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Nearly monotone and nearly convex approximation by smooth splines in $\mathbb{L}_p$, $p > 0$

K.A. Kopotun[a,1], D. Leviatan[b,*], A.V. Prymak[c,2]

[a] *Department of Mathematics, University of Manitoba, Winnipeg, Manitoba, Canada R3T 2N2*
[b] *Raymond and Beverly Sackler School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel*
[c] *Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB, Canada T6G 2G1*

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Abstract

Given a monotone or convex function on a finite interval we construct splines of arbitrarily high order having maximum smoothness which are "nearly monotone" or "nearly convex" and provide the rate of $\mathbb{L}_p$-approximation which can be estimated in terms of the third or fourth (classical or Ditzian–Totik) moduli of smoothness (for uniformly spaced or Chebyshev knots). It is known that these estimates are impossible in terms of higher moduli and are no longer true for "purely monotone" and "purely convex" spline approximation.

Keywords: Monotone and convex approximation by piecewise polynomials and splines; Degree of approximation; Jackson type estimates

1. Introduction and main results

Let $\mathcal{S}_r(\mathbf{z}_n)$ be the space of all piecewise polynomial functions (splines) of degree r (order $r+1$) with the knots $\mathbf{z}_n := (z_i)_{i=0}^n := \left(z_i^{(n)}\right)_{i=0}^n$, $-1 =: z_0 < z_1 < \cdots < z_{n-1} < z_n := 1$. In other words, if $s \in \mathcal{S}_r(\mathbf{z}_n)$ then, on each interval (z_{i-1}, z_i), $1 \leqslant i \leqslant n$, it is a polynomial of degree $\leqslant r$,

* Corresponding author.
E-mail addresses: kopotunk@cc.umanitoba.ca (K.A. Kopotun), leviatan@post.tau.ac.il (D. Leviatan), prymak@gmail.com (A.V. Prymak).

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i.e., $s|_{(z_{i-1},z_i)} \in \Pi_r$, where Π_r denotes the space of algebraic polynomials of degree $\leqslant r$. We denote by $\mathbf{u}_n$ and $\mathbf{t}_n$ the sets of knots for the uniform and Chebyshev partitions, respectively, i.e., $\mathbf{u}_n := (-1+2i/n)_{i=0}^n$ and $\mathbf{t}_n := (\cos((n-i)\pi/n))_{i=0}^n$.

Given $q \geqslant 0$ and an interval $J \subseteq [-1,1]$, a function f is said to be q-monotone on J if its divided differences of order q, $[x_0,\dots,x_q;f]$, are non-negative for all choices of $(q+1)$ distinct points $x_0,\dots,x_q$ in J. We denote the class of all such functions by $\mathcal{M}^q(J)$, so that, in particular, $\mathcal{M}^1(J)$ and $\mathcal{M}^2(J)$ are collections of all monotone and convex functions on J, respectively.

We denote by $\|\cdot\|_{\mathbb{L}_p(J)}$, $0<p\leqslant\infty$, the $\mathbb{L}_p$-(quasi)norm on J, and write $\|\cdot\|_p := \|\cdot\|_{\mathbb{L}_p[-1,1]}$. For a function $f \in \mathbb{L}_p := \mathbb{L}_p[-1,1]$, $0<p\leqslant\infty$, let

$$E(f,\mathcal{F})_p := \inf_{s\in\mathcal{F}} \|f-s\|_p$$

be the error of $\mathbb{L}_p$-approximation of f by elements from the set $\mathcal{F} \subset \mathbb{L}_p$. In particular, denote by

$$\mathcal{E}_r^{(q)}(f,\mathbf{z}_n,J)_p := E(f,\mathcal{S}_r(\mathbf{z}_n)\cap\mathcal{M}^q(J))_p$$

and

$$\widetilde{\mathcal{E}}_r^{(q)}(f,\mathbf{z}_n,J)_p := E(f,\mathcal{S}_r(\mathbf{z}_n)\cap\mathcal{M}^q(J)\cap\mathbb{C}^{r-1})_p$$

the errors of $\mathbb{L}_p$-approximation of f by splines from $\mathcal{S}_r(\mathbf{z}_n)$ and from $\mathcal{S}_r(\mathbf{z}_n)\cap\mathbb{C}^{r-1}$ (i.e., having maximum smoothness without becoming polynomials), respectively, which are q-monotone on $J \subseteq [-1,1]$.

If

$$\Delta_h^k(f,x) := \begin{cases} \displaystyle\sum_{i=0}^k \binom{k}{i}(-1)^{k-i} f(x-kh/2+ih) & \text{if } |x\pm kh/2|<1, \\ 0 & \text{otherwise} \end{cases}$$

denotes the kth symmetric difference, then the kth modulus of smoothness of a function $f \in \mathbb{L}_p[-1,1]$ is defined by $\omega_k(f,t)_p := \sup_{0<h\leqslant t}\|\Delta_h^k(f,\cdot)\|_p$, and the Ditzian–Totik kth modulus is $\omega_k^\varphi(f,t)_p := \sup_{0<h\leqslant t}\|\Delta_{h\varphi(\cdot)}^k(f,\cdot)\|_p$, where $\varphi(x) := \sqrt{1-x^2}$. (It is well known that $\omega_k^\varphi(f,t)_p \leqslant c\omega_k(f,t)_p$.) Finally, the (classical) kth modulus of smoothness on an interval $J \subset [-1,1]$, is defined by $\omega_k(f,t,J)_p := \sup_{0<h\leqslant t}\|\Delta_h^k(f,\cdot,J)\|_{\mathbb{L}_p(J)}$, where $\Delta_h^k(f,x,J) := \Delta_h^k(f,x)$ if $x\pm kh/2 \in J$, and $\Delta_h^k(f,x,J) := 0$ otherwise.

It is well known (see, e.g., [10] ($q=1$, $p=\infty$), [2] ($q=1$, $0<p<\infty$), [4, Theorem 1.1], [5] and [8, Section 4, Construction of the Convex Spline] ($q=2$, $p=\infty$), [1, Theorem 1.2] ($q=2$, $0<p<\infty$)) that for a function $f \in \mathbb{L}_p \cap \mathcal{M}^q[-1,1]$, $q=1,2$,

$$\widetilde{\mathcal{E}}_q^{(q)}(f,\mathbf{z}_n,[-1,1])_p \leqslant c \begin{cases} \omega_{q+1}(f,1/n)_p & \text{if } \mathbf{z}_n = \mathbf{u}_n, \\ \omega_{q+1}^\varphi(f,1/n)_p & \text{if } \mathbf{z}_n = \mathbf{t}_n, \end{cases} \tag{1}$$

where c are constants which are independent of f and n, but may depend on p when $p \to 0$. (Throughout the paper, c denotes positive constants which are not necessarily the same even when they occur on the same line.) Moreover, these estimates are best possible in the sense that one cannot replace the $(q+1)$st moduli by $\omega_k(f,1)_p$ for any $k > q+1$ even if the order of approximating splines is allowed to be increased (see [12]).

In [6], we showed that estimates (1) may be improved in the case $q = 1$ if we relax the constraints on the approximating splines by allowing them not to be monotone in some small parts of the interval $[-1, 1]$. (In the case $q = 1$ and $p = \infty$ this was also shown in [9].) Namely, we showed that, there exists an absolute constant $\kappa > 0$ such that, if $f \in \mathbb{L}_p$ is any non-decreasing function, then

$$\widetilde{\mathcal{E}}_2^{(1)}(f, \mathbf{u}_n, [-1 + \kappa n^{-1}, 1 - \kappa n^{-1}])_p \leqslant c\omega_3(f, 1/n)_p$$

and

$$\widetilde{\mathcal{E}}_2^{(1)}(f, \mathbf{t}_n, [-1 + \kappa n^{-2}, 1 - \kappa n^{-2}])_p \leqslant c\omega_3^{\varphi}(f, 1/n)_p.$$

Two natural questions now emerge. First, is it possible to improve (1) in the case $q = 2$ and $p < \infty$? (In the case $p = \infty$ this is indeed so as was shown by Shevchuk [11]). Second, are these improved estimates valid for smooth splines of higher orders and not just piecewise quadratic or cubic polynomials? The main purpose of this paper is to answer these questions in the affirmative.

The following theorem is our main result.

Theorem 1.1. *Let $f \in \mathcal{M}^q([-1, 1]) \cap \mathbb{L}_p$, $0 < p \leqslant \infty$, $q = 1, 2$, and $r \geqslant q + 1$. Then, there exists a constant $\kappa = \kappa(r) > 0$, such that for every $n \in \mathbb{N}$,*

$$\widetilde{\mathcal{E}}_r^{(q)}(f, \mathbf{u}_n, [-1 + \kappa n^{-1}, 1 - \kappa n^{-1}])_p \leqslant c\omega_{q+2}(f, 1/n)_p \tag{2}$$

and

$$\widetilde{\mathcal{E}}_r^{(q)}(f, \mathbf{t}_n, [-1 + \kappa n^{-2}, 1 - \kappa n^{-2}])_p \leqslant c\omega_{q+2}^{\varphi}(f, 1/n)_p, \tag{3}$$

where c are constants independent of f and n which may depend on r and on p as $p \to 0$.

Remark 1.2. We follow the usual convention that $[a, b] := \emptyset$ if $b < a$. Hence, if κ is big enough, the shape restriction on the approximating splines disappears for small n ($n \leqslant \kappa$ in (2) and $n \leqslant \sqrt{\kappa}$ in (3)), and Theorem 1.1 becomes a well known result on unconstrained spline approximation. Hence, in the sequel, we do not worry about small values of n in our proofs.

Note that while we require the approximating splines to be monotone or convex in a smaller interval than $[-1, 1]$, we do demand good approximation throughout the whole interval $[-1, 1]$. As was mentioned above, Theorem 1.1 is known for $q = 1$ and $r = 2$ (see [9] for $p = \infty$ and [6] for $0 < p < \infty$), and for $q = 2$, $r = 3$ and $p = \infty$ (see [11]). It is new in all other cases.

One may not replace the $(q + 2)$-nd order moduli of smoothness in Theorem 1.1 by any moduli of smoothness of higher order (see [6, Theorem 3.1]). Moreover, [6, Theorem 3.2] implies that in order to achieve the above improvement on the order of approximation, the intervals near the endpoints where the approximating splines are allowed to be "non-shape-preserving" may not be much smaller than nearby intervals produced by $\mathbf{u}_n$ or $\mathbf{t}_n$. In particular, it also implies that we will not get any improvement in the orders of approximation by relaxing the shape constraint on the splines, instead of near the end points, somewhere inside the interval $[-1, 1]$.

In Section 2, we construct a continuous nearly convex cubic spline, adapting ideas of Shevchuk [11] to the $\mathbb{L}_p$ situation, and proceeding similarly to [6]. In Section 3, we show how to construct approximating splines of higher orders having maximum smoothness (minimum defect) while preserving their shape properties as well as the approximation orders.

2. Construction of convex cubic spline

In this section we combine ideas of DeVore et al. [1] with a construction by Shevchuk [11]. The construction of a convex piecewise cubic polynomial is more complicated than the construction of a continuous monotone piecewise quadratic polynomial that we used in [6]. It is not enough to obtain a spline which is convex in each subinterval and interpolates at the endpoints of that subinterval. Additionally, the pieces have to join in a convex manner, namely, the first derivative of this spline has to be non-decreasing on the whole interval.

Let $J_j := [z_j, z_{j+1}]$, $\delta_j := |J_j|/3$, $0 \leqslant j \leqslant n-1$, and denote $\hat{J}_j := (z_j, z_j + \delta_j)$, $0 \leqslant j \leqslant n-1$, and $\hat{J}_n := (1 - \delta_{n-1}, 1)$. It is also convenient to denote $z_j := 1$, $j > n$ and $z_j := -1$, $j < 0$.

The proof of the following lemma is exactly the same as in [1, Lemma 2.1] (see also [6, Lemma 2.6]).

Lemma 2.1. *Given $f \in \mathbb{L}_p[-1, 1]$, $0 < p \leqslant \infty$, and $r \in \mathbb{N}$. There are points $\xi_j^{(r)} \in \hat{J}_j$, $0 \leqslant j \leqslant n$, such that, for $0 \leqslant i \leqslant n-r$, the polynomial $Q_{r,i} \in \Pi_r$ interpolating f at $\xi_j^{(r)}$, $j = i, i+1, \dots, i+r$, satisfies*

$$\|f - Q_{r,i}\|_{\mathbb{L}_p(\bar{J}_i)} \leqslant c\omega_{r+1}(f, |\bar{J}_i|, \bar{J}_i)_p, \tag{4}$$

where $\bar{J}_i := [z_{i-1}, z_{i+r+1}]$, and the constant c depends only on r, p (as $p \to 0$) and $\max_{0 \leqslant j \leqslant n-1} |J_{j\pm1}|/|J_j|$.

The following lemma is rather well known (see, e.g., [6, Lemma 2.3]).

Lemma 2.2. *Let $f \in \mathbb{L}_p[-1, 1]$, $0 < p \leqslant \infty$, and I and J be subintervals such that $I \subset J \subseteq [-1, 1]$. If $q_r \in \Pi_r$ is a polynomial satisfying $\|f - q_r\|_{\mathbb{L}_p(I)} \leqslant c_0\omega_{r+1}(f, |J|, J)_p$, then $\|f - q_r\|_{\mathbb{L}_p(J)} \leqslant c\omega_{r+1}(f, |J|, J)_p$, with constant c which depends only on c_0, r, the ratio $|J|/|I|$, and p as $p \to 0$.*

We now assume that n is sufficiently large ($n \geqslant 13$ will do) and, from now on, use c to denote constants which depend only on $\vartheta(\mathbf{z}_n) := \max_{0 \leqslant j \leqslant n-1} |J_{j\pm1}|/|J_j|$, and on p as $p \to 0$. (Note that $\vartheta(\mathbf{z}_n)$ is bounded for both the uniform and the Chebyshev partitions.)

Lemma 2.3. *Let $a := \zeta_0 < \zeta_1 < \cdots < \zeta_{n-1} < \zeta_n := b$, $\zeta_j := a$, $j < 0$, and $\zeta_j := b$, $j > n$. Given a convex $F \in \mathbb{L}_p[a, b]$, $0 < p \leqslant \infty$, assume that for $0 \leqslant j \leqslant n-3$, the cubic polynomials $Q_{3,j}$, which interpolate F at ζ_l, $l = j, j+1, j+2, j+3$, satisfy*

$$\|F - Q_{3,j}\|_{\mathbb{L}_p[\zeta_j, \zeta_{j+3}]} \leqslant E_j. \tag{5}$$

Then there exists a convex cubic piecewise polynomial function s_3 on $[\zeta_2, \zeta_{n-2}]$, such that if $s_3\big|_{[\zeta_j, \zeta_{j+1}]} =: q_{3,j}$, $2 \leqslant j \leqslant n-3$, then $q_{3,j}$ interpolates F at ζ_{j+l}, either for $l = 0, 1, 2$, or for $l = -1, 0, 1$, and satisfies

$$\|F - q_{3,j}\|_{\mathbb{L}_p[\zeta_j, \zeta_{j+1}]} \leqslant 2^{1/p} \max_{j-2 \leqslant i \leqslant j} E_i, \quad 2 \leqslant j \leqslant n-3. \tag{6}$$

Proof. This proof is a slight modification of what was used by Shevchuk in [11]. We adduce its main details here for readers' convenience since [11] may not be readily available.

Denote by

$$\Delta_j := [\zeta_j, \zeta_{j+1}, \zeta_{j+2}, \zeta_{j+3}; F], \quad 0 \leqslant j \leqslant n-3,$$

the divided differences of order 3 of F, and set

$$r_j(x) := (x-\zeta_j)(x-\zeta_{j+1})(x-\zeta_{j+2}), \quad 0 \leqslant j \leqslant n-2.$$

Let ℓ_j be the (unique) quadratic polynomial that interpolates F at ζ_l, $l = j, j+1, j+2$. Then Newton's formula for interpolating polynomials immediately yields the following representations of $Q_j := Q_{3,j}$:

$$Q_j(x) = r_j(x)\Delta_j + \ell_j(x) = r_{j+1}(x)\Delta_j + \ell_{j+1}(x), \quad 0 \leqslant j \leqslant n-3. \tag{7}$$

We are now ready to construct s_3 on $[\zeta_j, \zeta_{j+1}]$, $2 \leqslant j \leqslant n-3$, which we call $q_j := q_{3,j}$. Namely, depending on the signs and relations among Δ_{j-2}, Δ_{j-1} and Δ_j, the following five cases are possible:

(i) If $0 \leqslant \Delta_{j-2} \leqslant \Delta_{j-1}$, then $q_j := Q_{j-2}$.
(ii) If $\Delta_{j-2} < 0 \leqslant \Delta_{j-1}$, then $q_j := \ell_{j-1}$.
(iii) If $\Delta_{j-2} > \Delta_{j-1} \geqslant 0$, or if $\Delta_j \leqslant \Delta_{j-1} < 0$, then $q_j := Q_{j-1}$.
(iv) If $\Delta_{j-1} < \Delta_j < 0$, then $q_j := Q_j$.
(v) If $\Delta_{j-1} < 0 \leqslant \Delta_j$, then $q_j := \ell_j$.

Taking into account (7) as well as the observation that $r_{j-1}(x) \leqslant 0$ and $r_j(x) \geqslant 0$ for $x \in [\zeta_j, \zeta_{j+1}]$, we have

$$Q_{j-1}(x) = r_{j-1}(x)\Delta_{j-1} + \ell_{j-1}(x) \leqslant \ell_{j-1}(x) \leqslant r_{j-1}(x)\Delta_{j-2} + \ell_{j-1}(x) = Q_{j-2}(x)$$

in case (ii), and

$$Q_{j-1}(x) = r_j(x)\Delta_{j-1} + \ell_j(x) \leqslant \ell_j(x) \leqslant r_j(x)\Delta_j + \ell_j(x) = Q_j(x)$$

in case (v).

Therefore,

$$\|F - s_3\|_{\mathbb{L}_p[\zeta_j,\zeta_{j+1}]} \leqslant 2^{1/p} \max_{j-2 \leqslant i \leqslant j} \|F - Q_i\|_{\mathbb{L}_p[\zeta_j,\zeta_{j+1}]} \leqslant 2^{1/p} \max_{j-2 \leqslant i \leqslant j} E_i.$$

To show that s_3 is convex on $[\zeta_2, \zeta_{n-2}]$, taking into account that q_j'''s are polynomials of degree at most 1, it suffices to verify that

$$q_j''(\zeta_j) \geqslant 0 \quad \text{and} \quad q_j''(\zeta_{j+1}) \geqslant 0, \quad 2 \leqslant j \leqslant n-3 \tag{8}$$

and

$$q_j'(\zeta_{j+1}) \leqslant q_{j+1}'(\zeta_{j+1}), \quad 2 \leqslant j \leqslant n-4. \tag{9}$$

We note that ℓ_j is convex and so $\ell_j'' \equiv \text{const}_j \geqslant 0$, for all j. Hence, in cases (ii) and (v), inequalities (8) are obvious. Now, using the observation that $r_i''(x) > 0$, $x \geqslant \zeta_{i+2}$, and $r_i''(x) < 0$, $x \leqslant \zeta_i$, $0 \leqslant i \leqslant n-2$, fixing $2 \leqslant j \leqslant n-3$ and using (7) we conclude

$$q_j''(\zeta_i) = Q_{j-2}''(\zeta_i) = r_{j-2}''(\zeta_i)\Delta_{j-2} + \ell_{j-2}'' \geqslant 0, \quad i = j, j+1,$$

in case (i);

$$q_j''(\zeta_j) = Q_j''(\zeta_j) = r_j''(\zeta_j)\Delta_j + \ell_j'' \geqslant 0$$

and

$$q_j''(\zeta_{j+1}) = Q_j''(\zeta_{j+1}) = r_{j+1}''(\zeta_{j+1})\Delta_j + \ell_{j+1}'' \geqslant 0$$

in case (iv);

$$q_j''(\zeta_j) = Q_{j-1}''(\zeta_j) = r_j''(\zeta_j)\Delta_{j-1} + \ell_j'' \geqslant 0$$

in case (iii) with $\Delta_{j-1} < 0$;

$$q_j''(\zeta_{j+1}) = Q_{j-1}''(\zeta_{j+1}) = r_{j-1}''(\zeta_{j+1})\Delta_{j-1} + \ell_{j-1}'' \geqslant 0$$

in case (iii) with $\Delta_{j-1} \geqslant 0$.

Now, if $\Delta_{j-2} > \Delta_{j-1} \geqslant 0$ in case (iii), then

$$q_j''(\zeta_j) = Q_{j-1}''(\zeta_j) = r_{j-1}''(\zeta_j)\Delta_{j-1} + \ell_{j-1}'' \geqslant 0$$

if $r_{j-1}''(\zeta_j) \geqslant 0$, and

$$\begin{aligned} q_j''(\zeta_j) &= Q_{j-1}''(\zeta_j) - Q_{j-2}''(\zeta_j) + Q_{j-2}''(\zeta_j) \\ &= r_{j-1}''(\zeta_j)(\Delta_{j-1} - \Delta_{j-2}) + r_{j-2}''(\zeta_j)\Delta_{j-2} + \ell_{j-2}'' \geqslant 0, \end{aligned}$$

if $r_{j-1}''(\zeta_j) < 0$.

Similarly, if $\Delta_j \leqslant \Delta_{j-1} < 0$ in case (iii), then

$$q_j''(\zeta_{j+1}) = Q_{j-1}''(\zeta_{j+1}) = r_j''(\zeta_{j+1})\Delta_{j-1} + \ell_j'' \geqslant 0,$$

if $r_j''(\zeta_{j+1}) < 0$, and

$$\begin{aligned} q_j''(\zeta_{j+1}) &= Q_{j-1}''(\zeta_{j+1}) - Q_j''(\zeta_{j+1}) + Q_j''(\zeta_{j+1}) \\ &= r_j''(\zeta_{j+1})(\Delta_{j-1} - \Delta_j) + r_{j+1}''(\zeta_{j+1})\Delta_j + \ell_{j+1}'' \geqslant 0, \end{aligned}$$

if $r_j''(\zeta_{j+1}) \geqslant 0$. This completes the proof of (8).

For the proof of (9), note that $r_i'(\zeta_k) > 0$ if $k \neq i+1$, and $r_i'(\zeta_{i+1}) < 0$. Taking this into account and the identity

$$Q_{i+1}'(x) - Q_i'(x) = r_{i+1}'(x)(\Delta_{i+1} - \Delta_i)$$

which immediately follows from (7), we conclude that

$$\operatorname{sgn}\left(Q_{i+1}'(\zeta_{j+1}) - Q_i'(\zeta_{j+1})\right) = (-1)^{\delta_{i,j-1}} \operatorname{sgn}(\Delta_{i+1} - \Delta_i), \tag{10}$$

where $\delta_{kl} := 1$ if $k = l$, and $\delta_{kl} := 0$ if $k \neq l$, is the Kronecker symbol.

Similarly,

$$\operatorname{sgn}\left(Q_i'(\zeta_{j+1}) - l_i'(\zeta_{j+1})\right) = (-1)^{\delta_{i,j}} \operatorname{sgn}(\Delta_i) \tag{11}$$

and

$$\operatorname{sgn}\left(Q_i'(\zeta_{j+1}) - l_{i+1}'(\zeta_{j+1})\right) = (-1)^{\delta_{i,j-1}} \operatorname{sgn}(\Delta_i). \tag{12}$$

It remains to consider all possibilities for the values of $q_j'(\zeta_{j+1})$ and $q_{j+1}'(\zeta_{j+1})$, using the definition of s_3 above, and apply inequalities (10)–(12). For example, if $0\leqslant\Delta_{j-2} < \Delta_{j-1}$ and $\Delta_j < 0\leqslant\Delta_{j+1}$, then $q_j'(\zeta_{j+1}) = Q_{j-2}'(\zeta_{j+1})$ and $q_{j+1}'(\zeta_{j+1}) = l_{j+1}'(\zeta_{j+1})$, and we have

$$\begin{aligned} q_{j+1}'(\zeta_{j+1}) - q_j'(\zeta_{j+1}) = {} & l_{j+1}'(\zeta_{j+1}) - Q_j'(\zeta_{j+1}) + Q_j'(\zeta_{j+1}) - Q_{j-1}'(\zeta_{j+1}) \\ & + Q_{j-1}'(\zeta_{j+1}) - Q_{j-2}'(\zeta_{j+1}) \geqslant 0 \end{aligned}$$

by (12) with $i = j$, and (10) with $i = j-1$ and $i = j-2$. All other cases are completely analogous, and so we omit details. The proof of (9) is now complete. □

We are now ready to prove the following result which is interesting in its own right. It will be used in Section 3 to construct nearly convex smooth splines of higher orders.

Theorem 2.4. *Let $f \in \mathbb{L}_p[-1, 1]$, $0 < p\leqslant\infty$, be convex. Then there exists a continuous piecewise cubic polynomial function S_3 with knots at z_j, $j = 1, \dots, n-1$, which is convex in $[z_5, z_{n-6}]$ and such that*

$$\|f - S_3\|_{\mathbb{L}_p[z_j, z_{j+1}]} \leqslant c\omega_4(f, z_{j+6} - z_{j-6}, [z_{j-6}, z_{j+6}])_p, \quad 0\leqslant j\leqslant n-1. \tag{13}$$

Proof. We follow the proof of [6, Theorem 4.2]. First, we apply Lemma 2.1 with $r = 3$ to find the knots $\zeta_j := \xi_j^{(3)} \in \hat{J}_j$, $j = 0, \dots, n$, so that the cubic polynomials $Q_{3,j}$, $0\leqslant j\leqslant n-3$, interpolating f at $\xi_l^{(3)}$, $l = j, j+1, j+2, j+3$, satisfy

$$\|f - Q_{3,j}\|_{\mathbb{L}_p[\zeta_j, \zeta_{j+3}]} \leqslant \|f - Q_{3,j}\|_{\mathbb{L}_p(\bar{J}_j)} \leqslant c\omega_4(f, |\bar{J}_j|, \bar{J}_j)_p =: E_j.$$

Now, we apply Lemma 2.3 with $F := f$. The resulting spline s_3, with knots at ζ_j, is convex on $[\zeta_2, \zeta_{n-2}] \supset [z_3, z_{n-3}] =: [a, b]$, and

$$\|f - q_{3,j}\|_{\mathbb{L}_p[\zeta_j, \zeta_{j+1}]} = \|f - s_3\|_{\mathbb{L}_p[\zeta_j, \zeta_{j+1}]} \leqslant 2^{1/p} \max_{j-2\leqslant i\leqslant j} E_i, \quad 2\leqslant j\leqslant n-3, \tag{14}$$

recalling that $s_3|_{[\zeta_j, \zeta_{j+1}]} = q_{3,j}$.

However, s_3 has its knots in the "wrong" places, namely, not at the z_j. Thus, we apply Lemma 2.3 again, this time to $F := s_3$, in order to obtain an appropriate spline S_3 with knots z_j, $j = 3, \dots, n-3$. We need to establish the existence of the analogues of the polynomials $Q_{3,i}$ of (5). To this end, let $\tilde{Q}_{3,j}$, $3\leqslant j\leqslant n-6$, be the cubic polynomial that interpolates s_3 at z_i, $i = j, \dots, j+3$. We will show that

$$\|s_3 - \tilde{Q}_{3,j}\|_{\mathbb{L}_p[z_j, z_{j+3}]} \leqslant c\omega_4(f, z_{j+6} - z_{j-4}, [z_{j-4}, z_{j+6}])_p =: \tilde{E}_j. \tag{15}$$

Indeed, by its construction s_3 is the polynomial $q_{3,l-1}$ on the interval $[\zeta_{l-1}, \zeta_l]$, containing z_l, so that $\tilde{Q}_{3,j}(z_l) = q_{3,l-1}(z_l)$, $l = j, j+1, j+2, j+3$. Hence, for all $j\leqslant l\leqslant j+3$, we have

$$\begin{aligned} & \|q_{3,l-1} - \tilde{Q}_{3,j}\|_{\mathbb{L}_p[\zeta_{l-1}, \zeta_l]} \\ & \quad \leqslant c|\zeta_l - \zeta_{l-1}|^{1/p} \|q_{3,l-1} - \tilde{Q}_{3,j}\|_{\mathbb{L}_\infty[\zeta_{l-1}, \zeta_l]} \\ & \quad \leqslant c|z_{j+3} - z_j|^{1/p} \|q_{3,l-1} - \tilde{Q}_{3,j}\|_{\mathbb{L}_\infty[z_{j-1}, z_{j+4}]} \end{aligned}$$

$$\begin{aligned}
&\leqslant c|z_{j+3}-z_j|^{1/p}\max_{j\leqslant i\leqslant j+3}|q_{3,l-1}(z_i)-\tilde{Q}_{3,j}(z_i)|\\
&=c|z_{j+3}-z_j|^{1/p}\max_{j\leqslant i\leqslant j+3}|q_{3,l-1}(z_i)-q_{3,i-1}(z_i)|\\
&\leqslant c|z_{j+3}-z_j|^{1/p}\max_{j\leqslant i\leqslant j+3}\|q_{3,l-1}-q_{3,i-1}\|_{\mathbb{L}_\infty[z_j,z_{j+3}]}\\
&\leqslant c\max_{j\leqslant i\leqslant j+3}\|q_{3,l-1}-q_{3,i-1}\|_{\mathbb{L}_p[z_j,z_{j+3}]}\\
&\leqslant c\|f-q_{3,l-1}\|_{\mathbb{L}_p[z_j,z_{j+3}]}+c\max_{j\leqslant i\leqslant j+3}\|f-q_{3,i-1}\|_{\mathbb{L}_p[z_j,z_{j+3}]}\\
&\leqslant \tilde{E}_j,
\end{aligned}$$

where the last inequality follows from (14) and Lemma 2.2. This proves (15).

Now applying Lemma 2.3 to $F := s_3$ on $[a,b]=[z_3,z_{n-3}]$, we obtain a piecewise cubic polynomial S_3, with the knots at z_j, $6\leqslant j\leqslant n-6$, which is convex on $[z_5,z_{n-5}]$. By virtue of (15), we derive from (6) that

$$\|S_3-s_3\|_{\mathbb{L}_p[z_j,z_{j+1}]}\leqslant c\omega_4(f,z_{j+6}-z_{j-6},[z_{j-6},z_{j+6}])_p,\quad 5\leqslant j\leqslant n-6.$$

Combining this inequality with (14), we obtain (13) for $5\leqslant j\leqslant n-6$. We define S_3 on $[z_0,z_5]$ to be the polynomial $S_3|_{[z_5,z_6]}$, and on $[z_{n-5},z_n]$ to be the polynomial $S_3|_{[z_{n-6},z_{n-5}]}$. This extension may not be convex in $[z_0,z_n]$, but by virtue of Lemma 2.2 it satisfies (13) for $0\leqslant j\leqslant 4$ and for $n-5\leqslant j\leqslant n-1$. □

3. Spline smoothing and the proof of Theorem 1.1

We are now ready to prove Theorem 1.1. Taking into account the remark after Corollary 1.3 in [7], we can restate that corollary for $q=1,2$ as follows.

Lemma 3.1. *Let $q=1,2$, $r\geqslant q+1$, and let $\mathbf{z}_m=(z_i)_{i=0}^m:=\left(z_i^{(m)}\right)_{i=0}^m$ denote either $\mathbf{u}_m$ or $\mathbf{t}_m$, and $\mathcal{I}_j:=[(z_{j-1}+z_j)/2,(z_j+z_{j+1})/2]$. Then there is a constant $m_0=m_0(r)$ such that, for each $s\in\mathcal{S}_r(\mathbf{z}_m)\cap\mathcal{M}^q([-1,1])$, and any $n\geqslant m_0m$, there exists a spline $\tilde{s}\in\widetilde{\mathcal{S}}_r(\mathbf{z}_n)\cap\mathcal{M}^q([-1,1])$, where $\mathbf{z}_n$ is either $\mathbf{u}_n$ or $\mathbf{t}_n$, respectively, satisfying*

$$\|s-\tilde{s}\|_{\mathbb{L}_p(\mathcal{I}_j)}\leqslant c(p,r)\omega_{r+1}(s,|\mathcal{I}_j|,\mathcal{I}_j)_p,\quad 0\leqslant j\leqslant m$$

for all $0<p\leqslant\infty$.

3.1. Convex case ($q=2$)

According to Remark 1.2, we can choose $\kappa>0$ to be so large that only large values of $n\in\mathbb{N}$ have to be considered. In particular, we can assume that $m:=\lfloor n/m_0\rfloor\geqslant 13$, where $m_0=m_0(r)$ is the constant from the statement of Lemma 3.1.

Let $\mathbf{z}_m=(z_i)_{i=0}^m:=\left(z_i^{(m)}\right)_{i=0}^m$ be either $\mathbf{u}_m$ or $\mathbf{t}_m$. Applying Theorem 2.4 we obtain a piecewise cubic S_3, which is convex on $[z_5,z_{m-6}]$, and satisfies

$$\|f-S_3\|_{\mathbb{L}_p[z_j,z_{j+1}]}\leqslant c\omega_4(f,z_{j+6}-z_{j-6},[z_{j-6},z_{j+6}])_p,\quad 0\leqslant j\leqslant m-1. \tag{16}$$

We change S_3 outside of $[z_5, z_{m-6}]$ defining

$$s(x) := \begin{cases} S_3'(z_5+)(x - z_5) + S_3(z_5), & x \in [z_0, z_5), \\ S_3(x), & x \in [z_5, z_{m-6}], \\ S_3'(z_{m-6}-)(x - z_{m-6}) + S_3(z_{m-6}), & x \in (z_{m-6}, z_m], \end{cases} \tag{17}$$

and observe that $s \in \mathcal{S}_3(\mathbf{z}_m) \cap \mathcal{M}^2([-1, 1]) \subset \mathcal{S}_r(\mathbf{z}_n) \cap \mathcal{M}^2([-1, 1])$. Then Lemma 3.1 with $q = 2$ implies that there exists $\tilde{s} \in \widetilde{\mathcal{S}}_r(\mathbf{z}_n) \cap \mathcal{M}^2([-1, 1])$, satisfying

$$\|s - \tilde{s}\|_{\mathbb{L}_p(\mathcal{I}_j)} \leqslant c\omega_{r+1}(s, |\mathcal{I}_j|, \mathcal{I}_j)_p \leqslant c\omega_4(s, |\mathcal{I}_j|, \mathcal{I}_j)_p, \quad 0 \leqslant j \leqslant m.$$

In particular, we have

$$\|S_3 - \tilde{s}\|_{\mathbb{L}_p(\mathcal{I}_j)} \leqslant c\omega_4(S_3, |\mathcal{I}_j|, \mathcal{I}_j)_p, \quad 6 \leqslant j \leqslant m - 7,$$

which combined with (16) yields

$$\|f - \tilde{s}\|_{\mathbb{L}_p[z_j, z_{j+1}]} \leqslant c\omega_4(f, z_{j+6} - z_{j-6}, [z_{j-6}, z_{j+6}])_p, \quad 6 \leqslant j \leqslant m - 8. \tag{18}$$

Observe that while $\tilde{s}$ is convex on $[-1, 1]$, it may not give the degree of approximation we require near the end points. Therefore, we sacrifice the shape near the endpoints to ensure the proper approximation properties by modifying $\tilde{s}$. We define $S \in \widetilde{\mathcal{S}}_r(\mathbf{z}_n) \cap \mathcal{M}^2([z_6, z_{m-7}])$ by

$$S(x) := \begin{cases} p(x), & x \in [z_0, z_6), \\ \tilde{s}, & x \in [z_6, z_{m-7}], \\ q(x), & x \in [z_{m-7}, z_m], \end{cases}$$

where p is the polynomial $\tilde{s}|_{[z_6, z_6+\varepsilon]}$ and q is the polynomial $\tilde{s}|_{[z_{m-7-\varepsilon}, z_{m-7}]}$, and $\varepsilon > 0$ is small enough so that the two intervals are contained in intervals of the partition $\mathbf{z}_n$. Lemma 2.2 and (18) now yield

$$\|f - S\|_{\mathbb{L}_p[z_j, z_{j+1}]} \leqslant c\omega_4(f, z_{j+13} - z_{j-13}, [z_{j-13}, z_{j+13}])_p, \quad 0 \leqslant j \leqslant m - 1. \tag{19}$$

The case $p = \infty$ in Theorem 1.1 immediately follows from this inequality. The proof for $0 < p < \infty$ of (2) is completed using the well-known fact that, for $\mathbf{z}_m = \mathbf{u}_m$,

$$\left(\sum_{j=0}^{m-1} \omega_4(f, z_{j+13} - z_{j-13}, [z_{j-13}, z_{j+13}])_p^p \right)^{1/p} \leqslant c\omega_4(f, 1/m)_p \leqslant c\omega_4(f, 1/n)_p,$$

and that of (3) by applying for $\mathbf{z}_m = \mathbf{t}_m$,

$$\left(\sum_{j=0}^{m-1} \omega_4(f, z_{j+13} - z_{j-13}, [z_{j-13}, z_{j+13}])_p^p \right)^{1/p} \leqslant c\omega_4^{\varphi}(f, 1/m)_p \leqslant c\omega_4^{\varphi}(f, 1/n)_p.$$

These bounds are due to the equivalence of the regular modulus of smoothness with the averaged modulus of smoothness, see [3, pp. 184–185] and (4.1), (4.5) and (4.6) in [2] for details.

3.2. Monotone case ($q = 1$)

The proof is completely analogous to the above proof in the case $q = 2$. The only difference is that, instead of Theorem 2.4, we apply [6, Theorem 4.2] to obtain a quadratic spline S_2, which is monotone on $[z_3, z_{m-2}]$, and provides good approximation to f on $[-1, 1]$ (estimated using the third modulus of smoothness). Then, instead of (17), we can use a simpler definition

$$s(x) := \begin{cases} S_2(z_3), & x \in [z_0, z_3), \\ S_2(x), & x \in [z_5, z_{m-2}], \\ S_2(z_{m-2}), & x \in (z_{m-2}, z_m], \end{cases}$$

and s is clearly monotone on $[-1, 1]$. The remaining modifications are obvious.

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