varphi^r(f, A_1, t)_{p,w} \leq \Omega_\varphi^r(f, A_2, t)_{p,w}$ and $\widetilde{\Omega}_\varphi^r(f, A_1, t)_{p,w} \leq \widetilde{\Omega}_\varphi^r(f, A_2, t)_{p,w}$ if $A_1 \geq A_2$ (since $\mathcal{I}_{A_1,h} \subset \mathcal{I}_{A_2,h}$).

We will also need the following auxiliary quantity ("restricted averaged main part modulus" would be a proper name for it) which will be quite helpful in our estimates:

$$\widetilde{\Omega}_\varphi^r(f, t)_{\mathbb{L}_p(S),w} := \left(\frac{1}{t} \int_0^t \int_S w(x) |\Delta_{h\varphi(x)}^r(f, x, S)|^p dx dh \right)^{1/p},$$

where S is some subset (a union of intervals) of $[-1, 1]$ (that does not depend on h).

Note that

$$\widetilde{\Omega}_\varphi^r(f, t)_{\mathbb{L}_p(\mathcal{I}_{A,t}),w} \leq \widetilde{\Omega}_\varphi^r(f, A, t)_{p,w}.$$

We also remark that since $\mathcal{I}_{A,h}$ consists of a number of disjoint intervals when h is small, it is possible to define a main part modulus taking supremum on each of these intervals. In other words, one can define

$$\Omega_\varphi^{*r}(f, A, t)_{p,w} := \sum_{j=0}^{M} \sup_{0<h\leq t} \left\| \Delta_{h\varphi}^r(f) \right\|_{\mathbb{L}_p(J_{A,h}^j),w},$$

where $z_0 := -1$, $z_{M+1} := 1$, and $J_{A,h}^j$'s denote components of $\mathcal{I}_{A,h}$, i.e.,

$$J_{A,h}^j := \begin{cases} \left[z_j + A\rho(h, z_j), z_{j+1} - A\rho(h, z_{j+1}) \right], & \text{if } 1 \leq j \leq M-1, \\ \left[-1, z_1 - A\rho(h, z_j) \right], & \text{if } j = 0 \text{ and } z_1 \neq -1, \\ \left[z_M + A\rho(h, z_j), 1 \right], & \text{if } j = M \text{ and } z_M \neq 1. \end{cases}$$

It is obvious that $\Omega_\varphi^r(f, A, t)_{p,w} \leq \Omega_\varphi^{*r}(f, A, t)_{p,w}$, and it is less obvious that this inequality can be reversed for any $f \in \mathbb{L}_p^w$, $0 < p < \infty$. Hence, we note that Ω_φ^{*r} could replace Ω_φ^r everywhere in the proofs below, and so using Corollaries 11.1 and 11.2 we could actually show that these moduli are equivalent (in the case $0 < p < 1$, t would have to be small). However, we are not discussing this further.

4. Degree of local approximation

Proposition 4.1. *Let $n \in \mathbb{N}$ and suppose that, for some $1 \leq \mu \leq n$, $I_\mu \subset J$, where $J \subset [-1, 1]$ is an interval such that $|J| \leq c_0|I_\mu|$. Then there exists $m \in \mathbb{N}$ depending only on c_0 (and independent of n) such that J has a nonempty intersection with at most m intervals I_i, $1 \leq i \leq n$.*

Proof. If $n = 1$, the statement is obvious, and so we assume that $n \geq 2$. Because of symmetry, we may assume that $1 \leq \mu \leq \lceil n/2 \rceil$. Now let $1 \leq i \leq n$, and compare the distance from x_i to x_μ to the length of the interval I_μ. Using the estimates $x/10 \leq \sin x \leq x$, $0 \leq x \leq 7\pi/8$, we have

$$\frac{|x_i - x_\mu|}{|I_\mu|} = \frac{\sin\left[(i+\mu)\pi/(2n)\right] \sin\left[|i-\mu|\pi/(2n)\right]}{\sin\left[(2\mu-1)\pi/(2n)\right] \sin\left[\pi/(2n)\right]} \geq \frac{|i^2 - \mu^2|}{100(2\mu-1)} \geq \frac{|i-\mu|}{200}.$$

If $x_i \in J$, then $|x_i - x_\mu| \leq |J| \leq c_0|I_\mu|$ and so $|i-\mu| \leq 200c_0$. This implies that J has empty intersection with all intervals I_i such that $\min\{|i-\mu|, |i-1-\mu|\} > 200c_0$, and so the number of intervals I_i having nonempty intersections with J is $m \leq 400c_0 + 2$. $\quad\square$

Recall now that $\omega_r(f, t, I)_p := \sup_{0<h\leq t} \left\| \Delta_h^r(f, x, I) \right\|_{\mathbb{L}_p(I)}$ is the usual rth modulus of smoothness on an interval I, and that the well-known Whitney's theorem (see e.g. [23, Theorem 7.1, p. 195]) implies that, for any $f \in \mathbb{L}_p[a, b]$, $0 < p < \infty$,

$$\inf_{q\in\Pi_r} \| f - q \|_{\mathbb{L}_p[a,b]} \leq c\omega_r(f, b-a, [a, b])_p.$$

Lemma 4.2. *Let w be a doubling weight from the class $\mathcal{W}(\mathcal{Z})$, $0 < p < \infty$, $f \in \mathbb{L}_p^w$, $n, r \in \mathbb{N}$, and let $A > 0$ and $\theta > 0$ be arbitrary. Also, let*

$$I^* := \left\{ 1 \leq i \leq n \mid I_i \cap \mathcal{Z}_{A,1/n} = \emptyset \right\},$$

and suppose that, for each $i \in I^*$, the interval J_i is such that $I_i \subset J_i \subset \mathcal{I}_{A,1/n}$ and $|J_i| \leq c_0|I_i|$. Then

$$\sum_{i \in I^*} w(x_i)\omega_r(f, |J_i|, J_i)_p^p \leq c \widetilde{\Omega}_\varphi^r(f, \theta/n)_{\mathbb{L}_p(\mathcal{I}_{A,1/n}),w}^p,$$

where the constant c depends only on r, p, c_0, θ, A and the weight w.

Proof. The proof is rather standard (see [6] or [16, Lemma 5.1]). In fact, it is possible to derive an analog of this lemma as a corollary of [16, Lemma 5.1] by replacing f by a function g which is identically zero near the points from $\mathcal{Z}$. However, this approach is not shorter, and we do not immediately get exactly what we need. Hence, we opted for a direct proof even though it is quite similar to that of [16, Lemma 5.1]. We adduce it here for completeness.

The main idea of the proof is the employment of the inequality (see [23, Lemma 7.2, p. 191])

$$\omega_r(f, t, [a,b])_p^p \leq \frac{c}{t} \int_0^t \int_a^b |\Delta_h^r(f, x, [a,b])|^p dx\, dh, \quad 0 < p < \infty. \tag{4.1}$$

Proposition 4.1 implies that each J_i has a nonempty intersection with at most m intervals I_j, $1 \leq j \leq n$, where m depends only on c_0. Since $|I_i| \sim |I_{i\pm1}| \sim \rho_n(x_i)$, this implies that $\rho_n(x) \sim \rho_n(y) \sim |I_i|$ for all $x, y \in J_i$, and so $|x - y| \leq c\rho_n(x)$, for all $x, y \in J_i$. Hence, since $J_i \subset \mathcal{I}_{A,1/n}$, Lemma 2.3(iv) implies that $w(x) \sim w(x_i)$, for all $x \in J_i$, where the equivalence constants depend only on w, A and c_0.

Taking this into account and using (4.1) we have, for each $i \in I^*$,

$$w(x_i)\omega_r(f, |J_i|, J_i)_p^p \leq cw(x_i)\omega_r(f, c^*|I_i|, J_i)_p^p$$

$$\leq c|I_i|^{-1} \int_0^{c^*|I_i|} \int_{J_i} w(x_i)|\Delta_h^r(f, x, J_i)|^p dx\, dh$$

$$\leq c \int_{J_i} \int_0^{c^*|I_i|/\varphi(x)} \frac{\varphi(x)}{|I_i|} w(x)|\Delta_{h\varphi(x)}^r(f, x, J_i)|^p dh\, dx,$$

where $0 < c^* < 1$ is a constant that we will choose later.

Now, $|I_i| \sim \rho_n(x) \sim \varphi(x)/n$ for $x \in J_i$, $i \in J^*$, where

$$J^* := \left\{ i \in I^* \mid J_i \cap (I_1 \cup I_n) = \emptyset \right\}.$$

Note that depending on whether or not $z_1 = -1$ and $z_M = 1$ the set J^* may or may not be the same as I^*.

Now, for $i \in J^*$, taking into account that $c^* \leq \sqrt{c^*}$, we have

$$w(x_i)\omega_r(f, |J_i|, J_i)_p^p \leq cn \int_{J_i} \int_0^{c\sqrt{c^*}/n} w(x)|\Delta_{h\varphi(x)}^r(f, x, J_i)|^p dh\, dx. \tag{4.2}$$

Suppose now that $i \in I^* \setminus J^*$ (we have already remarked that this set may be empty depending on w). Recall that $\Delta_h^r(f, x, J_i)$ is defined to be 0 if $x \pm rh/2 \notin J_i$ and, in particular, $\Delta_{h\varphi(x)}^r(f, x, J_i) = 0$ if $1 - |x| < rh\varphi(x)/2$. Therefore, since $\varphi(x)/|I_i| \leq cn\rho_n(x)/|I_i| \leq cn$, $x \in J_i$, for each fixed $x \in J_i$, we have

$$\int_0^{c^*|I_i|/\varphi(x)} \frac{\varphi(x)}{|I_i|} w(x)|\Delta_{h\varphi(x)}^r(f, x, J_i)|^p dh \leq cn \int_S w(x)|\Delta_{h\varphi(x)}^r(f, x, J_i)|^p dh,$$

where

$$S := \left\{ h \mid 0 < h \leq \min \left\{ \frac{c^* |I_i|}{\varphi(x)}, \frac{2(1 - |x|)}{r\varphi(x)} \right\} \right\}$$

$$\subset \left\{ h \mid 0 < h \leq c \min \left\{ \frac{c^*}{n^2 \sqrt{1 - |x|}}, \sqrt{1 - |x|} \right\} \right\} \subset \left\{ h \mid 0 < h \leq c\sqrt{c^*}/n \right\}.$$

Therefore, (4.2) is valid for $i \in I^* \setminus J^*$ as well. We now choose c^* to be such that $c\sqrt{c^*}$ in the upper limit of the inner integral in (4.2) is less than θ. Since each x belongs to finitely many J_i's by Proposition 4.1, we have

$$\sum_{i \in I^*} w(x_i) \omega_r(f, |J_i|, J_i)_p^p \leq cn \sum_{i \in I^*} \int_{J_i} \int_0^{\theta/n} w(x) |\Delta_{h\varphi(x)}^r(f, x, J_i)|^p \, dh \, dx$$

$$\leq cn \int_0^{\theta/n} \int_{\mathcal{I}_{A,1/n}} w(x) |\Delta_{h\varphi(x)}^r(f, x, \mathcal{I}_{A,1/n})|^p \, dx \, dh$$

$$\leq c\widetilde{\Omega}_\varphi^r(f, \theta/n)_{\mathbb{L}_p(\mathcal{I}_{A,1/n}), w}^p,$$

and the proof is complete. $\square$

5. Jackson type estimate

The following lemma follows from [16, Lemma 3.1].

Lemma 5.1. *Let* $1 \leq i \leq n$, *and let* $v_0, \mu \in \mathbb{N}_0$ *be such that* $\mu \geq c_* \max\{v_0, 1\}$, *where* c_* *is some sufficiently large absolute (positive) constant. Then there exists a polynomial* $T_i = T_i(n, \mu)$ *of degree* $\leq c(\mu)n$ *satisfying the following inequalities for all* $x \in [-1, 1]$:

$$|T_i(x) - \chi_i(x)| \leq c\psi_i(x)^\mu$$

and

$$\left| T_i^{(v)}(x) \right| \leq c|I_i|^{-v} \psi_i(x)^\mu, \quad 0 \leq v \leq v_0,$$

where constants c *depend only on* μ.

We are now ready to state and prove our main direct result.

Theorem 5.2. *Let* w *be a doubling weight from the class* $\mathcal{W}(\mathcal{Z})$, $r, v_0 \in \mathbb{N}$, $v_0 \geq r$, $0 < p < \infty$, *and* $f \in \mathbb{L}_p^w$. *Then, there exists* $N \in \mathbb{N}$ *depending on* r, v_0, p *and the weight* w, *such that for every* $n \geq N$, $\vartheta > 0$ *and* $A > 0$, *there exists a polynomial* $P_n \in \Pi_n$ *such that*

$$\|f - P_n\|_{p,w} \leq c\widetilde{\omega}_\varphi^r(f, A, \vartheta/n)_{p,w} \leq c\omega_\varphi^r(f, A, \vartheta/n)_{p,w}$$

and

$$\left\| \rho_n^v P_n^{(v)} \right\|_{p,w} \leq c\widetilde{\omega}_\varphi^r(f, A, \vartheta/n)_{p,w} \leq c\omega_\varphi^r(f, A, \vartheta/n)_{p,w}, \quad r \leq v \leq v_0,$$

where constants c *depend only on* r, v_0, p, ϑ, A *and the weight* w.

Proof. The idea of this proof is similar to that of [16, Theorem 5.3] where a Jackson type theorem was proved for the weights w_n with moduli of smoothness defined like Ω_φ^r but with $[-1, 1]$ instead of $\mathcal{I}_{A,h}$. However, there are some difficulties that we need to overcome now in order to get the right estimates near $\mathcal{Z}$.

Let $A > 0$ and $\vartheta > 0$ be given (without loss of generality, we can assume that $0 < \vartheta \leq 1$), and let $n \in \mathbb{N}$ be sufficiently large (so that each (nonempty) interval $[z_j, z_{j+1}]$, $0 \leq j \leq M$, contains at least 10 intervals I_i), and let $(x_i)_{i=0}^n$ be the Chebyshev partition of $[-1, 1]$. Recall that $I_i := [x_i, x_{i-1}]$, $1 \leq i \leq n$.

For each $1 \leq j \leq M$, let

$$\nu_j := i \quad \text{such that } z_j \in I_i$$

(recall that, if $z_j = x_i$, $1 \leq i \leq n$, then we pick the right interval containing z_j, i.e., $\nu_j = i$ in this case).

Now, we modify partition $(x_i)_{i=0}^n$ by replacing, for each $1 \leq j \leq M$, the knots x_{ν_j} and x_{ν_j-1} by $z_j - \sigma_j \rho_n(z_j)$ and $z + \sigma_j \rho_n(z_j)$, respectively (replacing only one of them if z_j is 1 or -1). More precisely, for some collection of M constants $0 < \sigma_j \leq 1/10$, $1 \leq j \leq M$, which we will choose later, define

$$\widetilde{x}_1 := 1 - \sigma_M/n^2, \quad \text{if } i = 1 \text{ and } z_M = 1,$$

and

$$\widetilde{x}_{n-1} := -1 + \sigma_1/n^2, \quad \text{if } z_1 = -1.$$

Now, for all $1 \leq i \leq n - 1$ where $\widetilde{x}_i$ has not been defined yet, we let

$$\widetilde{x}_i := \begin{cases} z_j - \sigma_j \rho_n(z_j), & \text{if } i = \nu_j, \ 1 \leq j \leq M, \\ z_j + \sigma_j \rho_n(z_j), & \text{if } i = \nu_j - 1, \ 1 \leq j \leq M, \\ x_i, & \text{otherwise.} \end{cases}$$

We now note that this new partition $(\widetilde{x}_i)_{i=0}^n$ has the same properties as the original Chebyshev partition (with constants than now depend on σ_j). In particular, if $\widetilde{I}_i := [\widetilde{x}_i, \widetilde{x}_{i-1}]$, then $|I_i| \sim |\widetilde{I}_i|$, $|\widetilde{I}_{i\pm 1}| \sim |\widetilde{I}_i|$, $\widetilde{\psi}_i(x) := |\widetilde{I}_i|/\left(|x - \widetilde{x}_i| + |\widetilde{I}_i|\right) \sim \psi_i(x)$ and $|\chi_{[\widetilde{x}_i, 1]}(x) - \chi_{[x_i, 1]}(x)| \leq c\psi_i(x)$ uniformly in x, etc. We now simplify our notation by dropping tilde and keeping in mind that, from now on in this proof, $(x_i)_{i=0}^n$ is the *modified* Chebyshev partition. Hence, z_j is now the center of I_{ν_j} (unless z_j is -1 or 1 in which case z_j is, respectively, the left or the right endpoint of I_{ν_j}).

It is convenient to denote

$$I_* := \left\{ 1 \leq i \leq n \mid i = \nu_j, 1 \leq j \leq M \right\} \quad \text{and} \quad I^* := \left\{ 1 \leq i \leq n \mid i \notin I_* \right\}.$$

For each $1 \leq i \leq n$, define $q_i \in \Pi_r$ to be a polynomial of near best approximation of f on I_i with the weight w, i.e.,

$$\|f - q_i\|_{\mathbb{L}_p(I_i),w} \leq c E_r(f)_{\mathbb{L}_p(I_i),w},$$

and define S_n to be a piecewise polynomial function such that $S_n|_{I_i} = q_i$, $1 \leq i \leq n$.

The following is a crucial observation that follows from Lemma 2.3(v) and properties of w_n:

$$w(x) \sim w_n(x) \sim w_n(x_i), \quad \text{for each } x \in I_i \text{ with } i \in I^*. \tag{5.1}$$

Now, using Whitney's inequality we get

$$\|f - S_n\|_{p,w}^p = \sum_{i \in I^*} \int_{I_i} w(x)|f(x) - S_n(x)|^p dx + \sum_{j=1}^M \int_{I_{\nu_j}} w(x)|f(x) - S_n(x)|^p dx$$

$$\leq c \sum_{i \in I^*} w_n(x_i) \int_{I_i} |f(x) - q_i(x)|^p dx + c \sum_{j=1}^M E_r(f)_{\mathbb{L}_p(\mathcal{Z}_{\sigma_j,1/n}^j),w}^p$$

$$\leq c \sum_{i \in I^*} w(x_i) \omega_r(f, |I_i|, I_i)_p^p + c \sum_{j=1}^{M} E_r(f)_{\mathbb{L}_p(\mathcal{Z}_{\sigma_j, 1/n}^j), w}^p$$

$$\leq c \widetilde{\Omega}_\varphi^r(f, \theta/n)_{\mathbb{L}_p(S), w}^p + c \sum_{j=1}^{M} E_r(f)_{\mathbb{L}_p(\mathcal{Z}_{\sigma_j, 1/n}^j), w}^p,$$

where $S := S(1/n) := [-1, 1] \setminus \cup_{j=1}^{M} \mathcal{Z}_{\sigma_j, 1/n}^j$. In the last estimate, we took into account that $I_i \subset S(1/n)$, $i \in I^*$.

It is easy to check that S_n can be written as

$$S_n(x) = q_n(x) + \sum_{i=1}^{n-1} \left[q_i(x) - q_{i+1}(x) \right] \chi_i(x),$$

and define

$$P_n(x) := q_n(x) + \sum_{i=1}^{n-1} \left[q_i(x) - q_{i+1}(x) \right] T_i(x),$$

where $T_i = T_i(n, \mu)$ are the polynomials from Lemma 5.1 with a sufficiently large μ (we will prescribe it later so that all restrictions below are satisfied).

Lemma 5.1 now implies

$$\|S_n - P_n\|_{p, w}^p \leq \int_{-1}^{1} w(x) \left[\sum_{i=1}^{n-1} |q_i(x) - q_{i+1}(x)| \cdot |\chi_i(x) - T_i(x)| \right]^p dx$$

$$\leq c \int_{-1}^{1} w(x) \left[\sum_{i=1}^{n-1} \|q_i - q_{i+1}\|_\infty \psi_i(x)^\mu \right]^p dx.$$

Using the Lagrange interpolation formula and [7, Theorem 4.2.7] we have, for all $q \in \Pi_r$ and $0 \leq l \leq r - 1$,

$$\left\| q^{(l)} \right\|_\infty \leq c \psi_i^{-r+l+1} \left\| q^{(l)} \right\|_{\mathbb{C}(I_i)} \leq c \psi_i^{-r+l+1} |I_i|^{-l-1/p} \|q\|_{\mathbb{L}_p(I_i)}, \tag{5.2}$$

and so it yields (with $l = 0$)

$$\|S_n - P_n\|_{p, w}^p \leq c \int_{-1}^{1} w(x) \left[\sum_{i=1}^{n-1} \|q_i - q_{i+1}\|_{\mathbb{L}_p(I_i)} |I_i|^{-1/p} \psi_i(x)^{\mu-r+1} \right]^p dx.$$

Now, if $1 \leq p < \infty$, since $\sum_{i=1}^{n-1} \psi_i(x)^2 \leq c$, we have by Jensen's inequality

$$\left(\sum_{i=1}^{n-1} |\gamma_i| \psi_i(x)^2 \right)^p \leq c \sum_{i=1}^{n-1} |\gamma_i|^p \psi_i(x)^2 \leq c \sum_{i=1}^{n-1} |\gamma_i|^p,$$

and if $0 < p < 1$, then

$$\left(\sum_{i=1}^{n-1} |\gamma_i| \psi_i(x)^2 \right)^p \leq \sum_{i=1}^{n-1} |\gamma_i|^p \psi_i(x)^{2p} \leq c \sum_{i=1}^{n-1} |\gamma_i|^p.$$

Therefore,

$$\|S_n - P_n\|_{p,w}^p \le c \int_{-1}^1 \sum_{i=1}^{n-1} \|q_i - q_{i+1}\|_{\mathbb{L}_p(I_i)}^p |I_i|^{-1} w(x) \psi_i(x)^{(\mu-r-1)p} dx$$

$$\le c \left(\int_{[-1,1]\setminus\cup_{j=1}^M I_{\nu_j}} + \sum_{j=1}^M \int_{I_{\nu_j}} \right) \sum_{i=1}^{n-1} \|q_i - q_{i+1}\|_{\mathbb{L}_p(I_i)}^p |I_i|^{-1} w(x) \psi_i(x)^{(\mu-r-1)p} dx$$

$$=: \mathfrak{J}^* + \sum_{j=1}^M \mathfrak{J}_j.$$

Hence, since by (5.1), $w(x) \sim w_n(x) \le c\psi_i(x)^{-s} w_n(x_i)$, for $x \in [-1,1] \setminus \cup_{j=1}^M I_{\nu_j}$, we have

$$\mathfrak{J}^* \le c \int_{[-1,1]\setminus\cup_{j=1}^M I_{\nu_j}} \sum_{i=1}^{n-1} \|q_i - q_{i+1}\|_{\mathbb{L}_p(I_i)}^p |I_i|^{-1} w_n(x_i) \psi_i(x)^{(\mu-r-1)p-s} dx$$

$$\le c \sum_{i=1}^{n-1} \|q_i - q_{i+1}\|_{\mathbb{L}_p(I_i)}^p |I_i|^{-1} w_n(x_i) \int_{-1}^1 \psi_i(x)^{(\mu-r-1)p-s} dx$$

$$\le c \sum_{i=1}^{n-1} \|q_i - q_{i+1}\|_{\mathbb{L}_p(I_i)}^p w_n(x_i),$$

if $(\mu - r - 1)p - s \ge 2$, since $\int_{-1}^1 \psi(x)^\alpha dx \le c|I_i|$ if $\alpha \ge 2$.

Also, for each $1 \le j \le M$, taking into account that $|x - x_i| + |I_i| \sim |z_j - x_i| + |I_i|$ and so $\psi_i(x) \sim \psi_i(z_j)$ uniformly for $x \in I_{\nu_j}$, we have

$$\mathfrak{J}_j \le c \sum_{i=1}^{n-1} \|q_i - q_{i+1}\|_{\mathbb{L}_p(I_i)}^p |I_i|^{-1} \int_{I_{\nu_j}} w(x) \psi_i(x)^{(\mu-r-1)p} dx$$

$$\le c \sum_{i=1}^{n-1} \|q_i - q_{i+1}\|_{\mathbb{L}_p(I_i)}^p |I_i|^{-1} \psi_i(z_j)^{(\mu-r-1)p} \int_{I_{\nu_j}} w(x) dx$$

$$\le c \sum_{i=1}^{n-1} \|q_i - q_{i+1}\|_{\mathbb{L}_p(I_i)}^p |I_i|^{-1} \psi_i(z_j)^{(\mu-r-1)p} \rho_n(z_j) w_n(z_j)$$

$$\le c \sum_{i=1}^{n-1} \|q_i - q_{i+1}\|_{\mathbb{L}_p(I_i)}^p w_n(x_i) |I_i|^{-1} \psi_i(z_j)^{(\mu-r-1)p-s} \rho_n(z_j).$$

Now, using the inequality $\rho_n(x)^2 \le 4\rho_n(y)(|x-y| + \rho_n(y))$ we have

$$|I_i|^{-1} \psi_i(z_j)^{(\mu-r-1)p-s} \rho_n(z_j) \sim \psi_i(z_j)^{(\mu-r-1)p-s} \frac{\rho_n(z_j)}{\rho_n(x_i)}$$

$$\le c\psi_i(z_j)^{(\mu-r-1)p-s} \left[\frac{|x_i - z_j| + \rho_n(x_i)}{\rho_n(x_i)} \right]^{1/2}$$

$$\sim c\psi_i(z_j)^{(\mu-r-1)p-s-1/2} \le c,$$

provided $(\mu - r - 1)p - s - 1/2 \ge 0$. Note also that we could alternatively estimate this quantity as follows.

$$|I_i|^{-1} \psi_i(z_j)^{(\mu-r-1)p-s} \rho_n(z_j) \sim |I_i|^{-1} \int_{I_{v_j}} \psi_i(z_j)^{(\mu-r-1)p-s} dx$$

$$\sim |I_i|^{-1} \int_{I_{v_j}} \psi_i(x)^{(\mu-r-1)p-s} dx$$

$$\leq c|I_i|^{-1} \int_{-1}^{1} \psi_i(x)^{(\mu-r-1)p-s} dx \leq c,$$

provided $(\mu - r - 1)p - s \geq 2$.

Combining the above estimates we conclude that

$$\|S_n - P_n\|_{p,w}^p \leq c \sum_{i=1}^{n-1} \|q_i - q_{i+1}\|_{\mathbb{L}_p(I_i)}^p w_n(x_i).$$

Now, for each $1 \leq j \leq M$, let $L_j := [z_j - \sigma_j \rho_n(z_j), z_j - c_0 \sigma_j \rho_n(z_j)]$ and $R_j := [z_j + c_0 \sigma_j \rho_n(z_j), z_j + \sigma_j \rho_n(z_j)]$ (note that if $z_1 = -1$, then L_1 is not defined, and if $z_M = 1$, then R_M is not defined, but these intervals are not needed in these cases), where $c_0 \in (0, 1)$ is a constant that we will choose later (it will be 0.9 but we will keep writing "c_0" in order not to distract from the proof). Then, $L_j \cup R_j \subset I_{v_j}$, $|L_j| \sim |R_j| \sim |I_{v_j}|$, and $\text{dist}(L_j, z_j) = \text{dist}(R_j, z_j) = c_0 \sigma_j \rho_n(z_j)$, for all $1 \leq j \leq M$.

We continue estimating as follows

$$\|S_n - P_n\|_{p,w}^p \leq c \left(\sum_{i,i+1 \in I^*} + \sum_{i \in I_*} + \sum_{i+1 \in I_*} \right) \|q_i - q_{i+1}\|_{\mathbb{L}_p(I_i)}^p w_n(x_i)$$

$$\leq c \sum_{i,i+1 \in I^*} \|q_i - q_{i+1}\|_{\mathbb{L}_p(I_i)}^p w_n(x_i) + c \sum_{j=1}^{M} \|q_{v_j} - q_{v_j+1}\|_{\mathbb{L}_p(I_{v_j})}^p w_n(x_{v_j})$$

$$+ c \sum_{j=1}^{M} \|q_{v_j-1} - q_{v_j}\|_{\mathbb{L}_p(I_{v_j-1})}^p w_n(x_{v_j-1})$$

$$\leq c \sum_{i,i+1 \in I^*} \|q_i - q_{i+1}\|_{\mathbb{L}_p(I_i)}^p w_n(x_i) + c \sum_{j=1}^{M} \|q_{v_j} - q_{v_j+1}\|_{\mathbb{L}_p(L_j)}^p w_n(x_{v_j})$$

$$+ c \sum_{j=1}^{M} \|q_{v_j-1} - q_{v_j}\|_{\mathbb{L}_p(R_j)}^p w_n(x_{v_j-1}),$$

since $\|q\|_{\mathbb{L}_p(I)} \sim \|q\|_{\mathbb{L}_p(J)}$, for any polynomial $q \in \Pi_r$ and any intervals I and J of comparable length which are either next to each other or are such that one interval is a subset of the other one.

Now using Lemma A.1 (that implies that q_i's are polynomials of near best approximation of f on intervals which are slightly bigger than I_i), Whitney's inequality, (5.1) and the fact that $w(x) \sim w_n(x) \sim w_n(x_{v_j})$ for each $x \in L_j$ and $w(x) \sim w_n(x) \sim w_n(x_{v_j-1})$ for each $x \in R_j$, we have

$$\|S_n - P_n\|_{p,w}^p \leq c \sum_{i,i-1 \in I^*} \|f - q_i\|_{\mathbb{L}_p(I_i \cup I_{i-1})}^p w_n(x_i) + c \sum_{j=1}^{M} \|q_{v_j} - f\|_{\mathbb{L}_p(L_j)}^p w_n(x_{v_j})$$

$$+ c \sum_{j=1}^{M} \|f - q_{v_j+1}\|_{\mathbb{L}_p(L_j)}^p w_n(x_{v_j}) + c \sum_{j=1}^{M} \|q_{v_j-1} - f\|_{\mathbb{L}_p(R_j)}^p w_n(x_{v_j-1})$$

$$+ c \sum_{j=1}^{M} \left\| f - q_{\nu_j} \right\|_{\mathbb{L}_p(R_j)}^{p} w_n(x_{\nu_j - 1})$$

$$\leq c \sum_{i, i-1 \in I^*} \omega_r(f, |I_i \cup I_{i-1}|, I_i \cup I_{i-1})_p^p w(x_i)$$

$$+ c \sum_{j=1}^{M} \omega_r(f, |I_{\nu_j+1} \cup L_j|, I_{\nu_j+1} \cup L_j)_p^p w(x_{\nu_j})$$

$$+ c \sum_{j=1}^{M} \omega_r(f, |I_{\nu_j-1} \cup R_j|, I_{\nu_j-1} \cup R_j)_p^p w(x_{\nu_j-1}) + c \sum_{j=1}^{M} \left\| f - q_{\nu_j} \right\|_{\mathbb{L}_p(I_{\nu_j}), w}^{p}$$

$$\leq c \widetilde{\Omega}_\varphi^r(f, \theta/n)_{\mathbb{L}_p(\widetilde{S}), w}^{p} + c \sum_{j=1}^{M} E_r(f)_{\mathbb{L}_p(\mathcal{Z}_{\sigma_j, 1/n}^{j}), w}^{p},$$

where $\widetilde{S} := \widetilde{S}(1/n) := [-1, 1] \setminus \cup_{j=1}^{M} \mathcal{Z}_{c_0 \sigma_j, 1/n}^{j}$ (note that $S(1/n) \subset \widetilde{S}(1/n)$ and so $\widetilde{\Omega}_\varphi^r(f,$ $\theta/n)_{\mathbb{L}_p(S), w}^{p} \leq \widetilde{\Omega}_\varphi^r(f, \theta/n)_{\mathbb{L}_p(\widetilde{S}), w}^{p}$).

Now,

$$P_n^{(\nu)}(x) = p_n^{(\nu)}(x) + \sum_{i=1}^{n-1} \sum_{l=0}^{\nu} \binom{\nu}{l} \left[q_i^{(l)}(x) - q_{i+1}^{(l)}(x) \right] T_i^{(\nu-l)}(x),$$

and so, for $r \leq \nu \leq \nu_0$ (which guarantees that $p_n^{(\nu)} \equiv 0$), we have using Lemma 5.1 and estimate (5.2)

$$\left\| \rho_n^\nu P_n^{(\nu)} \right\|_{p,w}^{p} \leq \int_{-1}^{1} w(x) \rho_n(x)^{\nu p} \left[\sum_{i=1}^{n-1} \sum_{l=0}^{\nu} \binom{\nu}{l} \left| q_i^{(l)}(x) - q_{i+1}^{(l)}(x) \right| \cdot \left| T_i^{(\nu-l)}(x) \right| \right]^{p} dx$$

$$\leq c \int_{-1}^{1} w(x) \rho_n(x)^{\nu p} \left[\sum_{i=1}^{n-1} \sum_{l=0}^{\nu} \left\| q_i^{(l)} - q_{i+1}^{(l)} \right\|_\infty |I_i|^{-\nu+l} \psi_i(x)^{\mu} \right]^{p} dx$$

$$\leq c \int_{-1}^{1} w(x) \rho_n(x)^{\nu p} \left[\sum_{i=1}^{n-1} \sum_{l=0}^{\nu} \left\| q_i - q_{i+1} \right\|_{\mathbb{L}_p(I_i)} |I_i|^{-\nu-1/p} \psi_i(x)^{\mu-r+l+1} \right]^{p} dx$$

$$\leq c \int_{-1}^{1} w(x) \rho_n(x)^{\nu p} \left[\sum_{i=1}^{n-1} \left\| q_i - q_{i+1} \right\|_{\mathbb{L}_p(I_i)} |I_i|^{-\nu-1/p} \psi_i(x)^{\mu-r+1} \right]^{p} dx$$

$$\leq c \int_{-1}^{1} w(x) \rho_n(x)^{\nu p} \sum_{i=1}^{n-1} \left\| q_i - q_{i+1} \right\|_{\mathbb{L}_p(I_i)}^{p} |I_i|^{-\nu p - 1} \psi_i(x)^{(\mu-r-1)p} dx.$$

Now, since $\rho_n(x)^2 \leq c\rho_n(x_i) (|x - x_i| + \rho_n(x_i))$ and $|I_i| \sim \rho_n(x_i)$, we have

$$\left\| \rho_n^\nu P_n^{(\nu)} \right\|_{p,w}^{p} \leq c \int_{-1}^{1} w(x) \sum_{i=1}^{n-1} \left\| q_i - q_{i+1} \right\|_{\mathbb{L}_p(I_i)}^{p} [\rho_n(x_i) (|x - x_i| + \rho_n(x_i))]^{\nu p/2}$$

$$\times |I_i|^{-\nu p - 1} \psi_i(x)^{(\mu-r-1)p} dx$$

$$\leq c \int_{-1}^{1} w(x) \sum_{i=1}^{n-1} \left\| q_i - q_{i+1} \right\|_{\mathbb{L}_p(I_i)}^{p} |I_i|^{-1} \psi_i(x)^{(\mu-r-1-\nu/2)p} dx,$$

and exactly the same sequence of inequalities as above (only the power of ψ_i is different) yields

$$\left\| \rho_n^\nu P_n^{(\nu)} \right\|_{p,w} \le c \widetilde{\Omega}_\varphi^r(f, \theta/n)_{\mathbb{L}_p(\widetilde{S}),w}^p + c \sum_{j=1}^M E_r(f)_{\mathbb{L}_p(\mathcal{Z}_{\sigma_j,1/n}^j),w}^p,$$

provided $(\mu - r - 1 - \nu_0/2)p - s \ge 2$.

Thus, if we pick $\mu = \mu(r, \nu_0, p, s)$ so that this (the most restrictive in this proof) inequality is satisfied then, for each $m \in \mathbb{N}$, we constructed a polynomial P_m of degree $< n_0 m$ with some $n_0 \in \mathbb{N}$ depending only on r, ν_0, p and the doubling constant of the weight w, such that

$$\max \left\{ \left\| \rho_m^\nu P_m^{(\nu)} \right\|_{p,w}, \|f - P_m\|_{p,w} \right\}$$
$$\le c \widetilde{\Omega}_\varphi^r(f, \theta/m)_{\mathbb{L}_p(\widetilde{S}(1/m)),w}^p + c \sum_{j=1}^M E_r(f)_{\mathbb{L}_p(\mathcal{Z}_{\sigma_j,1/m}^j),w}^p.$$

Suppose now that $n \ge D n_0 =: N$, where D is a natural number ≥ 10 that will be picked in a moment. Then there exists $m \in \mathbb{N}$ such that $m n_0 \le n < (m+1) n_0$ (note that $m \ge D$ and so $n_0 \le n/m \le (1 + 1/D) n_0$). Then the polynomial P_m is of degree $< n_0 m \le n$ (i.e., $P_m \in \Pi_n$).

Now, we need to pick θ, σ_j's, c_0 and D so that

$$\widetilde{\Omega}_\varphi^r(f, \theta/m)_{\mathbb{L}_p(\widetilde{S}(1/m)),w}^p + c \sum_{j=1}^M E_r(f)_{\mathbb{L}_p(\mathcal{Z}_{\sigma_j,1/m}^j),w}^p \le c \widetilde{\omega}_\varphi^r(f, A, \vartheta/n)_{p,w}. \qquad (5.3)$$

This will complete the proof since $\rho_m(x) \sim \rho_n(x)$.

The estimate (5.3) is satisfied if, in particular, for $1 \le j \le M$,

$$\mathcal{Z}_{A,\vartheta/n}^j \subset \mathcal{Z}_{c_0 \sigma_j, 1/m}^j, \qquad \mathcal{Z}_{\sigma_j, 1/m}^j \subset \mathcal{Z}_{2A, \vartheta/n}^j \quad \text{and} \quad \theta/m \le \vartheta/n$$

(see properties of the moduli in Section 3). We pick θ so that $\theta \le \vartheta/(2n_0)$, and to finish the proof we need to make sure that the following holds:

$$c_0 \sigma_j \rho_m(z_j) \ge A \rho(\vartheta/n, z_j) \quad \text{and} \quad \sigma_j \rho_m(z_j) \le 2A \rho(\vartheta/n, z_j), \quad 1 \le j \le M. \qquad (5.4)$$

Recall that σ_j is assumed to be $\le 1/10$, and that it cannot depend on m or n (but can depend on n_0). We also note that we can assume that ϑ is small since

$$\widetilde{\omega}_\varphi^r(f, A, \vartheta_1/n)_{p,w} \le c \widetilde{\omega}_\varphi^r(f, A, \vartheta_2/n)_{p,w}, \quad \text{if } \vartheta_1 \le \vartheta_2.$$

So we assume that $\vartheta \le 1$ is such that it guarantees that $\sigma_j \le 1/10$ (see the estimates below). Alternatively, we can guarantee this by letting n_0 be sufficiently large.

Hence, if $z_j = \pm 1$ the inequalities in (5.4) become

$$A \vartheta^2 \le c_0 \sigma_j \frac{n^2}{m^2} \quad \text{and} \quad \sigma_j \frac{n^2}{m^2} \le 2A \vartheta^2,$$

and recalling that $n_0 \le n/m \le (1 + 1/D) n_0$, we now pick σ_j so that

$$\frac{A \vartheta^2}{c_0 n_0^2} \le \sigma_j \le \frac{2A \vartheta^2}{(1 + 1/D)^2 n_0^2}.$$

For example, with $c_0 := 0.9$ we set $\sigma_j := A \vartheta^2/(0.9 n_0^2)$ (recall that $D \ge 10$).

We now let $D \geq 10$ be so large that $D \geq 10/\varphi(z_j)$ for all $1 \leq j \leq M$, for which $z_j \neq \pm 1$ (so, clearly, D depends only on the weight w). Recalling that $n \geq m \geq D$, this implies that, if $z_j \neq \pm 1$, then

$$\varphi(z_j)/m \leq \rho(1/m, z_j) \leq 1.1\varphi(z_j)/m \quad \text{and} \quad \vartheta\varphi(z_j)/n \leq \rho(\vartheta/n, z_j) \leq 1.1\vartheta\varphi(z_j)/n.$$

Therefore, to guarantee that the inequalities in (5.4) hold it is sufficient to pick σ_j so that

$$\frac{1.1A\vartheta}{c_0} \leq \sigma_j \frac{n}{m} \quad \text{and} \quad 1.1\sigma_j \frac{n}{m} \leq 2A\vartheta,$$

which, in turn, follows from

$$\frac{1.1A\vartheta}{c_0 n_0} \leq \sigma_j \leq \frac{2A\vartheta}{1.1(1 + 1/D)n_0}.$$

Now, recall that we already picked $c_0 = 0.9$, and let

$$\sigma_j := \frac{1.1A\vartheta}{0.9n_0},$$

for all $1 \leq j \leq M$ such that $z_j \neq \pm 1$. $\square$

6. Remez and Markov–Bernstein type theorems and applications

Most results in this section are based on a well known idea to use Remez type results to go back and forth between $\varphi(x)$ and $\varphi(x) + 1/n$ in various estimates involving polynomials and on the fact that $\|P_n\|_{p,w} \sim \|P_n\|_{p,w_n}$ for polynomials from Π_n (G. Mastroianni and V. Totik deserve most credit for this observation). Note that most of them are given for general doubling weights without the requirement that they belong to $\mathcal{W}(\mathcal{Z})$ (but see a comment following the statement of Corollary 6.3).

6.1. Remez type theorems and applications

We start with the following crucial lemma that states that the norms of polynomials of degree $< n$ are essentially the same irrespectively of whether the weight w or the weight w_n is used (where w is a doubling weight).

Lemma 6.1. *Let w be a doubling weight on $[-1, 1]$. Then for every $0 < p < \infty$ there is a constant c_0 depending only on p and the doubling constant of w such that, for every polynomial $P_n \in \Pi_n$,*

$$c_0^{-1} \|P_n\|_{p,w} \leq \|P_n\|_{p,w_n} \leq c_0 \|P_n\|_{p,w}.$$

In the case $1 \leq p < \infty$, this is [20, Theorem 7.2]. It is obtained in [20] as a corollary of an analogous result for trigonometric polynomials (see [20, Theorem 3.1]) with a method that does not depend on whether or not p is greater or less than 1. Since the result for trigonometric polynomials holds for all $0 < p < \infty$ (see [13, Theorem 2.1]), we conclude that Lemma 6.1 is valid.

The following Remez inequality for doubling weights holds.

Theorem 6.2 (*[13,20]*)**.** *Let W be a 2π-periodic function which is a doubling weight on $[0, 2\pi]$, and let $0 < p < \infty$ be arbitrary. Then there is a constant $C > 0$ depending only on p and on the doubling constant of W so that if T_n is a trigonometric polynomial of degree at most n and E is a measurable subset of $[0, 2\pi]$ of measure at most Λ/n, $1 \leq \Lambda \leq n$, that is a union of intervals of length at least c/n, then*

$$\int_{-\pi}^{\pi} |T_n(u)|^p W(u) du \leq \left(\frac{C}{c}\right)^{\Lambda} \int_{[0,2\pi]\setminus E} |T_n(u)|^p W(u) du.$$

The following is a corollary for algebraic polynomials (see [20] in the case $1 \leq p < \infty$, the case $0 < p < 1$ is analogous).

Corollary 6.3. *Let w be a doubling weight and $0 < p < \infty$. If $E \subset [-1, 1]$ is a union of at most K intervals and $\int_E (1 - x^2)^{-1/2} dx \leq \Lambda/n$, $\Lambda \leq n$, then for each $p_n \in \Pi_n$, we have*

$$\int_{-1}^{1} |p_n(x)|^p w(x) dx \leq C \int_{[-1,1]\setminus E} |p_n(x)|^p w(x) dx,$$

where the constant C depends only on Λ, K, p and the doubling constant of w.

We note that there is a simple proof showing that Corollary 6.3 is satisfied for doubling weights from the class $\mathcal{W}(\mathcal{Z})$. This follows from the usual unweighted Remez inequality (i.e., Corollary 6.3 with $w \equiv 1$) and the fact that $w_n(x) \sim \mathcal{Q}_n(x)^p$, where $0 < p < \infty$ and $\mathcal{Q}_n \in \Pi_n$ (see [20, (7.34)–(7.36)] or [16, Theorem 4.1]).

Indeed, suppose that $E \subset [-1, 1]$ is a union of at most K intervals and $\int_E (1 - x^2)^{-1/2} dx \leq c/n$. We enlarge E to $E \cup \widetilde{E}$, where $\widetilde{E} := \mathcal{Z}_{1,1/n} = [-1, 1] \cap \cup_{j=1}^{M} [z_j - \rho_n(z_j), z_j + \rho_n(z_j)]$ and note that

$$\int_{\widetilde{E}} (1 - x^2)^{-1/2} dx \leq \sum_{j=1}^{M} \int_{z_j - \rho_n(z_j)}^{z_j + \rho_n(z_j)} (1 - x^2)^{-1/2} dx \leq c/n.$$

Then, using Lemma 6.1 we have

$$\begin{aligned}
\|P_n\|_{p,w} &\sim \|P_n\|_{p,w_n} \leq c \|P_n \mathcal{Q}_n\|_p \leq c \|P_n \mathcal{Q}_n\|_{\mathbb{L}_p([-1,1]\setminus(E\cup\widetilde{E}))} \\
&\leq c \|P_n\|_{\mathbb{L}_p([-1,1]\setminus(E\cup\widetilde{E})),w_n} \\
&\leq c \|P_n\|_{\mathbb{L}_p([-1,1]\setminus(E\cup\widetilde{E})),w} \leq c \|P_n\|_{\mathbb{L}_p([-1,1]\setminus E),w},
\end{aligned}$$

since $w \sim w_n$ on $[-1, 1] \setminus \widetilde{E}$ by Lemma 2.3(v).

One of the applications of Corollary 6.3 is the following result which is quite useful in the proofs.

Theorem 6.4. *Let w be a doubling weight, $0 < p < \infty$, $n \in \mathbb{N}$, $0 \leq \mu \leq n$. Then, for any $P_n \in \Pi_n$,*

$$\left\|\varphi^{\mu} P_n\right\|_{p,w} \sim \left\|\varphi^{\mu} P_n\right\|_{p,w_n} \tag{6.1}$$

and

$$\left\|\lambda_n^{\mu} P_n\right\|_{p,w} \sim \left\|\lambda_n^{\mu} P_n\right\|_{p,w_n}, \tag{6.2}$$

where $\lambda_n(x) := \max\left\{\sqrt{1-x^2}, 1/n\right\}$, and the equivalence constants depend only on p and the doubling constant of w, and are independent of μ.

Proof. The idea used in this proof is well known. Remez type results allow us to replace $[-1, 1]$ by $[-1+n^{-2}, 1-n^{-2}]$ where $\lambda_n \sim \varphi$. We have to be careful with the constants though making sure that they do not depend on μ.

We start with the equivalence (6.1). Note that if μ is an even integer, then this equivalence immediately follows from Lemma 6.1 since $\varphi^\mu P_n \in \Pi_{n+\mu} \subset \Pi_{2n}$ and $w_n \sim w_{2n}$. It is now clear how to proceed. We let $m := 2\lfloor \mu/2 \rfloor$. Then m is an even integer such that $\mu - 2 < m \le \mu$ (note that $m = 0$ if $\mu < 2$), and $Q_{n+m} := \varphi^m P_n \in \Pi_{n+m} \subset \Pi_{2n}$.

Since w is a doubling weight, then $w\varphi^{\gamma p}$, $\gamma > 0$, is also a doubling weight (with a doubling constant depending on $\lceil \gamma \rceil$, p and the doubling constant of w) and (see also [20, Lemma 4.5 and p. 65])

$$(w\varphi^{\gamma p})_n \sim w_n \varphi_n^{\gamma p},$$

where $\varphi_n(x) \sim \varphi(x) + 1/n$, and the equivalence constants depend on $\lceil \gamma \rceil$, p and the doubling constant of w.

Hence, denoting $\mathcal{J}_n := [-1+n^{-2}, 1-n^{-2}]$, $\eta := \mu - m$, noting that $0 \le \eta < 2$ (and so $\lceil \eta \rceil$ is either 1 or 2 allowing us to replace constant that depend on $\lceil \eta \rceil$ by those independent of η), and using Lemma 6.1 and Corollary 6.3 we have

$$\left\| \varphi^\mu P_n \right\|_{p,w} = \left\| \varphi^\eta Q_{n+m} \right\|_{p,w} = \left\| Q_{n+m} \right\|_{p,w\varphi^{\eta p}} \sim \left\| Q_{n+m} \right\|_{p,(w\varphi^{\eta p})_n}$$
$$\sim \left\| Q_{n+m} \right\|_{\mathbb{L}_p(\mathcal{J}_n),(w\varphi^{\eta p})_n} \sim \left\| Q_{n+m} \right\|_{\mathbb{L}_p(\mathcal{J}_n),w_n\varphi_n^{\eta p}} \sim \left\| Q_{n+m} \right\|_{\mathbb{L}_p(\mathcal{J}_n),w_n\varphi^{\eta p}}.$$

Now, since the weight $w_n\varphi^{\eta p}$ is doubling with the doubling constant depending only on the doubling constant of w and p, we can continue as follows.

$$\left\| Q_{n+m} \right\|_{\mathbb{L}_p(\mathcal{J}_n),w_n\varphi^{\eta p}} \sim \left\| Q_{n+m} \right\|_{p,w_n\varphi^{\eta p}} = \left\| \varphi^\eta Q_{n+m} \right\|_{p,w_n} = \left\| \varphi^\mu P_n \right\|_{p,w_n}.$$

Note that none of the constants in the equivalences above depend on μ. This completes the proof of (6.1).

Now, let $\mathcal{E}_n := \left\{ x \mid \sqrt{1-x^2} \le 1/n \right\}$ and note that $\lambda_n(x) = 1/n$ if $x \in \mathcal{E}_n$, and $\lambda_n(x) = \varphi(x)$ if $x \in [-1, 1] \setminus \mathcal{E}_n$. Using (6.1) we have

$$2^{\min\{0, 1-1/p\}} \left\| \lambda_n^\mu P_n \right\|_{p,w} \le \left\| \lambda_n^\mu P_n \right\|_{\mathbb{L}_p(\mathcal{E}_n),w} + \left\| \lambda_n^\mu P_n \right\|_{\mathbb{L}_p([-1,1]\setminus\mathcal{E}_n),w}$$
$$= n^{-\mu} \left\| P_n \right\|_{\mathbb{L}_p(\mathcal{E}_n),w} + \left\| \varphi^\mu P_n \right\|_{\mathbb{L}_p([-1,1]\setminus\mathcal{E}_n),w}$$
$$\le n^{-\mu} \left\| P_n \right\|_{p,w} + \left\| \varphi^\mu P_n \right\|_{p,w}$$
$$\le c_0 \left(n^{-\mu} \left\| P_n \right\|_{p,w_n} + \left\| \varphi^\mu P_n \right\|_{p,w_n} \right)$$
$$\le 2c_0 \left\| \lambda_n^\mu P_n \right\|_{p,w_n}.$$

In the other direction, the sequence of inequalities is exactly the same (switching w and w_n). This verifies (6.2). $\square$

If we allow constants to depend on μ, then we have the following result.

Corollary 6.5. *Let w be a doubling weight, $0 < p < \infty$, $n \in \mathbb{N}$ and $\mu \geq 0$. Then, for any $P_n \in \Pi_n$,*

$$\left\| \varphi_n^\mu P_n \right\|_{p,w} \sim \left\| \varphi^\mu P_n \right\|_{p,w} \sim \left\| \varphi^\mu P_n \right\|_{p,w_n} \sim \left\| \varphi_n^\mu P_n \right\|_{p,w_n},$$

where the equivalence constants depend only on p, μ and the doubling constant of w.

Proof. Since $\lambda_n(x) \sim \varphi_n(x) \sim \varphi(x) + 1/n$ and $\varphi(x) \leq \varphi(x) + 1/n \sim \varphi_n(x)$, we immediately get from Theorem 6.4

$$\left\| \varphi^\mu P_n \right\|_{p,w} \sim \left\| \varphi^\mu P_n \right\|_{p,w_n} \leq c \left\| \varphi_n^\mu P_n \right\|_{p,w_n} \sim \left\| \varphi_n^\mu P_n \right\|_{p,w}.$$

The following sequence finishes the proof:

$$\left\| \varphi^\mu P_n \right\|_{p,w} = \| P_n \|_{p,w\varphi^{\mu p}} \sim \| P_n \|_{p,(w\varphi^{\mu p})_n} \sim \| P_n \|_{p,w_n \varphi_n^{\mu p}} = \left\| \varphi_n^\mu P_n \right\|_{p,w_n}. \quad \square$$

6.2. Markov–Bernstein type theorems

In this subsection, we continue with the applications of the results presented in the first part of this section and discuss several Markov–Bernstein estimates for doubling weights.

We note that the following theorem can be obtained from [20, Theorem 4.1] and [13, Theorem 3.1] (Markov–Bernstein estimate for trigonometric polynomials) with the same proof as that of [20, Theorem 7.3, (7.10) and (7.12)]. However, we provide an alternative proof using the equivalence results from the previous section.

Theorem 6.6. *Let w be a doubling weight, $0 < p < \infty$ and $r \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$ and $P_n \in \Pi_n$,*

$$n^{-r} \left\| \varphi^r P_n^{(r)} \right\|_{p,w} \sim n^{-r} \left\| \varphi^r P_n^{(r)} \right\|_{p,w_n} \sim \left\| \rho_n^r P_n^{(r)} \right\|_{p,w_n} \sim \left\| \rho_n^r P_n^{(r)} \right\|_{p,w}$$
$$\leq c \| P_n \|_{p,w} \sim \| P_n \|_{p,w_n},$$

where the constant c and the equivalence constants depend only on r, p and the doubling constant of w.

Proof. The statement of the lemma is an immediate consequence of Corollary 6.5 and the following estimate (see [16, Lemma 6.1], for example)

$$\left\| \rho_n^r P_n^{(r)} \right\|_{p,w_n} \leq c \| P_n \|_{p,w_n},$$

where the constant c depends only on r, p and the doubling constant of w. $\quad \square$

In the proof of inverse results for $0 < p < 1$ we need to know how the constants in Markov–Bernstein estimates depend on the order of derivatives.

We start with the following result that was proved in [16] (see Corollaries 6.4 and 6.6 there).

Lemma 6.7. *Let w be a doubling weight and $0 < p < 1$. Then, for all $n, r \in \mathbb{N}$ and $l \in \mathbb{N}_0$ such that $l \leq r \leq n - 1$, and $P_n \in \Pi_n$,*

$$\left\| \delta_n^r P_n^{(r)} \right\|_{p,w_n} \leq (c_*)^{r-l} \frac{r!}{l!} \left\| \delta_n^l P_n^{(l)} \right\|_{p,w_n}$$

and

$$\left\| \varphi^r P_n^{(r)} \right\|_{p, w_n} \leq (c_*)^{r-l} \frac{r!}{l!} n^{r-l} \left\| \varphi^l P_n^{(l)} \right\|_{p, w_n},$$

where $\delta_n(x) := \max\left\{ \sqrt{1-x^2}/n, 1/n^2 \right\}$, and the constant c_* depends only on p and the doubling constant of w.

We remark that if we are not interested in the exact dependence of the constants on l (the order of the lower derivative in the estimates), then the first estimate in Lemma 6.7 and Corollary 6.5 imply the following (weaker) analog of the second estimate in Lemma 6.7 which actually would have been sufficient for our purposes:

$$\left\| \varphi^r P_n^{(r)} \right\|_{p, w_n} \leq n^r \left\| \delta_n^r P_n^{(r)} \right\|_{p, w_n} \leq c(c_*)^r r! n^r \left\| \delta_n^l P_n^{(l)} \right\|_{p, w_n} \leq c(c_*)^r r! n^{r-l} \left\| \varphi^l P_n^{(l)} \right\|_{p, w_n},$$

where c is allowed to depend on l in addition to p and the doubling constant of w.

Taking into account Theorem 6.4 and observing that $\delta_n(x) = \lambda_n(x)/n$ we immediately get the following corollary (in order not to overcomplicate the notation we incorporate the extra constant into c_*, i.e., we emphasize once again that constants c_* in different statements are different).

Corollary 6.8. *Let w be a doubling weight and $0 < p < 1$. Then, for all $n, r \in \mathbb{N}$ and $l \in \mathbb{N}_0$ such that $l \leq r \leq n-1$, and $P_n \in \Pi_n$,*

$$\left\| \delta_n^r P_n^{(r)} \right\|_{p, w} \leq (c_*)^{r-l} \frac{r!}{l!} \left\| \delta_n^l P_n^{(l)} \right\|_{p, w}$$

and

$$\left\| \varphi^r P_n^{(r)} \right\|_{p, w} \leq (c_*)^{r-l} \frac{r!}{l!} n^{r-l} \left\| \varphi^l P_n^{(l)} \right\|_{p, w},$$

where the constant c_* depends only on p and the doubling constant of w.

Now, taking into account that $\delta_n(x) \leq \rho_n(x) \leq 2\delta_n(x)$, this immediately implies the following.

Corollary 6.9. *Let w be a doubling weight and $0 < p < 1$. Then, for all $n, r \in \mathbb{N}$ and $l \in \mathbb{N}_0$ such that $l \leq r \leq n-1$, and $P_n \in \Pi_n$,*

$$\left\| \rho_n^r P_n^{(r)} \right\|_{p, w} \leq 2^l (c_*)^{r-l} \frac{r!}{l!} \left\| \rho_n^l P_n^{(l)} \right\|_{p, w},$$

where the constant c_* depends only on p and the doubling constant of w.

7. Two crucial auxiliary lemmas

In the case $1 \leq p < \infty$, we have the following lemma.

Lemma 7.1. *Let w be a doubling weight, $1 \leq p < \infty$ and $A > 0$. Then for any $n, r \in \mathbb{N}$, $I := \mathcal{Z}_{A, 1/n}^j$, and any polynomials $Q_n \in \Pi_n$ and $q_r \in \Pi_r$ satisfying $Q_n^{(v)}(z_j) = q_r^{(v)}(z_j)$, $0 \leq v \leq r-1$, the following inequality holds*

$$\| Q_n - q_r \|_{\mathbb{L}_p(I), w} \leq c n^{-r} \left\| \varphi^r Q_n^{(r)} \right\|_{p, w},$$

where the constant c depends only on r, p, A and the doubling constant of w.

Remark 7.2. Using the same proof it is possible to show that, for any f such that $f^{(r-1)} \in AC(I)$,

$$E_r(f)_{\mathbb{L}_p(I),w} \le c \left\| \rho_n^r f^{(r)} \right\|_{\mathbb{L}_p(I),w_n}.$$

At the same time, w_n on the right-hand side of this estimate cannot be replaced with w since, otherwise, together with Lemma 8.2 and Theorem 5.2 we would get the estimate $E_n(f)_{p,w} \le c \left\| \rho_n^r f^{(r)} \right\|_{p,w}$ which is not valid for all doubling weights (see [18, Example 3.5]). In fact, even the estimate $E_r(f)_{\mathbb{L}_p(I),w} \le c \left\| \rho_n^r f^{(r)} \right\|_{p,w}$ is invalid in general.

Proof of Lemma 7.1. The proof is rather straightforward and relies on Taylor's theorem (see e.g. [4, Proposition 4.1]). However, since it is short and works for all doubling weights, we sketch it below for completeness. Denote $z := z_j$, and note that $(Q_n - q_r)^{(v)}(z) = 0$, $0 \le v \le r - 1$, and that we can assume that $n \ge r + 1$. Using Taylor's theorem with the integral remainder we have

$$Q_n(x) - q_r(x) = \frac{1}{(r-1)!} \int_z^x (x - u)^{r-1} Q_n^{(r)}(u) du.$$

Hence, using Hölder's inequality (with $1/p + 1/p' = 1$) we have

$$
\begin{aligned}
\|Q_n - q_r\|_{\mathbb{L}_p(I),w}^p &\le \int_I w(x) \left[\int_{[z,x]} |x - u|^{r-1} |Q_n^{(r)}(u)| du \right]^p dx \\
&\le \int_I w(x) \left(\int_{[z,x]} |x - u|^{(r-1)p'} du \right)^{p/p'} \int_{[z,x]} |Q_n^{(r)}(u)|^p du dx \\
&\le \int_I w(x) |x - z|^{rp-1} \int_{[z,x]} |Q_n^{(r)}(u)|^p du dx \\
&\le (A\rho_n(z))^{rp-1} \left\| Q_n^{(r)} \right\|_{\mathbb{L}_p(I)}^p \int_I w(x) dx.
\end{aligned}
$$

Now, using the fact that $w(I) \le cw([z - \rho_n(z), z + \rho_n(z)])$ with c depending only on A and the doubling constant of w, and the fact that $\rho_n(x) \sim \rho_n(z)$ and $w_n(x) \sim w_n(z)$, for $x \in I$, we have

$$
\begin{aligned}
\|Q_n - q_r\|_{\mathbb{L}_p(I),w}^p &\le cw_n(z) \left\| \rho_n^r Q_n^{(r)} \right\|_{\mathbb{L}_p(I)}^p \le c \left\| \rho_n^r Q_n^{(r)} \right\|_{\mathbb{L}_p(I),w_n}^p \le c \left\| \rho_n^r Q_n^{(r)} \right\|_{p,w_n}^p \\
&\le cn^{-r} \left\| \varphi^r Q_n^{(r)} \right\|_{p,w}^p,
\end{aligned}
$$

where the last estimate follows from Corollary 6.5. $\quad\square$

If $0 < p < 1$, we no longer can use Hölder's inequality in a straightforward way, and so it takes much more effort to get an analog of Lemma 7.1. If there is a simple proof of the following lemma, we were unable to find it.

Lemma 7.3. Let w be a doubling weight and $0 < p < 1$. Then there exists a positive constant $\theta \le 1$ depending only on p and the doubling constant of w such that, for $n, r \in \mathbb{N}$, $I := \mathcal{Z}_{\theta,1/n}^j$, and any polynomials $Q_n \in \Pi_n$ and $q_r \in \Pi_r$ satisfying $Q_n^{(v)}(z_j) = q_r^{(v)}(z_j)$, $0 \le v \le r - 1$, the

following inequality holds

$$\|Q_n - q_r\|_{\mathbb{L}_p(I),w} \le cn^{-r} \left\| \varphi^r Q_n^{(r)} \right\|_{p,w},$$

where the constant c depends only on r, p and the doubling constant of w.

Proof. We use the approach from [11, Section 6]. Denote $g := Q_n - q_r$ and $z := z_j$, and note that $g^{(\nu)}(z) = 0$, $0 \le \nu \le r-1$, and $g^{(r)} = Q_n^{(r)}$. Using Taylor's theorem with the integral remainder we have

$$g(x) = \frac{1}{(r-1)!} \int_z^x (x-u)^{r-1} g^{(r)}(u) du.$$

Hence,

$$\|Q_n - q_r\|_{\mathbb{L}_p(I),w}^p = \|g\|_{\mathbb{L}_p(I),w}^p = \int_I |g(x)|^p w(x) dx$$

$$\le \int_I \left| \int_z^x (x-u)^{r-1} g^{(r)}(u) w(x)^{1/p} du \right|^p dx$$

$$\le \int_I \left| \int_z^x \left| (x-u)^{r-1} g^{(r)}(u) \right|^{1-p} w(x)^{-1+1/p} \times \left| (x-u)^{r-1} g^{(r)}(u) \right|^p w(x) du \right|^p dx$$

$$\le \int_I \left\| \left| (x-u)^{r-1} g^{(r)}(u) \right|^{1-p} w(x)^{-1+1/p} \right\|_{\mathbb{L}_\infty[z,x]}^p$$

$$\times \left| \int_z^x \left| (x-u)^{r-1} g^{(r)}(u) \right|^p w(x) du \right|^p dx.$$

Now, using Hölder's inequality with $\sigma_1 = 1/(1-p)$ and $\sigma_2 = 1/p$ (note that $1/\sigma_1 + 1/\sigma_2 = 1$) we have

$$\|g\|_{\mathbb{L}_p(I),w}^p \le \left[\int_I \left\| \left| (x-u)^{r-1} g^{(r)}(u) \right|^{1-p} w(x)^{-1+1/p} \right\|_{\mathbb{L}_\infty[z,x]}^{p/(1-p)} dx \right]^{1/\sigma_1}$$

$$\times \left[\int_I \left| \int_z^x \left| (x-u)^{r-1} g^{(r)}(u) \right|^p w(x) du \right| dx \right]^{1/\sigma_2}$$

$$\le \left[\int_I \left\| \left| (x-u)^{r-1} g^{(r)}(u) \right|^p w(x) \right\|_{\mathbb{L}_\infty[z,x]} dx \right]^{1-p}$$

$$\times \left[\int_I \left| \int_z^x \left| (x-u)^{r-1} g^{(r)}(u) \right|^p w(x) du \right| dx \right]^p =: T_1^p \times T_2^p.$$

To estimate T_2 we recall that $[z,x] := [x,z]$ if $x < z$ and write

$$T_2 = \int_I w(x) \int_{[z,x]} |x-u|^{(r-1)p} |g^{(r)}(u)|^p du dx$$

$$\le \int_I \left| g^{(r)}(u) \right|^p du \int_I w(x) |x-z|^{(r-1)p} dx$$

$$\le (\theta \rho_n(z))^{(r-1)p} \left\| g^{(r)} \right\|_{\mathbb{L}_p(I)}^p \int_I w(x) dx$$

$$\le (\theta \rho_n(z))^{(r-1)p} \rho_n(z) w_n(z) \left\| g^{(r)} \right\|_{\mathbb{L}_p(I)}^p .$$

Since $w_n(x) \sim w_n(z)$ and $\rho_n(x) \sim \rho_n(z)$, $x \in I$, this implies

$$T_2 \le c\rho_n(z)^{1-p} \left\| \rho_n^r g^{(r)} \right\|_{\mathbb{L}_p(I), w_n}^p.$$

Now, we need to estimate

$$T_1^{p/(1-p)} = \int_I \left\| \left| (x-u)^{r-1} g^{(r)}(u) \right|^p w(x) \right\|_{\mathbb{L}_\infty[z,x]} dx.$$

For u between z and x we have

$$
|x-u|^{(r-1)p} |g^{(r)}(u)|^p = |x-u|^{(r-1)p} \left| \sum_{\nu=r}^{n-1} \frac{g^{(\nu)}(x)}{(\nu-r)!} (u-x)^{\nu-r} \right|^p
$$
$$
\le \sum_{\nu=r}^{n-1} |g^{(\nu)}(x)|^p \left[\frac{|x-z|^{\nu-1}}{(\nu-r)!} \right]^p,
$$

and so

$$
T_1^{p/(1-p)} \le \int_I w(x) \sum_{\nu=r}^{n-1} |g^{(\nu)}(x)|^p \left[\frac{|x-z|^{\nu-1}}{(\nu-r)!} \right]^p dx
$$
$$
\le c \int_I w(x) \sum_{\nu=r}^{n-1} |\rho_n(x)^\nu g^{(\nu)}(x)|^p \left[\frac{|x-z|^{\nu-1}}{\rho_n(z)^\nu (\nu-r)!} \right]^p dx
$$
$$
\le c \sum_{\nu=r}^{n-1} \left[\frac{(\theta \rho_n(z))^{\nu-1}}{\rho_n(z)^\nu (\nu-r)!} \right]^p \left\| \rho_n^\nu g^{(\nu)} \right\|_{\mathbb{L}_p(I), w}^p
$$
$$
\le c \sum_{\nu=r}^{n-1} \left[\frac{\theta^{\nu-1}}{\rho_n(z)(\nu-r)!} \right]^p \left\| \rho_n^\nu g^{(\nu)} \right\|_{p, w}^p.
$$

We now use Corollary 6.9 to conclude

$$
T_1^{p/(1-p)} \le c \sum_{\nu=r}^{n-1} \left[\frac{\theta^{\nu-1}}{\rho_n(z)(\nu-r)!} 2^r (c_*)^{\nu-r} \frac{\nu!}{r!} \right]^p \left\| \rho_n^r g^{(r)} \right\|_{p, w}^p
$$
$$
\le c\rho_n(z)^{-p} \left\| \rho_n^r g^{(r)} \right\|_{p, w}^p \sum_{\nu=r}^{\infty} (\theta c_*)^{\nu p} \left[\binom{\nu}{r} \right]^p
$$
$$
\le c\rho_n(z)^{-p} \left\| \rho_n^r g^{(r)} \right\|_{p, w}^p,
$$

provided $\theta c_* \le 1/2$. Therefore,

$$T_1 \le c\rho_n(z)^{p-1} \left\| \rho_n^r g^{(r)} \right\|_{p, w}^{1-p}.$$

Combining estimates of T_1 and T_2, we have

$$
\|g\|_{\mathbb{L}_p(I)} \le c \left\| \rho_n^r g^{(r)} \right\|_{\mathbb{L}_p(I), w_n}^p \left\| \rho_n^r g^{(r)} \right\|_{p, w}^{1-p} \le c \left\| \rho_n^r g^{(r)} \right\|_{p, w_n}^p \left\| \rho_n^r g^{(r)} \right\|_{p, w}^{1-p}
$$
$$
\le cn^{-r} \left\| \varphi^r g^{(r)} \right\|_{p, w},
$$

noting that the last estimate immediately follows from Corollary 6.5. $\square$

8. Preliminary results for inverse theorems

Lemma 8.1. *If w is a doubling weight from the class $\mathcal{W}(\mathcal{Z})$, $0 < p < \infty$, $f \in \mathbb{L}_p^w$, $r \in \mathbb{N}$, and $A, t > 0$, then*

$$\omega_\varphi^r(f, A, t)_{p,w} \leq c \, \|f\|_{p,w} \,,$$

where c depends only on r, p, A and the weight w.

Proof. First of all, it is clear that

$$\sum_{j=1}^{M} E_r(f)_{\mathbb{L}_p(\mathcal{Z}_{2A,t}^j),w} \leq \sum_{j=1}^{M} \|f\|_{\mathbb{L}_p(\mathcal{Z}_{2A,t}^j),w} \leq M \, \|f\|_{p,w} \,.$$

Now, recall that $\Delta_{h\varphi(x)}^r(f, x, \mathcal{I}_{A,h}) = 0$ if $x \notin S_{A,h} \subset \mathfrak{D}_{rh/2}$, where

$$S_{A,h} := \left\{ x \mid [x - rh\varphi(x)/2, x + rh\varphi(x)/2] \subset \mathcal{I}_{A,h} \right\}$$

and

$$\begin{aligned}
\mathfrak{D}_{rh/2} &:= \left\{ x \mid x \neq \pm 1 \text{ and } x \pm rh\varphi(x)/2 \in [-1, 1] \right\} \\
&= \left\{ x \mid |x| \leq (4 - r^2 h^2)/(4 + r^2 h^2) \right\},
\end{aligned}$$

and so

$$\Omega_\varphi^r(f, A, t)_{p,w} = \sup_{0 < h \leq t} \left\| \Delta_{h\varphi}^r(f) \right\|_{\mathbb{L}_p(S_{A,h}),w} \,.$$

Let $h \in (0, t]$ be fixed, $x \in S_{A,h}$ and denote $y_i(x) := x + (i - r/2)h\varphi(x)$, $0 \leq i \leq r$. Then, $[x, y_i(x)] \subset \mathcal{I}_{A,h}$ and $|x - y_i(x)| \leq (r/2)\rho(h, x)$, and so Lemma 2.3(iv) implies that $w(x) \sim w(y_i(x))$.

Now, taking into account that $1/2 \leq y_i'(x) \leq 3/2$, $x \in \mathfrak{D}_{rh/2}$, we have

$$\begin{aligned}
\left\| \Delta_{h\varphi}^r(f) \right\|_{\mathbb{L}_p(S_{A,h}),w}^p &\leq \int_{S_{A,h}} w(x) \left(\sum_{i=0}^{r} \binom{r}{i} |f(x + (i - r/2)h\varphi(x))| \right)^p dx \\
&\leq c \sum_{i=0}^{r} \int_{S_{A,h}} w(y_i(x)) \, |f(y_i(x))|^p \, dx \\
&\leq c \int_{-1}^{1} w(y) \, |f(y)|^p \, dy \leq c \, \|f\|_{p,w}^p \,. \qquad \square
\end{aligned}$$

Lemma 8.2. *Let w be a doubling weight from the class $\mathcal{W}(\mathcal{Z})$, $1 \leq p < \infty$ $n, r \in \mathbb{N}$ and $A, t > 0$. If f is such that $f^{(r-1)} \in \mathrm{AC}_{\mathrm{loc}}((-1, 1) \setminus \mathcal{Z})$ and $\left\| \varphi^r f^{(r)} \right\|_{p,w} < \infty$, then*

$$\Omega_\varphi^r(f, A, t)_{p,w} \leq ct^r \left\| \varphi^r f^{(r)} \right\|_{p,w} \,,$$

where c depends only on r, A, p and the weight w.

Proof. Recall that

$$\Omega_\varphi^r(f, A, t)_{p,w} = \sup_{0 < h \leq t} \left\| \Delta_{h\varphi}^r(f) \right\|_{\mathbb{L}_p(S_{A,h}),w} \,,$$

where

$$S_{A,h} := \left\{ x \mid [x - rh\varphi(x)/2, x + rh\varphi(x)/2] \subset \mathcal{I}_{A,h} \right\}.$$

Since $\mathcal{I}_{A,h}$ has at most $M + 1$ components, it is sufficient (and necessary) to verify the lemma for each of them. We have two different types of components: when a component is "in the middle" of $[-1, 1]$, i.e.,

$$J_{A,h}^j := [z_j + A\rho(h, z_j), z_{j+1} - A\rho(h, z_{j+1})], \quad \text{where } 1 \leq j \leq M - 1,$$

and when a component is near the endpoints of $[-1, 1]$. Note that there is a component of this type only when $z_1 \neq -1$ and $z_M \neq 1$. More precisely, define

$$J_{A,h}^0 := [-1, z_1 - A\rho(h, z_1)] \quad \text{if } z_1 \neq -1$$

and

$$J_{A,h}^M := [z_M + A\rho(h, z_j), 1] \quad \text{if } z_M \neq 1.$$

Recall that $\Delta_{h\varphi(x)}^r(f, x, \mathcal{I}_{A,h}) = 0$ if $x \in J_{A,h}^j$ and $[x - rh\varphi(x)/2, x + rh\varphi(x)/2] \not\subset J_{A,h}^j$, and so we also denote

$$S_{A,h}^j := \left\{ x \mid [x - rh\varphi(x)/2, x + rh\varphi(x)/2] \subset J_{A,h}^j \right\}, \quad 0 \leq j \leq M.$$

Suppose now that $1 \leq p < \infty$ and let $h \in (0, t]$ be fixed. Since f has the $(r - 1)$st locally absolutely continuous derivative inside each $S_{A,h}^j$, we have for any $x \in S_{A,h}^j$

$$\Delta_{h\varphi(x)}^r(f, x) = \int_{-h\varphi(x)/2}^{h\varphi(x)/2} \cdots \int_{-h\varphi(x)/2}^{h\varphi(x)/2} f^{(r)}(x + t_1 + \cdots + t_r) dt_r \cdots dt_1,$$

and, by Lemma 2.3(iv), $w(x) \sim w(u)$, for $u \in [x - rh\varphi(x)/2, x + rh\varphi(x)/2]$.

Therefore,

$$\left(\int_{S_{A,h}^j} w(x) |\Delta_{h\varphi(x)}^r(f, x)|^p dx \right)^{1/p}$$

$$\leq \left(\int_{S_{A,h}^j} \left[\int_{-h\varphi(x)/2}^{h\varphi(x)/2} \cdots \int_{-h\varphi(x)/2}^{h\varphi(x)/2} w(x)^{1/p} |f^{(r)}(x + t_1 + \cdots + t_r)| dt_r \cdots dt_1 \right]^p dx \right)^{1/p}$$

$$\leq c \left(\int_{S_{A,h}^j} \left[\int_{-h\varphi(x)/2}^{h\varphi(x)/2} \cdots \int_{-h\varphi(x)/2}^{h\varphi(x)/2} w(x + t_1 + \cdots + t_r)^{1/p} \right. \right.$$

$$\left. \left. \times |f^{(r)}(x + t_1 + \cdots + t_r)| dt_r \cdots dt_1 \right]^p dx \right)^{1/p}.$$

By Hölder's inequality, for each u satisfying $[x + u - h\varphi(x)/2, x + u + h\varphi(x)/2] \subset S_{A,h}^j$, we have

$$\int_{-h\varphi(x)/2}^{h\varphi(x)/2} w(x + u + t_r)^{1/p} |f^{(r)}(x + u + t_r)| dt_r = \int_{x+u-h\varphi(x)/2}^{x+u+h\varphi(x)/2} w(v)^{1/p} |f^{(r)}(v)| dv$$

$$\leq \left\| w^{1/p} \varphi^r f^{(r)} \right\|_{\mathbb{L}_p(\mathcal{A}(x,u))} \left\| \varphi^{-r} \right\|_{\mathbb{L}_{p'}(\mathcal{A}(x,u))},$$

where $1/p + 1/p' = 1$ and

$$\mathcal{A}(x, u) := [x + u - h\varphi(x)/2, \, x + u + h\varphi(x)/2].$$

The needed estimate now follows from

$$\int_{S_{A,h}^j} \left[\int_{-h\varphi(x)/2}^{h\varphi(x)/2} \cdots \int_{-h\varphi(x)/2}^{h\varphi(x)/2} \left\| \varphi^{-r} \right\|_{\mathbb{L}_{p'}(\mathcal{A}(x, t_1 + \cdots + t_{r-1}))} \right.$$

$$\left. \times \left\| w^{1/p} \varphi^r f^{(r)} \right\|_{\mathbb{L}_p(\mathcal{A}(x, t_1 + \cdots + t_{r-1}))} dt_{r-1} \cdots dt_1 \right]^p dx \le ch^{rp} \left\| w^{1/p} \varphi^r f^{(r)} \right\|_p^p,$$

$$1 \le p < \infty. \tag{8.1}$$

Note that, in the case $r = 1$, (8.1) is understood as

$$\int_{S_{A,h}^j} \left\| \varphi^{-1} \right\|_{\mathbb{L}_{p'}(\mathcal{A}(x,0))}^p \left\| w^{1/p} \varphi f' \right\|_{\mathbb{L}_p(\mathcal{A}(x,0))}^p dx \le ch^p \left\| w^{1/p} \varphi f' \right\|_p^p, \quad 1 \le p < \infty. \tag{8.2}$$

Estimates (8.1) and (8.2) are proved in exactly the same way as [17, (4.2)–(4.4)]. $\quad\square$

The following lemma can be proved using exactly the same sequence of estimates that were used to prove [16, Lemma 6.9] with the only difference that the second estimate of Corollary 6.8 should be used instead of [16, Corollary 6.6].

Lemma 8.3. *Let w be a doubling weight, $0 < p < 1$ and $n, r \in \mathbb{N}$. Then, there exists a positive constant ϑ depending only on r, p and the doubling constant of w, such that, for any $P_n \in \Pi_n$ and $0 < h \le \vartheta/n$,*

$$(1/2)^{1/p} h^r \left\| \varphi^r P_n^{(r)} \right\|_{p,w} \le \left\| \Delta_{h\varphi}^r (P_n) \right\|_{p,w} \le (3/2)^{1/p} h^r \left\| \varphi^r P_n^{(r)} \right\|_{p,w}.$$

Taking into account that

$$\Omega_\varphi^r (P_n, A, t)_{p,w} = \sup_{0 < h \le t} \left\| \Delta_{h\varphi}^r (P_n) \right\|_{\mathbb{L}_p(\mathcal{I}_{A,h}), w} \le \sup_{0 < h \le t} \left\| \Delta_{h\varphi}^r (P_n) \right\|_{p,w}$$

we immediately get the following corollary.

Corollary 8.4. *Let w be a doubling weight, $0 < p < 1$ and $n, r \in \mathbb{N}$. Then, there exists a positive constant $\vartheta \le 1$ depending only on r, p and the doubling constant of w, such that, for any $P_n \in \Pi_n$, $A > 0$ and $0 < t \le \vartheta/n$,*

$$\Omega_\varphi^r (P_n, A, t)_{p,w} \le n^{-r} \left\| \varphi^r P_n^{(r)} \right\|_{p,w}.$$

9. Inverse theorem for $1 \le p < \infty$

Theorem 9.1. *Suppose that w is a doubling weight from the class $\mathcal{W}(\mathcal{Z})$, $r \in \mathbb{N}$, $1 \le p < \infty$, and $f \in \mathbb{L}_p^w$. Then*

$$\omega_\varphi^r (f, A, n^{-1})_{p,w} \le cn^{-r} \sum_{k=1}^n k^{r-1} E_k(f)_{p,w},$$

where the constant c depends only on r, p, A and the weight w.

Proof. Let $P_n^* \in \Pi_n$ denote a polynomial of (near) best approximation to f with weight w, i.e.,

$$c \left\| f - P_n^* \right\|_{p,w} \leq \inf_{P_n \in \Pi_n} \| f - P_n \|_{p,w} =: E_n(f)_{p,w}.$$

We let $N \in \mathbb{N}$ be such that $2^N \leq n < 2^{N+1}$. To estimate $\Omega_\varphi^r(f, A, n^{-1})_{p,w}$, using Lemma 8.1 we have

$$
\begin{aligned}
\Omega_\varphi^r(f, A, n^{-1})_{p,w} &\leq \Omega_\varphi^r(f, A, 2^{-N})_{p,w} \\
&\leq \Omega_\varphi^r(f - P_{2^N}^*, A, 2^{-N})_{p,w} + \Omega_\varphi^r(P_{2^N}^*, A, 2^{-N})_{p,w} \\
&\leq c \left\| f - P_{2^N}^* \right\|_{p,w} + \Omega_\varphi^r(P_{2^N}^*, A, 2^{-N})_{p,w} \\
&\leq c E_{2^N}(f)_{p,w} + \Omega_\varphi^r(P_{2^N}^*, A, 2^{-N})_{p,w}.
\end{aligned}
$$

Now, using

$$P_{2^N}^* = P_1^* + \sum_{i=0}^{N-1} (P_{2^{i+1}}^* - P_{2^i}^*) \tag{9.1}$$

as well as Lemma 8.2 we have

$$
\begin{aligned}
\Omega_\varphi^r(P_{2^N}^*, A, 2^{-N})_{p,w} &\leq \sum_{i=0}^{N-1} \Omega_\varphi^r \left(P_{2^{i+1}}^* - P_{2^i}^*, A, 2^{-N} \right)_{p,w} \\
&\leq c 2^{-Nr} \sum_{i=0}^{N-1} \left\| \varphi^r \left(P_{2^{i+1}}^* - P_{2^i}^* \right)^{(r)} \right\|_{p,w}.
\end{aligned}
$$

Now, for each $1 \leq j \leq M$, taking into account that $\mathcal{Z}_{2A,t_1}^j \subset \mathcal{Z}_{2A,t_2}^j$ if $t_1 \leq t_2$, we have

$$
\begin{aligned}
E_r(f)_{\mathbb{L}_p(\mathcal{Z}_{2A,1/n}^j),w} &\leq \inf_{q \in \Pi_r} \| f - q \|_{\mathbb{L}_p(\mathcal{Z}_{2A,2^{-N}}^j),w} \\
&\leq \left\| f - P_{2^N}^* \right\|_{\mathbb{L}_p(\mathcal{Z}_{2A,2^{-N}}^j),w} + \inf_{q \in \Pi_r} \left\| P_{2^N}^* - q \right\|_{\mathbb{L}_p(\mathcal{Z}_{2A,2^{-N}}^j),w} \\
&\leq c E_{2^N}(f)_{p,w} + \left\| P_{2^N}^* - q_r(P_{2^N}^*) \right\|_{\mathbb{L}_p(\mathcal{Z}_{2A,2^{-N}}^j),w},
\end{aligned}
$$

where $q_r(g)$ denotes the Taylor polynomial of degree $< r$ at z_j for g. Using (9.1) again, noting that

$$q_r(P_{2^N}^*) = P_1^* + \sum_{i=0}^{N-1} q_r(P_{2^{i+1}}^* - P_{2^i}^*), \tag{9.2}$$

and taking Lemma 7.1 into account we have

$$
\begin{aligned}
\left\| P_{2^N}^* - q_r(P_{2^N}^*) \right\|_{\mathbb{L}_p(\mathcal{Z}_{2A,2^{-N}}^j),w} &\leq \sum_{i=0}^{N-1} \left\| (P_{2^{i+1}}^* - P_{2^i}^*) - q_r(P_{2^{i+1}}^* - P_{2^i}^*) \right\|_{\mathbb{L}_p(\mathcal{Z}_{2A,2^{-N}}^j),w} \\
&\leq c \sum_{i=0}^{N-1} 2^{-Nr} \left\| \varphi^r (P_{2^{i+1}}^* - P_{2^i}^*)^{(r)} \right\|_{p,w}.
\end{aligned}
$$

Hence,

$$\omega_\varphi^r(f, A, n^{-1})_{p,w} \leq c E_{2^N}(f)_{p,w} + c 2^{-Nr} \sum_{i=0}^{N-1} \left\| \varphi^r \left(P_{2^{i+1}}^* - P_{2^i}^* \right)^{(r)} \right\|_{p,w}.$$

Now, using Theorem 6.6 we have

$$
\omega_\varphi^r(f, A, n^{-1})_{p,w} \leq c E_{2^N}(f)_{p,w} + c 2^{-Nr} \sum_{i=0}^{N-1} 2^{ir} \left\| P_{2^{i+1}}^* - P_{2^i}^* \right\|_{p,w}
$$

$$
\leq c 2^{-Nr} \sum_{i=0}^{N} 2^{ir} E_{2^i}(f)_{p,w}
$$

$$
\leq c n^{-r} \left(E_1(f)_{p,w} + \sum_{i=1}^{N} \sum_{k=2^{i-1}+1}^{2^i} k^{r-1} E_k(f)_{p,w} \right)
$$

$$
\leq c n^{-r} \sum_{k=1}^{n} k^{r-1} E_k(f)_{p,w}
$$

with all constants c depending only on r, p, A and the weight w. $\square$

10. Inverse theorem for $0 < p < 1$

Theorem 10.1. *Suppose that w is a doubling weight from the class $\mathcal{W}(\mathcal{Z})$, and let $r \in \mathbb{N}$, $A > 0$, $0 < p < 1$, and $f \in \mathbb{L}_p^w$. Then there exists a positive constant $\vartheta \leq 1$ depending only on p, r, A and the doubling constant of w, and such that*

$$
\omega_\varphi^r(f, A, \vartheta n^{-1})_{p,w}^p \leq c n^{-rp} \sum_{k=1}^{n} k^{rp-1} E_k(f)_{p,w}^p,
$$

where the constant c depends only on r, p, A and the weight w.

Proof. The method of the proof is standard and well known (see [10] or [16]). With the same notation as in the proof of Theorem 9.1 (i.e., P_n^* is a polynomial of (near) best weighted approximation to f and $2^N \leq n < 2^{N+1}$), we have using Lemma 8.1 (note that we will be putting restrictions on ϑ as we go along)

$$
\Omega_\varphi^r(f, A, \vartheta n^{-1})_{p,w}^p \leq c E_{2^N}(f)_{p,w}^p + \Omega_\varphi^r(P_{2^N}^*, A, \vartheta 2^{-N})_{p,w}^p
$$

and, using (9.1),

$$
\Omega_\varphi^r(P_{2^N}^*, A, \vartheta 2^{-N})_{p,w}^p \leq \sum_{i=0}^{N-1} \Omega_\varphi^r \left(P_{2^{i+1}}^* - P_{2^i}^*, A, \vartheta 2^{-N} \right)_{p,w}^p.
$$

Lemma 8.2 can no longer be used, and so we employ Corollary 8.4 (we assume that the current constant $\vartheta \leq 1$ is not bigger than ϑ from Corollary 8.4) which implies

$$
\Omega_\varphi^r(P_{2^N}^*, A, \vartheta 2^{-N})_{p,w}^p \leq 2^{-Nrp} \sum_{i=0}^{N-1} \left\| \varphi^r \left(P_{2^{i+1}}^* - P_{2^i}^* \right)^{(r)} \right\|_{p,w}^p.
$$

For each $1 \leq j \leq M$, recalling that $q_r(g)$ denotes the Taylor polynomial of degree $< r$ at z_j for g, we have

$$
E_r(f)_{\mathbb{L}_p(\mathcal{Z}_{2A,\vartheta/n}^j),w}^p \leq \inf_{q \in \Pi_r} \| f - q \|_{\mathbb{L}_p(\mathcal{Z}_{2A,\vartheta 2^{-N}}^j),w}^p
$$

$$
\leq c E_{2^N}(f)_{p,w}^p + \left\| P_{2^N}^* - q_r(P_{2^N}^*) \right\|_{\mathbb{L}_p(\mathcal{Z}_{2A,\vartheta 2^{-N}}^j),w}^p.
$$

Now, we make sure that ϑ is so small that

$$\mathcal{Z}^j_{2A,\vartheta 2^{-N}} \subset \mathcal{Z}^j_{\theta,2^{-N}}, \quad 1 \le j \le M,$$

where θ is the constant from Lemma 7.3. This is achieved if $\vartheta \le \theta/(2A)$. Therefore, Lemma 7.3 implies

$$\|Q - q_r(Q)\|_{\mathbb{L}_p(\mathcal{Z}^j_{2A,\vartheta 2^{-N}}),w} \le c2^{-Nr} \left\| \varphi^r Q^{(r)} \right\|_{p,w}, \quad \text{for any } Q \in \Pi_{2^N},$$

with c depending only on r, p and the doubling constant of w. Hence, using (9.1) and (9.2) we obtain

$$\left\| P^*_{2^N} - q_r(P^*_{2^N}) \right\|^p_{\mathbb{L}_p(\mathcal{Z}^j_{2A,\vartheta 2^{-N}}),w} \le \sum_{i=0}^{N-1} \left\| (P^*_{2^{i+1}} - P^*_{2^i}) - q_r(P^*_{2^{i+1}} - P^*_{2^i}) \right\|^p_{\mathbb{L}_p(\mathcal{Z}^j_{2A,\vartheta 2^{-N}}),w}$$

$$\le c \sum_{i=0}^{N-1} 2^{-Nrp} \left\| \varphi^r (P^*_{2^{i+1}} - P^*_{2^i})^{(r)} \right\|^p_{p,w}.$$

Therefore,

$$\omega^r_\varphi(f, A, \vartheta n^{-1})^p_{p,w} \le c E_{2^N}(f)^p_{p,w} + c2^{-Nrp} \sum_{i=0}^{N-1} \left\| \varphi^r \left(P^*_{2^{i+1}} - P^*_{2^i} \right)^{(r)} \right\|^p_{p,w}.$$

Now, using Theorem 6.6, similarly to the case $1 \le p < \infty$, we get

$$\omega^r_\varphi(f, A, \vartheta n^{-1})^p_{p,w} \le c E_{2^N}(f)^p_{p,w} + c2^{-Nrp} \sum_{i=0}^{N-1} 2^{irp} \left\| P^*_{2^{i+1}} - P^*_{2^i} \right\|^p_{p,w}$$

$$\le c2^{-Nrp} \sum_{i=0}^{N} 2^{irp} E_{2^i}(f)^p_{p,w}$$

$$\le cn^{-rp} \sum_{k=1}^{n} k^{rp-1} E_k(f)^p_{p,w}$$

with all constants c depending only on r, p and the weight w. $\quad\square$

11. Equivalence of moduli and realization functionals

Let w be a doubling weight from the class $\mathcal{W}(\mathcal{Z})$, $r \in \mathbb{N}$, $0 < p < \infty$ and $f \in \mathbb{L}^w_p$.

Realization functionals were introduced by Hristov and Ivanov in [15]. They became a quite useful tool to measure smoothness of functions once it was observed that certain K-functionals are zero if $0 < p < 1$ and so cannot serve this purpose if $p < 1$ (see [9]). See [9,11,8] for more discussions.

We define the following realization functionals as follows

$$R_{r,\varphi}(f, t, \Pi_n)_{p,w} := \inf_{P_n \in \Pi_n} \left(\|f - P_n\|_{p,w} + t^r \left\| \varphi^r P_n^{(r)} \right\|_{p,w} \right).$$

Clearly, $R_{r,\varphi}(f, t_1, \Pi_n)_{p,w} \sim R_{r,\varphi}(f, t_2, \Pi_n)_{p,w}$ if $t_1 \sim t_2$.

Theorem 5.2 implies that, for every $n \geq N$ (with N depending only on r, p and w), $\vartheta > 0$ and $A > 0$, there exists a polynomial $P_n \in \Pi_n$ such that

$$R_{r,\varphi}(f, 1/n, \Pi_n)_{p,w} \leq c\widetilde{\omega}_\varphi^r(f, A, \vartheta/n)_{p,w}, \tag{11.1}$$

where constants c depend only on r, p, ϑ, A and the weight w.

Lemma 8.1 implies that, for any $0 < p < \infty$, $f \in \mathbb{L}_p^w$, $r \in \mathbb{N}$ and $A, t > 0$, and any $g \in \mathbb{L}_p^w$,

$$\Omega_\varphi^r(f, A, t)_{p,w} \leq c\Omega_\varphi^r(f - g, A, t)_{p,w} + c\Omega_\varphi^r(g, A, t)_{p,w}$$

$$\leq c\|f - g\|_{p,w} + c\Omega_\varphi^r(g, A, t)_{p,w}, \tag{11.2}$$

where c depends only on r, p, A and the weight w.

Now, in the case $1 \leq p < \infty$, Lemma 8.2 additionally yields that, if g is such that $g^{(r-1)} \in AC_{loc}((-1, 1) \setminus \mathcal{Z})$ and $\left\|\varphi^r g^{(r)}\right\|_{p,w} < \infty$, then

$$\Omega_\varphi^r(f, A, t)_{p,w} \leq c\|f - g\|_{p,w} + ct^r\left\|\varphi^r g^{(r)}\right\|_{p,w}.$$

This, in particular, implies that, if $1 \leq p < \infty$, then for any $n \in \mathbb{N}$, $\vartheta > 0$, $A > 0$ and $0 < t \leq \vartheta/n$,

$$\Omega_\varphi^r(f, A, t)_{p,w} \leq c\|f - P_n\|_{p,w} + cn^{-r}\left\|\varphi^r P_n^{(r)}\right\|_{p,w}, \tag{11.3}$$

where c depends only on r, p, ϑ, A and the weight w.

If we use Corollary 8.4 instead of Lemma 8.2 then we conclude that (11.3) is valid if $0 < p < 1$ as well, but now $0 < \vartheta \leq 1$ is some fixed constant that depends on r, p and the doubling constant of w.

Now, using Lemma 7.1 we have, for $1 \leq p < \infty$, any $\vartheta > 0$ and $0 < t \leq \vartheta/n$ (taking into account that $\mathcal{Z}_{2A,t}^j \subset \mathcal{Z}_{2A,\vartheta/n}^j \subset \mathcal{Z}_{2A\vartheta\max\{\vartheta,1\},1/n}^j$),

$$\sum_{j=1}^M \inf_{q \in \Pi_r} \|f - q\|_{\mathbb{L}_p(\mathcal{Z}_{2A,t}^j),w} \leq c\|f - P_n\|_{p,w} + \sum_{j=1}^M \inf_{q \in \Pi_r} \|P_n - q\|_{\mathbb{L}_p(\mathcal{Z}_{2A\vartheta\max\{\vartheta,1\},1/n}^j),w}$$

$$\leq c\|f - P_n\|_{p,w} + cn^{-r}\left\|\varphi^r P_n^{(r)}\right\|_{p,w}, \tag{11.4}$$

where constants c depend on r, p, A, ϑ and the doubling constant of w.

In the case $0 < p < 1$, using Lemma 7.3 we conclude that there exists $0 < \vartheta \leq 1$ depending only on p, A and the doubling constant of w such that, for $0 < t \leq \vartheta/n$, (11.4) is satisfied with constants c that depend on r, p, A, and the doubling constant of w. Note that this follows from the observation that $\mathcal{Z}_{2A,t}^j \subset \mathcal{Z}_{2A,\vartheta/n}^j \subset \mathcal{Z}_{2A\vartheta,1/n}^j \subset \mathcal{Z}_{\theta,1/n}^j$, where θ is the constant from the statement of Lemma 7.3 and $\vartheta := \min\{\theta/(2A), 1\}$.

Hence, we actually verified the validity of the following two corollaries. First, (11.1), (11.3) and (11.4) yield the following result.

Corollary 11.1. *Let w be a doubling weight from the class $\mathcal{W}(\mathcal{Z})$, $r \in \mathbb{N}$, $1 \leq p < \infty$ and $f \in \mathbb{L}_p^w$. Then there exists a constant $N \in \mathbb{N}$ depending on r, p and the weight w such that, for any $\vartheta_2 \geq \vartheta_1 > 0$, $n \geq N$, $\vartheta_1/n \leq t \leq \vartheta_2/n$, and $A > 0$, we have*

$$R_{r,\varphi}(f, t, \Pi_n)_{p,w} \sim \widetilde{\omega}_\varphi^r(f, A, t)_{p,w} \sim \omega_\varphi^r(f, A, t)_{p,w}.$$

Corollary 11.1 implies, in particular, that $\omega_\varphi^r(f, A_1, t_1)_{p,w} \sim \omega_\varphi^r(f, A_2, t_2)_{p,w}$ if $A_1 \sim A_2$ and $t_1 \sim t_2$.

In the case $0 < p < 1$, we have

Corollary 11.2. *Let w be a doubling weight from the class $\mathcal{W}(\mathcal{Z})$, $r \in \mathbb{N}$, $0 < p < 1$, $A > 0$, and $f \in \mathbb{L}_p^w$. Then there exist $N \in \mathbb{N}$ depending on r, p and the weight w, and $\vartheta > 0$ depending on r, p, A, and the doubling constant of w, such that, for any $\vartheta_1 \in (0, \vartheta]$, $n \geq N$, $\vartheta_1/n \leq t \leq \vartheta/n$, we have*

$$R_{r,\varphi}(f, t, \Pi_n)_{p,w} \sim \widetilde{\omega}_\varphi^r(f, A, t)_{p,w} \sim \omega_\varphi^r(f, A, t)_{p,w}.$$

Corollary 11.2 implies that, for $A_1, A_2 > 0$, $A_1 \sim A_2$, there exists $t_0 > 0$ such that $\omega_\varphi^r(f, A_1, t_1)_{p,w} \sim \omega_\varphi^r(f, A_2, t_2)_{p,w}$ for $0 < t_1, t_2 \leq t_0$ such that $t_1 \sim t_2$.

Acknowledgment

Appendix A

Lemma A.1. *Suppose that w is a doubling weight from the class $\mathcal{W}(\mathcal{Z})$, $0 < p < \infty$, $f \in \mathbb{L}_p^w$, and suppose that intervals I and J are such that $I \subset J \subset [-1, 1]$ and $|J| \leq c_0|I|$. Then, for any $r \in \mathbb{N}$, if $q \in \Pi_r$ is a polynomial of near best approximation to f on I in the $\mathbb{L}_p$ (quasi)norm with weight w, i.e.,*

$$\|f - q\|_{\mathbb{L}_p(I),w} \leq c_1 E_r(f)_{\mathbb{L}_p(I),w},$$

then q is also a polynomial of near best approximation to f on J. In other words,

$$\|f - q\|_{\mathbb{L}_p(J),w} \leq c E_r(f)_{\mathbb{L}_p(J),w},$$

where the constant c depends only on p, c_0, c_1 and the weight w.

Proof. First, we assume that $|I| \leq \mathcal{D}/2$, and so I may contain at most one z_j from $\mathcal{Z}$. Now, we denote by a the midpoint of I and let $n \in \mathbb{N}$ be such that

$$\rho_{n+1}(a) < |I|/1000 \leq \rho_n(a).$$

Then $|I|/1000 \leq \rho_n(a) \leq |I|/250$.

We recall again that $\rho_n(x) \leq |I_i| \leq 5\rho_n(x)$ for $x \in I_i$, and $|I_{i\pm1}| \leq 3|I_i|$. Hence, if $a \in I_\nu$, for some ν, then

$$\sum_{i=0}^2 |I_{\nu-i}| \leq (1 + 3 + 9)|I_\nu| = 13|I_\nu| \leq 65\rho_n(a) < |I|/2,$$

and so $I_{\nu-1} \cup I_{\nu-2} \subset I$. Similarly,

$$\sum_{i=0}^2 |I_{\nu+i}| \leq (1 + 3 + 9)|I_\nu| = 13|I_\nu| \leq 65\rho_n(a) < |I|/2,$$

and so $I_{\nu+1} \cup I_{\nu+2} \subset I$.

In other words, I contains at least 5 adjacent intervals $I_{\nu+i}$, $i = 2, 1, 0, -1, -2$. Since I contains at most one z_j, we now can pick one of these 5 intervals in such a way that there is another interval I_i between it and z_j (if I does not contain any z_j's, we pick one of the intervals "in the middle" of I, for example I_ν). Suppose that the interval that we picked is I_μ. Then,

$$|I| \geq |I_\mu| \geq |I_\nu|/9 \geq \rho_n(a)/9 \geq |I|/9000,$$

i.e., $|I_\mu| \sim |I|$. Also, $I_\mu \subset \mathcal{I}_{c,1/n}$ with some absolute constant c, and Lemma 2.3(iv) implies that $w(x) \sim w(y)$, for $x, y \in I_\mu$, with equivalence constants depending only on w.

Suppose now that $\widetilde{q}$ is a polynomial of near best approximation of f on J, i.e.,

$$\|f - \widetilde{q}\|_{\mathbb{L}_p(J),w} \leq c E_r(f)_{\mathbb{L}_p(J),w}.$$

Then, taking into account that $|I_\mu| \sim |I| \sim |J|$ and using properties of doubling weights (see [20, Lemma 2.1(vi) and Lemma 7.1], for example), we have

$$
\begin{aligned}
\|\widetilde{q} - q\|^p_{\mathbb{L}_p(J),w} &= \int_J w(x)|\widetilde{q}(x) - q(x)|^p dx \leq \|\widetilde{q} - q\|^p_{\mathbb{C}(J)} \int_J w(x)dx \\
&\leq c\,\|\widetilde{q} - q\|^p_{\mathbb{C}(I_\mu)} \int_{I_\mu} w(x)dx \leq c|I_\mu|^{-1} \|\widetilde{q} - q\|^p_{\mathbb{L}_p(I_\mu)} \int_{I_\mu} w(x_\mu)dx \\
&\leq c \int_{I_\mu} |\widetilde{q}(x) - q(x)|^p w(x_\mu)dx \leq c\,\|\widetilde{q} - q\|^p_{\mathbb{L}_p(I_\mu),w} \\
&\leq c\,\|\widetilde{q} - q\|^p_{\mathbb{L}_p(I),w}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|f - q\|_{\mathbb{L}_p(J),w} &\leq c\,\|f - \widetilde{q}\|_{\mathbb{L}_p(J),w} + c\,\|\widetilde{q} - q\|_{\mathbb{L}_p(J),w} \\
&\leq c\,\|f - \widetilde{q}\|_{\mathbb{L}_p(J),w} + c\,\|\widetilde{q} - q\|_{\mathbb{L}_p(I),w} \\
&\leq c\,\|f - \widetilde{q}\|_{\mathbb{L}_p(J),w} + c\,\|\widetilde{q} - f\|_{\mathbb{L}_p(I),w} + c\,\|f - q\|_{\mathbb{L}_p(I),w} \\
&\leq c\,\|f - \widetilde{q}\|_{\mathbb{L}_p(J),w} + c\,\|f - q\|_{\mathbb{L}_p(I),w} \\
&\leq c E_r(f)_{\mathbb{L}_p(J),w} + c E_r(f)_{\mathbb{L}_p(I),w} \\
&\leq c E_r(f)_{\mathbb{L}_p(J),w},
\end{aligned}
$$

and the proof is complete if $|I| \leq \mathcal{D}/2$.

If $|I| > \mathcal{D}/2$, then $|I| \sim |J| \sim 1$, and we take $n \in \mathbb{N}$ to be such I contains at least $4M + 4$ intervals I_i. Then I contains 4 adjacent intervals I_i not containing any points from $\mathcal{Z}$, and we can use the same argument as above. $\quad\square$

Appendix B. Definitions, notation and glossary of symbols

In this section, we repeat the main definitions and list the main notation used throughout this paper (or provide references to appropriate statements/formulas).

$\mathbb{N}$	$\{1, 2, 3, \ldots\}$
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$\mathbb{C}(I)$	Space of continuous functions on I
$\|f\|_{\mathbb{C}(I)}$	$\max_{x \in I} |f(x)|$

$\|f\|_{\mathbb{L}_p(I),w}$	$\left(\int_I |f(u)|^p w(u)du\right)^{1/p}$, $0 < p < \infty$
$\|f\|_{\mathbb{L}_p(I)}$	$\|f\|_{\mathbb{L}_p(I),1}$
$\|\cdot\|_{p,w}$	$\|\cdot\|_{\mathbb{L}_p[-1,1],w}$
$\|\cdot\|_p$	$\|\cdot\|_{p,1}$
$\mathbb{L}_p^w$	All measurable functions on $[-1,1]$ such that $\|f\|_{p,w} < \infty$
Π_r	Space of algebraic polynomials of degree $\leq r - 1$
$E_r(f)_{\mathbb{L}_p(I),w}$	$\inf_{q \in \Pi_r} \|f - q\|_{\mathbb{L}_p(I),w}$
$[x, y]$, $x > y$	$[y, x]$
$\|[x, y]\|$	$\|x - y\|$
$2I$	The interval of length $2|I|$ with the same center as I
$\varphi(x)$	$\sqrt{1 - x^2}$
$\rho(h, x)$	$h\varphi(x) + h^2$
$\rho_n(x)$	$n^{-1}\varphi(x) + n^{-2}$ $(=\rho(1/n, x))$
$\lambda_n(x)$	$\max\{\varphi(x), 1/n\}$
$w(I)$	$\int_{I \cap [-1,1]} w(u)du$
$w_n(x)$	$\rho_n(x)^{-1} w([x - \rho_n(x), x + \rho_n(x)])$
$\mathcal{Z}$	$(z_j)_{j=1}^M$, $-1 \leq z_1 < \cdots < z_{M-1} < z_M \leq 1$
$\mathcal{W}(\mathcal{Z})$	See Definition 2.1
$\mathrm{pmin}(S)$	The smallest *positive* number from the set S of nonnegative reals
$\mathcal{D} := \mathcal{D}(w)$	$\mathrm{pmin}\left\{|z_j - z_{j-1}| \mid 1 \leq j \leq M + 1\right\}$, where $z_0 := -1$, $z_{M+1} := 1$
$\mathcal{Z}_{A,h}^j$	$\left\{x \in [-1,1] \mid |x - z_j| \leq A\rho(h, z_j)\right\}$, $1 \leq j \leq M$
$\mathcal{Z}_{A,h}$	$\cup_{j=1}^M \mathcal{Z}_{A,h}^j$
$\mathcal{I}_{A,h}$	$\left([-1,1] \setminus \mathcal{Z}_{A,h}\right)^{cl} =$ $\left\{x \in [-1,1] \mid |x - z_j| \geq A\rho(h, z_j), \text{ for all } 1 \leq j \leq M\right\}$
$J_{A,h}^j$	$[z_j + A\rho(h, z_j), z_{j+1} - A\rho(h, z_{j+1})]$, $1 \leq j \leq M - 1$
$J_{A,h}^0$	$[-1, z_1 - A\rho(h, z_1)]$ if $z_1 \neq -1$
$J_{A,h}^M$	$[z_M + A\rho(h, z_j), 1]$ if $z_M \neq 1$
$S_{A,h}$	$\left\{x \mid [x - rh\varphi(x)/2, x + rh\varphi(x)/2] \subset \mathcal{I}_{A,h}\right\}$
$S_{A,h}^j$	$\left\{x \mid [x - rh\varphi(x)/2, x + rh\varphi(x)/2] \subset J_{A,h}^j\right\}$, $0 \leq j \leq M$
x_i	$\cos(i\pi/n)$
I_i	$[x_i, x_{i-1}]$
$\psi_i := \psi_i(x)$	$|I_i| / (|x - x_i| + |I_i|)$
$\chi_i(x) := \chi_{[x_i,1]}(x)$	1 if $x_i \leq x \leq 1$, and 0 otherwise
$\Delta_h^r(f, x, S)$	$\sum_{i=0}^r \binom{r}{i}(-1)^{r-i} f(x - rh/2 + ih)$ if $[x - rh/2, x + rh/2] \subset S$, and 0 otherwise (rth symmetric difference on S)
$\omega_r(f, t, I)_p$	$\sup_{0 < h \leq t} \left\|\Delta_h^r(f, x, I)\right\|_{\mathbb{L}_p(I)}$ (usual rth modulus of smoothness on an interval I)
$\Omega_\varphi^r(f, A, t)_{p,w}$	$\sup_{0 < h \leq t} \left\|\Delta_{h\varphi(\cdot)}^r(f, \cdot, \mathcal{I}_{A,h})\right\|_{\mathbb{L}_p(\mathcal{I}_{A,h}),w}$ (main part weighted modulus)
$\omega_\varphi^r(f, A, t)_{p,w}$	$\Omega_\varphi^r(f, A, t)_{p,w} + \sum_{j=1}^M E_r(f)_{\mathbb{L}_p(\mathcal{Z}_{2A,t}^j),w}$ (complete weighted modulus)

$\widetilde{\Omega}_\varphi^r(f, A, t)_{p,w}$ $\qquad$ $\left(t^{-1} \int_0^t \left\| \Delta_{h\varphi(\cdot)}^r(f, \cdot, \mathcal{I}_{A,h}) \right\|_{\mathbb{L}_p(\mathcal{I}_{A,h}),w}^p dh\right)^{1/p}$ (averaged main part weighted modulus)

$\widetilde{\omega}_\varphi^r(f, A, t)_{p,w}$ $\qquad$ $\widetilde{\Omega}_\varphi^r(f, A, t)_{p,w} + \sum_{j=1}^M E_r(f)_{\mathbb{L}_p(\mathcal{Z}_{2A,t}^j),w}$ (averaged weighted modulus)

$\widetilde{\Omega}_\varphi^r(f, t)_{\mathbb{L}_p(S),w}$ $\qquad$ $\left(t^{-1} \int_0^t \int_S w(x) |\Delta_{h\varphi(x)}^r(f, x, S)|^p dx dh\right)^{1/p}$, $S \subset [-1, 1]$ (restricted averaged main part modulus)

$R_{r,\varphi}(f, t, \Pi_n)_{p,w}$ $\qquad$ See Section 11 (realization functional)

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