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ON MONOTONE AND CONVEX APPROXIMATION BY ALGEBRAIC POLYNOMIALS

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The following results are obtained: If $\alpha > 0$, $\alpha \neq 2$, $\alpha \bar{\in} [3, 4]$, and f is a nondecreasing (convex) function on $[-1, 1]$ such that $E_n(f) \le n^{-\alpha}$ for any $n > \alpha$, then $E_n^{(1)}(f) \le Cn^{-\alpha}$ ($E_n^{(2)}(f) \le Cn^{-\alpha}$) for $n > \alpha$, where $C = C(\alpha)$, $E_n(f)$ is the best uniform approximation of a continuous function by polynomials of degree $(n-1)$, and $E_n^{(1)}(f)$ ($E_n^{(2)}(f)$) are the best monotone and convex approximations, respectively. For $\alpha = 2$ ($\alpha \in [3, 4]$), this result is not true.

1. Introduction and Principal Results

Recall that coapproximation (or shape-preserving approximation) is the approximation of functions f such that $\overline{\Delta}_h^q(f, x) \ge 0$ for given $q \in N$, all $0 \le h \le 2/q$, and $x \in [-1, 1]$, by polynomials with nonnegative qth derivatives. Here,

$$\overline{\Delta}_h^q(f, x) = \begin{cases} \sum_{i=0}^{q} (-1)^{q-i} \binom{q}{i} f(x + (i - q/2)h) & \text{if } |x \pm qh/2| \le 1, \\ 0, & \text{otherwise,} \end{cases}$$

is the qth symmetric difference.

Let Δ^q be the set of such functions f. Note that if $f \in C^q[a, b]$, then $f \in \Delta^q$ if and only if $f^{(q)}(x) \ge 0$, $x \in [-1, 1]$.

In the present paper, we consider the monotone and convex approximations by algebraic polynomials, i.e., the cases of $q = 1$ and $q = 2$, respectively. These kinds of coapproximation were extensively investigated in recent years. Many estimates of the degree of coapproximation were obtained in the cases of the uniform metric and L_p-metric, $0 < p < \infty$. The order of these estimates is often the same as in the case of unconstrained approximation.

The following theorem is a result of this type:

Theorem A. *Let* $\alpha > 0$. *If, for a nondecreasing (convex) function* $f = f(x)$ *on* $[-1, 1]$ *and any integer* $n > \alpha$, *there exists an algebraic polynomial* $p_{n-1} = p_{n-1}(x)$ *of the* $(n-1)$*th degree such that*

$$|f(x) - p_{n-1}(x)| \le \left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2} \right)^{\alpha}, \quad x \in [-1, 1],$$

then, for any $n > \alpha - 1$, *there is a nondecreasing (convex) polynomial* $p_{n-1}^* = p_{n-1}^*(x)$ *such that, for* $x \in [-1, 1]$,

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$$|f(x) - p^*_{n-1}(x)| \le C\left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right)^{\alpha},$$

where $C = C(\alpha)$, i.e., the constant C depends on α and is independent of n and f.

Theorem A is a consequence of classical inverse theorems (see, e.g., Dzyadyk [1, p. 263]) and the results of De Vore and Yu [2] (for $f \in \Delta^1$ and $0 < \alpha < 2$), Shevchuk [3] (for $f \in \Delta^1$ and $\alpha \ge 2$), Leviatan [4] (for $f \in \Delta^2$ and $0 < \alpha < 2$), Manya and Shevchuk (for $f \in \Delta^2$ and $\alpha > 2$; see, e.g., [5]), and Kopotun [6] (for $f \in \Delta^2$ and $\alpha = 2$).

It is clear that one of the advantages of Theorem A is the possibility of application of its inverse results. It is well known that, in the case of algebraic polynomial approximation in the uniform metric, inverse theorems in terms of the standard modulus of smoothness

$$\omega_k(f, t) := \sup_{0 < h \le t} \left\| \overline{\Delta}_h^k(f, x) \right\|_\infty$$

must be pointwise. This explains why Theorem A contains pointwise estimates.

The situation changed when Ditzian and Totik [7] suggested a new modulus of smoothness

$$\omega_\varphi^k(f, t)_p := \sup_{0 < h \le t} \left\| \overline{\Delta}_{h\varphi(x)}^k(f, x) \right\|_p,$$

which differently reflects the behavior of functions near the endpoints of an interval and inside it. On the basis of the Ditzian–Totik modulus, it became possible to obtain inverse results in terms of uniform estimates. Together with some direct results for shape-preserving approximation in terms of ω_φ^k (see, e.g., [6, 8]), this enables one to characterize functions by uniform estimates of algebraic polynomial approximation.

In this paper, we investigate the relationship between the rates of shape-preserving and unconstrained approximations on the basis of uniform estimates (instead of pointwise ones as in Theorem A).

Let $E_n^{(1)}(f)$ and $E_n^{(2)}(f)$ be the best $(n-1)$th-degree polynomial approximations, monotone and convex, respectively, of monotone and convex functions on $[-1, 1]$. Denote by $E_n(f)$ the best $(n-1)$th-degree unconstrained polynomial approximation of f, i.e.,

$$E_n(f) := \inf_{p_{n-1} \in P_{n-1}} \|f - p_{n-1}\|_\infty$$

and

$$E_n^{(q)}(f) := \inf_{p_{n-1} \in P_{n-1} \cap \Delta^q} \|f - p_{n-1}\|_\infty, \qquad q = 1, 2,$$

where P_n is the set of algebraic polynomials whose degree does not exceed n.

The principal results of this paper are formulated in the theorems below.

Theorem 1. *Suppose that $\alpha > 0$, $\alpha \ne 2$, and f is a nondecreasing function on $[-1, 1]$ such that $E_n(f) \le n^{-\alpha}$ for every $n > \alpha$. Then $E_n^{(1)}(f) \le Cn^{-\alpha}$ for $n > \alpha$, where $C = C(\alpha)$.*

Theorem 2. *For $\alpha = 2$, the assertion of Theorem 1 is not true.*

Theorem 3. *If* $\alpha \in (0,3) \cup (4,+\infty)$ *and* f *is a convex function on* $[-1,1]$ *such that* $E_n(f) \le n^{-\alpha}$ *for every* $n > \alpha$, *then* $E_n^{(2)}(f) \le C n^{-\alpha}$ *for* $n > \alpha$, *where* $C = C(\alpha)$.

Theorem 4. *For* $\alpha \in [3,4]$, *the assertion of Theorem 3 is not true.*

In particular, Theorems 2 means that, for $\alpha = 2$, the constant C in Theorem 1 cannot be independent of n and f. The same is true for $\alpha \in [3,4]$ in the case of the convex approximation. We do not know whether these negative statements will remain true if we weaken the conditions imposed on C, for example, if we assume that C depends on α and f but does not depend on n.

Let us recall some useful definitions and notation (see [7, 5]).

Let $\varphi(x) := \sqrt{1-x^2}$ and let B^r, $r \in \mathbb{N}$, be the space of all functions f continuous on $[-1,1]$ and such that their $(r-1)$th derivatives $f^{(r-1)}$ are absolutely locally continuous on $(-1,1)$ and $\left|(\varphi(x))^r f^{(r-1)}(x)\right| < \infty$ almost everywhere on $(-1,1)$.

For a function $f \in C(-1,1)$, the weighted Ditzian-Totik modulus of smoothness is defined as follows:

$$\overline{\omega}_{\varphi,r}^{k}(f,t) := \sup_{0<h\le t} \max_{x\in(-1,1)} \left| \left(1 - \frac{k}{2} h\varphi(x) - x\right)^{r/2} \left(1 - \frac{k}{2} h\varphi(x) + x\right)^{r/2} \overline{\Delta}_{h\varphi(x)}^{k}(f,x) \right|.$$

Obviously, $\overline{\omega}_{\varphi,0}^{k}(f,t) = \omega_{\varphi}^{k}(f,t)_\infty$. For $k=0$, we set

$$\overline{\omega}_{\varphi,r}^{0}(f,t) := \operatorname*{ess\,sup}_{x\in(-1,1)} \left|(\varphi(x))^r f(x)\right|.$$

Clearly, the function $\overline{\omega}_{\varphi,r}^{k}(f,t)$ can be unbounded. As was shown in [5], the necessary and sufficient condition for $\overline{\omega}_{\varphi,r}^{k}(f,t)$ to be bounded for all $t>0$ is $\left|(\varphi(x))^r f(x)\right| < M$, $x \in (-1,1)$, where $M = \text{const} < \infty$. This implies that

$$\overline{\omega}_{\varphi,r}^{k}(f^{(r)},t) < \infty, \quad t>0 \quad \Leftrightarrow \quad f \in B^r.$$

For a function $f \in B^r \cap C^r(-1,1)$, $r \ge 1$, and $k \ge 0$, the following inequality is true (see, e.g., [5]):

$$\overline{\omega}_{\varphi,l}^{k+r-l}(f^{(l)},t) \le C t^{r-l} \overline{\omega}_{\varphi,r}^{k}(f^{(r)},t), \qquad t>0, \tag{1}$$

where $0 \le l \le r-1$.

Let $B^r \overline{H}[k,\psi]$ be the set of functions $f \in B^r \cap C^r(-1,1)$ such that $\overline{\omega}_{\varphi,r}^{k}(f^{(r)},t) \le \psi(t)$, where $\psi \in \Phi^k$ (we have $\psi \in \Phi^k$ if $\psi(0)=0$, $\psi=\psi(t)$ is a continuous and nondecreasing function for $t \ge 0$, and $t^{-k}\psi(t)$ does not increase).

We can now define an analog of the class $\text{Lip}^*\alpha := \left\{ f \mid \omega_2(f^{(r)},t) = O(t^\beta), \text{ where } \alpha > 0, \ \alpha = r+\beta, \ r \in N \cup \{0\}, \text{ and } 0 < \beta \le 1 \right\}$ in terms of weighted Ditzian-Totik moduli of smoothness as follows:

$$\hat{H}^\alpha = \begin{cases} B^r \overline{H}[1,t^\beta] & \text{if } \alpha \notin N, \text{ where } r := [\alpha] \text{ and } \beta := \alpha - r, \\ B^r \overline{H}[2,t] & \text{if } \alpha \in N, \text{ where } r := \alpha - 1. \end{cases}$$

Note that the equivalence $f \in \hat{H}^{\alpha} \Leftrightarrow E_n(f) \le Cn^{-\alpha}$ for any $\alpha < 0$ and $n > \alpha$, where $C = C(\alpha)$ is a constant depending only on α, is a consequence of the following direct and inverse theorems:

Direct Theorem (see, e.g., [7, 5]). *Suppose that* $k \in N$, $(r+1) \in N$, *and* $\psi \in \Phi^k$. *Then, for a given function* $f \in B^r \overline{H}[k, \psi]$ *on* $[-1, 1]$ *and every* $n \ge k + r$, *the following inequality is true:*

$$E_n(f) \le Cn^{-r}\psi(n^{-1}), \qquad C = C(r, k).$$

Inverse Theorem ([7, 5]). *Suppose that* $k \in N$, $(r+1) \in N$, *and* $\psi \in \Phi^k$. *If, for a given function* f *on* $[-1, 1]$ *and every* $n \ge k + r$, *the inequality*

$$E_n(f) \le n^{-r}\psi(n^{-1})$$

holds, then

$$\overline{\omega}^k_{\varphi,r}(f^{(r)}, t) = C\left\{ r\int_0^t \psi(u)u^{-1}\,du + t^k\int_t^1 \psi(u)u^{-k-1}\,du \right\}, \qquad C = C(r, k).$$

In view of these facts, one can conclude that Theorems 1–4 are consequences of the following theorems:

Theorem 5. *Let* $\alpha > 0$, $\alpha \ne 2$. *Then, for a given nondecreasing function* $f \in \hat{H}^{\alpha}$ *on* $[-1, 1]$ *and every* $n \in N$, $n > \alpha$, *the following inequality is true:*

$$E_n^{(1)}(f) \le Cn^{-\alpha}, \qquad C = C(\alpha).$$

For $\alpha = 2$, *this implication is false.*

Theorem 6. *Let* $\alpha \in (0, 3) \cup (4, +\infty)$. *Then, for a given convex function* $f \in \hat{H}^{\alpha}$ *on* $[-1, 1]$ *and every* $n \in N$, $n > \alpha$, *the following inequality is true:*

$$E_n^{(2)}(f) \le Cn^{-\alpha}, \qquad C = C(\alpha).$$

For $\alpha \in [3, 4]$, *this implication is false.*

2. Proof of the Negative Results

Below, for arbitrary n and $\alpha = 2$ and $\alpha \in [3, 4]$ in the monotone and convex cases, respectively, we construct the sequences of functions $\{g_b\} \subset \hat{H}^{\alpha}$ and $\{f_b\} \subset \hat{H}^{\alpha}$ such that $E_n^{(1)}(g_b) \to \infty$ and $E_n^{(2)}(f_b) \to \infty$ as $b \to \infty$. This will prove the negative statements of Theorems 5 and 6.

We need the following lemma:

Lemma 1 [9]. *For arbitrary* $n \in N$ *and* $M = \text{const}$, *there exists a convex function* f_b *on* $[-1, 1]$, $f_b''(x) = bx + b - \ln b - \ln(1 + x)$, $b \in R$, *such that, for any convex polynomial* p_n *of degree* n *on* $[-1, 1]$, *the inequality* $\|f - p_n\| > M$ *is true.*

By using the same method, one can easily prove a similar result for the monotone case (see also [9]).

Lemma 2. *For arbitrary* $n \in N$ *and* $M = \text{const}$, *there exists a nondecreasing function* g_b *on* $[-1, 1]$, $g_b'(x) = bx + b - \ln b - \ln(1+x)$, $b \in R$, *such that, for any nondecreasing polynomial* $\tilde{p}_n$ *of degree* n, *the inequality* $\|f - \tilde{p}_n\| > M$ *holds on* $[-1, 1]$.

Let us determine classes that contain the functions f_b and g_b.

Lemma 3. *For any* $b \in R$, *the functions* f_b *and* g_b *belong to the classes* $B^3\overline{H}[1, Ct]$ *and* $B^1\overline{H}[2, Ct]$, *respectively, where* C *is an absolute constant.*

Proof. For any real number b and functions $f_b \in B^4$ and $g_b \in B^2$, it follows from inequality (1) that

$$\overline{\omega}^1_{\varphi,3}\left(f_b^{(3)}, t\right) \le Ct\,\overline{\omega}^0_{\varphi,4}\left(f_b^{(4)}, t\right) = Ct \operatorname*{ess\,sup}_{x \in (-1,1)} \left|(\varphi(x))^4 (1+x)^{-2}\right| \le Ct, \qquad t > 0,$$

and

$$\overline{\omega}^1_{\varphi,2}\left(g_b'', t\right) \le \sup_{0<h\le t} \sup_{x \pm (1/2)h\varphi(x) \in (-1,1)} \left|\left(1 - \frac{1}{2}h\varphi(x) - x\right)\right.$$

$$\times \left.\left(1 - \frac{1}{2}h\varphi(x) + x\right)\left(g_b''\left(x - \frac{1}{2}h\varphi(x)\right) - g_b''\left(x + \frac{1}{2}h\varphi(x)\right)\right)\right|$$

$$\le 2 \sup_h \sup_x \left|\frac{h\varphi(x)}{1 + x + (1/2)h\varphi(x)}\right| \le 4.$$

This implies that $\overline{\omega}^2_{\varphi,1}\left(g_b', t\right) \le Ct\overline{\omega}^1_{\varphi,2}\left(g_b'', t\right) \le Ct$, $t > 0$. Thus, Lemma 3 is proved.

To complete the proof of the negative statements, it suffices to note that $B^1\overline{H}[2, t] = \hat{H}^2$ and $B^3\overline{H}[1, t] \subset \hat{H}^\alpha$ for $\alpha \in [3, 4]$.

For $0 < \alpha < 2$, Theorems 5 and 6 follow from the results obtained by Leviatan [8]. For $2 \le \alpha < 3$, Theorem 6 follows from [6].

The direct statements of Theorems 5 and 6 for other α can be proved as in [3, 9, 10–12]. The corresponding proofs involve nonlinear techniques and are quite cumbersome. At the same time, these theorems are intermediate steps in the investigation of degrees of coapproximation of functions from the classes $B^k\overline{H}[k, \psi]$ being, thus, only of relative value. This is why we omit detailed proofs of these assertions.

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