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NEARLY MONOTONE SPLINE APPROXIMATION IN $\mathbb{L}_p$

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ABSTRACT. It is shown that the rate of $\mathbb{L}_p$-approximation of a non-decreasing function in $\mathbb{L}_p$, $0 < p < \infty$, by "nearly non-decreasing" splines can be estimated in terms of the third classical modulus of smoothness (for uniformly spaced knots) and third Ditzian-Totik modulus (for Chebyshev knots), and that estimates in terms of higher moduli are impossible. It is known that these estimates are no longer true for "purely" monotone spline approximation, and properties of intervals where the monotonicity restriction can be relaxed in order to achieve better approximation rate are investigated.

1. INTRODUCTION AND THE MAIN RESULTS

Throughout this paper, we denote by Π_r the space of algebraic polynomials of degree $\le r$, and by $\mathcal{S}_r(\mathbf{z}_n)$ the (linear) space of all piecewise polynomial functions (which we refer to as "splines") of degree r (order $r+1$) with the knots $\mathbf{z}_n := (z_i)_0^n$, $-1 =: z_0 < z_1 < \cdots < z_{n-1} < z_n := 1$. In other words, $s \in \mathcal{S}_r(\mathbf{z}_n)$ if, on each interval (z_{i-1}, z_i), $1 \le i \le n$, it is a polynomial of degree $\le r$, *i.e.*, $s|_{(z_{i-1},z_i)} \in \Pi_r$. Note that we do not put any restrictions on smoothness (or even continuity) of splines at the knots $\mathbf{z}_n$. We assume that a spline s and its derivatives are defined at the knots in $\mathbf{z}_n$ by continuity, if possible, and not defined otherwise. We also denote by $\mathbf{u}_n$ and $\mathbf{t}_n$ the sets of knots for the uniform and Chebyshev partitions, *i.e.*, $\mathbf{u}_n := \left(-1 + \frac{2i}{n}\right)_{i=0}^n$ and $\mathbf{t}_n := \left(\cos \frac{(n-i)\pi}{n}\right)_{i=0}^n$.

Given $q \ge 0$ and a set $J \subseteq [-1,1]$, a function f is said to be q-monotone on J if its qth divided differences $[x_0, \ldots, x_q]f$ are nonnegative for all choices of $(q+1)$ distinct points $x_0, \ldots, x_q$ in J. We denote the class of all such functions by $\mathcal{M}^q(J)$, and note that $\mathcal{M}^1(J)$ is the collection of all non-decreasing functions on J.

If $J \subseteq [-1,1]$, we denote by $\|\cdot\|_{\mathbb{L}_p(J)}$, $0 < p \le \infty$, the $\mathbb{L}_p$-(quasi)norm on J, and write $\|\cdot\|_p := \|\cdot\|_{\mathbb{L}_p[-1,1]}$. For a function $f \in \mathbb{L}_p := \mathbb{L}_p[-1,1]$, $0 < p \le \infty$, we denote by

$$E(f, \mathcal{F})_p := \inf_{s \in \mathcal{F}} \|f - s\|_p$$

the error of $\mathbb{L}_p$-approximation of f by elements from the set $\mathcal{F} \subset \mathbb{L}_p$. In particular,

$$\mathcal{E}_r^{(q)}(f, \mathbf{z}_n, J)_p := E(f, \mathcal{S}_r(\mathbf{z}_n) \cap \mathcal{M}^q(J))_p$$

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and

$$\widetilde{\mathcal{E}}_r^{(q)}(f, \mathbf{z}_n, J)_p := E(f, \mathcal{S}_r(\mathbf{z}_n) \cap \mathcal{M}^q(J) \cap \mathbb{C}^{r-1})_p$$

are the errors of $\mathbb{L}_p$-approximation of f by splines from $\mathcal{S}_r(\mathbf{z}_n)$ and from $\mathcal{S}_r(\mathbf{z}_n) \cap \mathbb{C}^{r-1}$ (*i.e.,* having maximum smoothness) which are q-monotone on $J \subseteq [-1, 1]$.

It is well known (see, *e.g.,* [2, 7]) that for a function $f \in \mathbb{L}_p \cap \mathcal{M}^1[-1, 1]$,

$$\widetilde{\mathcal{E}}_1^{(1)}(f, \mathbf{u}_n, [-1, 1])_p \le c\omega_2(f, 1/n)_p \quad \text{and} \quad \widetilde{\mathcal{E}}_1^{(1)}(f, \mathbf{t}_n, [-1, 1])_p \le c\omega_2^\varphi(f, 1/n)_p\,,$$

where c are constants which are independent of f and n, but dependent on p when $p \to 0$. (Throughout the paper, c denotes positive constants which are not necessarily the same even when they occur on the same line. For the definition of $\omega_k(f, 1/n)_p$ and $\omega_k^\varphi(f, 1/n)_p$ see below.) Moreover, these estimates are best possible in the sense that one cannot replace $\omega_2(f, 1/n)_p$ and $\omega_2^\varphi(f, 1/n)_p$ by $\omega_m(f, 1)_p$ for any $m > 2$ (see [10]).

It is natural to ask whether it is possible to improve the above estimates by relaxing the constraints on the approximating splines, for instance, by allowing them not to be non-decreasing in some small parts of the interval. We know (see [5]) that for non-decreasing $f \in \mathbb{C}[-1, 1]$ (*i.e.,* in the case $p = \infty$) this is indeed so. Namely, if we allow the splines not to be non-decreasing in small neighborhoods of the endpoints ± 1, then these inequalities with $p = \infty$ can be improved by considering quadratic splines instead of linear ones and replacing the right-hand sides with $\omega_3(f, 1/n)_\infty$ and $\omega_3^\varphi(f, 1/n)_\infty$, respectively. This problem remained unresolved for $p < \infty$, and the main purpose of this paper is to close this gap by investigating the relaxed constrained approximation of non-decreasing functions in $\mathbb{L}_p$, $0 < p < \infty$, by nearly non-decreasing splines.

Let

$$\Delta_h^k(f, x) := \begin{cases} \displaystyle\sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f(x - kh/2 + ih), & \text{if } |x \pm kh/2| < 1, \\ 0, & \text{otherwise}, \end{cases}$$

be the kth symmetric difference. Then the (classical) kth modulus of smoothness of a function $f \in \mathbb{L}_p[-1, 1]$ is defined by $\omega_k(f, t)_p := \sup_{0<h\le t} \|\Delta_h^k(f, \cdot)\|_p$, and the Ditzian-Totik kth modulus of smoothness is $\omega_k^\varphi(f, t)_p := \sup_{0<h\le t} \|\Delta_{h\varphi(\cdot)}^k(f, \cdot)\|_p$, where $\varphi(x) := \sqrt{1 - x^2}$. (It is well known that $\omega_k^\varphi(f, t)_p \le c\omega_k(f, t)_p$.) Finally, the kth modulus of smoothness on a subinterval $J \subset [-1, 1]$ is defined by $\omega_k(f, t, J)_p := \sup_{0<h\le t} \|\Delta_h^k(f, \cdot, J)\|_{\mathbb{L}_p(J)}$, where $\Delta_h^k(f, x, J) := \Delta_h^k(f, x)$ if $x \pm kh/2 \in J$, and $:= 0$ otherwise.

Theorem 1.1. *Let $f \in \mathbb{L}_p[-1, 1] \cap \mathcal{M}^1[-1, 1]$, $0 < p \le \infty$ (i.e., $f \in \mathbb{L}_p$ is a non-decreasing function on $[-1, 1]$). Then there exists an absolute constant $\kappa > 0$ such that, for every $n \in \mathbb{N}$,*

$$\widetilde{\mathcal{E}}_2^{(1)}(f, \mathbf{u}_n, [-1 + \kappa n^{-1}, 1 - \kappa n^{-1}])_p \le c\omega_3(f, 1/n)_p$$

and

$$\widetilde{\mathcal{E}}_2^{(1)}(f, \mathbf{t}_n, [-1 + \kappa n^{-2}, 1 - \kappa n^{-2}])_p \le c\omega_3^\varphi(f, 1/n)_p\,,$$

where c are constants independent of f and n which may depend on p as $p \to 0$.

In Section 2, we introduce the notation to be used throughout the paper, recall some well-known properties of algebraic polynomials and discuss properties of splines from $\mathcal{S}_r(\mathbf{z}_n)$. Then, in Section 3, we provide counterexamples that show that the estimates and assumptions in Theorem 1.1 are exact in some sense and cannot be improved. In Section 4, we provide construction of a nearly non-decreasing continuous quadratic spline and, in Section 5, we show how this spline can be "smoothed" to become continuously differentiable. Finally, Theorem 1.1 is proved in Section 6.

2. Notation and auxiliary results

Let $\mathbf{z}_n := \{z_0, \dots, z_n | -1 =: z_0 < z_1 < \cdots < z_n := 1\}$ be a partition of $[-1,1]$, and extend the notation by setting $z_j := -1$, $j < 0$, and $z_j := 1$, $j > n$. Throughout this paper, we use the notation $J_j := [z_j, z_{j+1}]$ and denote the scale of the partition $\mathbf{z}_n$ by

$$\vartheta := \vartheta(\mathbf{z}_n) := \max_{0 \le j \le n-1} \frac{|J_{j\pm 1}|}{|J_j|}, \tag{2.1}$$

where $|J|$ denotes the length of the interval J.

We now recall several well-known facts about algebraic polynomials which will be frequently used in the sequel. The first lemma is merely the equivalence of norms in a finite dimensional space and the well-known Markov's inequality.

Lemma 2.1. *For any polynomial $q_r \in \Pi_r$ and an interval J,*

$$\|q_r\|_{\mathbb{L}_p(J)} \le |J|^{1/p} \|q_r\|_{\mathbb{C}(J)} \le c\|q_r\|_{\mathbb{L}_p(J)}, \quad 0 < p \le \infty,$$

and

$$\|q_r'\|_{\mathbb{C}(J)} \le 2r^2 |J|^{-1} \|q_r\|_{\mathbb{C}(J)}.$$

Hence, in particular, for any $0 \le k \le r$,

$$\|q_r^{(k)}\|_{\mathbb{C}(J)} \le c|J|^{-k-1/p} \|q_r\|_{\mathbb{L}_p(J)}, \quad 0 < p \le \infty. \tag{2.2}$$

Constants c above depend only on r and p as $p \to 0$.

Lemma 2.2. *Let I and J be subintervals such that $I \subset J$. If $q_r \in \Pi_r$, then, for $0 < p \le \infty$, $\|q_r\|_{\mathbb{L}_p(J)} \le c\,(|J|/|I|)^{r+1/p} \|q_r\|_{\mathbb{L}_p(I)}$, where the constant c depends only on r and p as $p \to 0$.*

The following lemma now follows readily by Whitney's inequality

$$\inf_{p \in \Pi_r} \|f - p\|_{\mathbb{L}_p(I)} \le c\omega_{r+1}(f, |I|, I)_p.$$

Lemma 2.3. *Let $f \in \mathbb{L}_p[-1,1]$, $0 < p \le \infty$, and let I and J be subintervals such that $I \subset J \subseteq [-1,1]$. If $q_r \in \Pi_r$ is a polynomial satisfying $\|f - q_r\|_{\mathbb{L}_p(I)} \le c_0 \omega_{r+1}(f, |J|, J)_p$, then $\|f - q_r\|_{\mathbb{L}_p(J)} \le c\omega_{r+1}(f, |J|, J)_p$, with constant c which depends only on c_0, r, the ratio $|J|/|I|$, and p as $p \to 0$.*

We now present some properties of splines from $\mathcal{S}_r(\mathbf{z}_n)$.

Lemma 2.4. *For any $s \in \mathcal{S}_r(\mathbf{z}_n)$ and $0 \le k \le r$ we have*

$$|s^{(k)}(z_j+) - s^{(k)}(z_j-)| \le c|J_j|^{-k-1/p} \omega_{r+1}(s, |J_j|, [z_{j-1}, z_{j+1}])_p, \quad 1 \le j \le n-1,$$

where c depends on k, r, p and the scale $\vartheta(\mathbf{z}_n)$.

Proof. Denote $\mathrm{w} := \omega_{r+1}(s, |J_j|, [z_{j-1}, z_{j+1}])_p$. By Whitney's inequality, there is a polynomial $p \in \Pi_r$ such that $\|s - p\|_{\mathbb{L}_p[z_{j-1}, z_{j+1}]} \le c\mathrm{w}$. Inequality (2.2) implies

$$|s^{(k)}(z_j+) - p^{(k)}(z_j)| \le c|J_j|^{-k-1/p}\|s - p\|_{\mathbb{L}_p(J_j)} \le c|J_j|^{-k-1/p}\mathrm{w},$$

and, similarly, $|s^{(k)}(z_j-) - p^{(r)}(z_j)| \le c|J_{j-1}|^{-k-1/p}\mathrm{w}$. Hence,

$$\begin{aligned} |s^{(k)}(z_j+) - s^{(k)}(z_j-)| &\le |s^{(k)}(z_j+) - p^{(k)}(z_j)| + |s^{(k)}(z_j-) - p^{(k)}(z_j)| \\ &\le c|J_j|^{-k-1/p}\mathrm{w}. \end{aligned}$$

$\square$

Corollary 2.5. *Let $s \in \mathcal{S}_r(\mathbf{z}_n)$, and suppose that $s\big|_{J_j} =: p_j$, $0 \le j \le n-1$. Then, for $0 < p \le \infty$,*

$$\|p_j - p_{j-1}\|_{\mathbb{L}_p[z_{j-1}, z_{j+1}]} \le c\omega_{r+1}(s, z_{j+1} - z_{j-1}, [z_{j-1}, z_{j+1}])_p, \quad 1 \le j \le n-1,$$

where c depends on r, p (as $p \to 0$), and the scale $\vartheta(\mathbf{z}_n)$.

Proof. Since, by Taylor's formula,

$$p_j(x) - p_{j-1}(x) = \sum_{k=0}^{r} \frac{1}{k!}\left(p_j^{(k)}(z_j) - p_{j-1}^{(k)}(z_j)\right)(x - z_j)^k,$$

taking into account that $p_j^{(k)}(z_j) = s^{(k)}(z_j+)$ and $p_{j-1}^{(k)}(z_j) = s^{(k)}(z_j-)$, and using Lemma 2.4 we immediately get

$$\begin{aligned} \|p_j - p_{j-1}\|_{\mathbb{C}(J_j \cup J_{j-1})} &\le c\sum_{k=0}^{r} |J_j \cup J_{j-1}|^k \left|s^{(k)}(z_j+) - s^{(k)}(z_j-)\right| \\ &\le c|J_j \cup J_{j-1}|^{-1/p}\omega_{r+1}(s, z_{j+1} - z_{j-1}, [z_{j-1}, z_{j+1}])_p. \end{aligned}$$

Finally, Lemma 2.1 completes the proof. $\square$

Let $\delta_j := |J_j|/3$, $0 \le j \le n-1$, and denote $\hat{J}_j := (z_j, z_j + \delta_j)$, $0 \le j \le n-1$, and $\hat{J}_n := (1 - \delta_{n-1}, 1)$. The proof of the following lemma is exactly the same as in [1, Lemma 2.1].

Lemma 2.6. *Given $f \in \mathbb{L}_p[-1,1]$, $0 < p \le \infty$, and $r \in \mathbb{N}$. There are points $\xi_j^{(r)} \in \hat{J}_j$, $0 \le j \le n$, such that, for $0 \le j \le n-r$, the polynomial $L_{j,r} \in \Pi_r$ interpolating f at $\xi_i^{(r)}$, $i = j, j+1, \dots, j+r$, satisfies*

$$\|f - L_{j,r}\|_{\mathbb{L}_p(\bar{J}_j)} \le c\omega_{r+1}(f, |\bar{J}_j|, \bar{J}_j)_p, \tag{2.3}$$

where $\bar{J}_j := [z_{j-1}, z_{j+r+1}]$, and the constant c depends only on r, p (as $p \to 0$), and the scale $\vartheta(\mathbf{z}_n)$.

We now show that if a function f in the statement of Lemma 2.6 happens to be a spline from $\mathcal{S}_r(\mathbf{z}_n)$, then the inequality (2.3) is valid for arbitrary (but not too close to each other) points of interpolation.

Lemma 2.7. *Suppose that $r \in \mathbb{N}$, $s \in \mathcal{S}_r(\mathbf{z}_n)$, and $I := I_{\mu,\nu} := [z_\mu, z_\nu]$, where $0 \le \mu < \nu \le n$ and $\nu - \mu \le c_0$. Suppose further that the set $\{\xi_i\}_0^r \in I$ is such that $\min_{i \ne j} |\xi_i - \xi_j| \ge c_1|I|$. Then, the polynomial $L_r \in \Pi_r$ interpolating s at ξ_i, $0 \le i \le r$, satisfies*

$$\|s - L_r\|_{\mathbb{L}_p(I)} \le c\omega_{r+1}(s, |I|, I)_p \tag{2.4}$$

where the constant c depends only on r, c_0, c_1, p (as $p \to 0$), and the scale $\vartheta(\mathbf{z}_n)$.

Proof. We denote $s\,|_{J_i} =: p_i$, and note that, in order to prove (2.4) it suffices to estimate $\|p_l - L_r\|_{\mathbb{L}_p(I)}$ for $\mu \le l \le \nu - 1$. Taking into account that $L_r(\xi_i) = s(\xi_i) = p_{\nu_i}(\xi_i)$ for some $\mu \le \nu_i \le \nu - 1$, and using Lemmas 2.1 and 2.3 as well as the Lagrange interpolation formula (using $\min_{i\ne j} |\xi_i - \xi_j| \ge c_1|I|$) we have, for each $\mu \le l \le \nu - 1$,

$$\begin{aligned}
\|p_l - L_r\|_{\mathbb{L}_p(I)} &\le c|I|^{1/p}\|p_l - L_r\|_{\mathbb{C}(I)} \le c|I|^{1/p} \max_{0\le i\le r} |p_l(\xi_i) - L_r(\xi_i)| \\
&= c|I|^{1/p} \max_{0\le i\le r} |p_l(\xi_i) - p_{\nu_i}(\xi_i)| \le c|I|^{1/p} \max_{0\le i\le r} \|p_l - p_{\nu_i}\|_{\mathbb{C}(I)} \\
&\le c \max_{0\le i\le r} \|p_l - p_{\nu_i}\|_{\mathbb{L}_p(I)} \le c \sum_{i=\mu+1}^{\nu-1} \|p_i - p_{i-1}\|_{\mathbb{L}_p(I)} \\
&\le c\omega_{r+1}(s, |I|, I)_p\,,
\end{aligned}$$

where the last inequality follows from Corollary 2.5 and Lemma 2.2 (taking into account that $|[z_{i-1}, z_{i+1}]| \sim |I|$). ☐

3. Counterexamples

Theorem 3.1 implies that the third moduli of smoothness in the statement of Theorem 1.1 cannot be replaced with any moduli of higher order.

Theorem 3.1. *For any $k \in \mathbb{N}$, $A > 0$, $0 < p \le \infty$, $r \in \mathbb{N}$, $n \in \mathbb{N}$, and a partition $\mathbf{z}_n := \{z_0, \dots, z_n| -1 =: z_0 < z_1 < \cdots < z_n := 1\}$ of $[-1,1]$, there exists a function $f \in \mathbb{C}^k[-1,1] \cap \mathcal{M}^k[-1,1]$ such that*

$$\|f - q_r\|_{\mathbb{L}_p[z_\nu, z_{\nu+1}]} > A\omega_{k+3}(f,1)_p \tag{3.1}$$

for any $q_r \in \Pi_r$ satisfying $q^{(k)}(0) \ge 0$, where $0 \le \nu \le n-1$ is such that $z_\nu \le 0 < z_{\nu+1}$.

Proof. This proof is a modification of the proof of inequality (4.2) in [4] and, in fact, the idea can be traced back to the paper of Shvedov [10]. Let f be such that $f^{(k)}(x) := (x^2 - h^2)_+ := \max\{x^2 - h^2, 0\}$, where $h > 0$ is a constant to be prescribed. We now let a polynomial $Q \in \Pi_{k+2}$ be such that $Q^{(k)}(x) = x^2 - h^2$, and $Q^{(i)}(-1) = f^{(i)}(-1)$ for all $0 \le i \le k-1$. Then, since

$$f(x) - Q(x) = \frac{1}{(k-1)!}\int_{-1}^{x} (x-t)^{k-1}\left(f^{(k)}(t) - Q^{(k)}(t)\right) dt\,,$$

we have

$$\begin{aligned}
\|f - Q\|_{\mathbb{C}[-1,1]} &\le \frac{1}{(k-1)!}\int_{-1}^{1} (1-t)^{k-1}\left|f^{(k)}(t) - Q^{(k)}(t)\right| dt \\
&\le \frac{2^{k-1}}{(k-1)!}\int_{-h}^{h} (h^2 - t^2)\, dt = ch^3\,.
\end{aligned}$$

This implies that

$$\|f - Q\|_{\mathbb{L}_p[-1,1]} \le 2^{1/p}\,\|f - Q\|_{\mathbb{C}[-1,1]} \le ch^3$$

and

$$\omega_{k+3}(f,1)_p = \omega_{k+3}(f - Q, 1)_p \le c\,\|f - Q\|_{\mathbb{L}_p[-1,1]} \le ch^3\,.$$

Now, assume that (3.1) is not true, *i.e.*, that there exists a polynomial $P \in \Pi_r$ such that $P^{(k)}(0) \ge 0$ and $\|f-P\|_{\mathbb{L}_p[z_\nu, z_{\nu+1}]} \le A\omega_{k+3}(f,1)_p$. Then, for $0 \in J_\nu := [z_\nu, z_{\nu+1}]$, using (2.2), we have

$$\begin{aligned}\left|P^{(k)}(0)-Q^{(k)}(0)\right| &\le \left\|P^{(k)}-Q^{(k)}\right\|_{\mathbb{C}(J_\nu)} \le c\,\|P-Q\|_{\mathbb{L}_p(J_\nu)}\\ &\le c\left(\|P-f\|_{\mathbb{L}_p(J_\nu)} + \|f-Q\|_{\mathbb{L}_p(J_\nu)}\right)\\ &\le c\left(A\omega_{k+3}(f,1)_p + \|f-Q\|_{\mathbb{L}_p[-1,1]}\right) \le c_0 h^3\,,\end{aligned}$$

where c_0 depends on k, r, p, $|J_\nu|$, and A, and is independent of h. Finally,

$$P^{(k)}(0) \le Q^{(k)}(0) + \left|P^{(k)}(0)-Q^{(k)}(0)\right| \le -h^2 + c_0h^3 < 0\,,$$

for sufficiently small h, which is a contradiction. □

The following theorem shows that the intervals near the endpoints where approximating splines are allowed to be non-k-monotone cannot be much smaller than nearby intervals J_j produced by $\mathbf{z}_n$. (For the sake of simplicity we state it for the right-hand endpoint only.) It also implies that we will not get any improvement in orders of approximation if we relax the condition on k-monotonicity of the splines instead of near the endpoints, somewhere inside the interval $[-1,1]$.

Theorem 3.2. *Let $k \in \mathbb{N}$, $0 < p \le \infty$, $r \in \mathbb{N}$, and suppose that, for each $n \in \mathbb{N}$, $\xi_n \in (0,1)$ and partition $\mathbf{z}_n := \{z_0, \dots, z_n | -1 =: z_0 < z_1 < \cdots < z_n := 1\}$ are such that the number of indices in the set $\mathfrak{J} := \{j | J_j \cap [2\xi_n - 1, \xi_n] \ne \emptyset\}$ is bounded independently of n, i.e., $\operatorname{card}(\mathfrak{J}) \le c_0$, where c_0 is a constant independent of n. In addition, suppose that the scale of the partition $\mathbf{z}_n$ is bounded by an absolute constant ($\vartheta(\mathbf{z}_n) \le c_1$), and that*

$$\liminf_{n\to\infty} \frac{1-\xi_n}{|J_\nu|} = 0\,, \tag{3.2}$$

where $\nu := \min\{j | j \in \mathfrak{J}\}$. Then, for any $A > 0$, there exist an $n \in \mathbb{N}$ and a function $f \in \mathbb{C}^k[-1,1] \cap \mathcal{M}^k[-1,1]$ such that

$$\|f - q_r\|_{\mathbb{L}_p(J_\nu)} > A\omega_{k+2}(f,1)_p \tag{3.3}$$

for any $q_r \in \Pi_r$ satisfying $q_r^{(k)}(\xi_n) \ge 0$.

Proof. The idea is quite similar to the one used in the proof of Theorem 3.1 above. For convenience, we denote $d_n := 1-\xi_n$ everywhere in this proof. Let f be such that $f^{(k)}(x) := (1-2d_n-x)_+ := \max\{1-2d_n-x, 0\}$, and let a polynomial $Q \in \Pi_{k+1}$ be such that $Q^{(k)}(x) = 1-2d_n-x$, and $Q^{(i)}(-1) = f^{(i)}(-1)$ for all $0 \le i \le k-1$. Then, $f \equiv Q$ on $[-1, 1-2d_n]$ and, for any $x \in [1-2d_n, 1]$,

$$\begin{aligned}|f(x)-Q(x)| &\le \frac{1}{(k-1)!}\int_{-1}^{1}(1-t)^{k-1}\left|f^{(k)}(t)-Q^{(k)}(t)\right|\,dt\\ &\le \frac{2^{k-1}}{(k-1)!}d_n^{k-1}\int_{1-2d_n}^{1}(t-1+2d_n)\,dt\\ &\le c d_n^{k+1}\,.\end{aligned}$$

This implies that

$$\|f-Q\|_p = \|f-Q\|_{\mathbb{L}_p[1-2d_n,1]} \le (2d_n)^{1/p}\,\|f-Q\|_{\mathbb{C}[1-2d_n,1]} \le cd_n^{k+1+1/p}$$

and

$$\omega_{k+2}(f,1)_p = \omega_{k+2}(f-Q,1)_p \le c\,\|f-Q\|_{\mathbb{L}_p[-1,1]} \le cd_n^{k+1+1/p}\,.$$

Now, assume that (3.3) is not true, *i.e.,* that for any $n \in \mathbb{N}$, there exists a polynomial $P \in \Pi_r$ such that $P^{(k)}(\xi_n) \ge 0$ and $\|f-P\|_{\mathbb{L}_p(J_\nu)} \le A\omega_{k+2}(f,1)_p$. Then, letting $I(\mathfrak{J}) := \bigcup_{j\in\mathfrak{J}} J_j$, noting that $|J_\nu| \le |I(\mathfrak{J})| \le c_0 c_1^{c_0}|J_\nu|$, and using Lemmas 2.2 and 2.1 we have

$$\begin{aligned}
\left|P^{(k)}(\xi_n) - Q^{(k)}(\xi_n)\right| &\le \left\|P^{(k)} - Q^{(k)}\right\|_{\mathbb{C}(I(\mathfrak{J}))} \\
&\le c\left(|I(\mathfrak{J})|/|J_\nu|\right)^{\max\{r-k,1\}+1/p}\left\|P^{(k)} - Q^{(k)}\right\|_{\mathbb{C}(J_\nu)} \\
&\le c|J_\nu|^{-k-1/p}\,\|P-Q\|_{\mathbb{L}_p(J_\nu)} \\
&\le c|J_\nu|^{-k-1/p}\left(\|P-f\|_{\mathbb{L}_p(J_\nu)} + \|f-Q\|_{\mathbb{L}_p(J_\nu)}\right) \\
&\le c|J_\nu|^{-k-1/p}\left(A\omega_{k+2}(f,1)_p + \|f-Q\|_{\mathbb{L}_p[-1,1]}\right) \\
&\le c_2|J_\nu|^{-k-1/p} d_n^{k+1+1/p}\,,
\end{aligned}$$

where c_2 is independent of n. Finally, using (3.2), we get

$$\begin{aligned}
P^{(k)}(\xi_n) &\le Q^{(k)}(\xi_n) + \left|P^{(k)}(\xi_n) - Q^{(k)}(\xi_n)\right| \le -d_n + c_2|J_\nu|^{-k-1/p}d_n^{k+1+1/p} \\
&\le d_n\left(-1 + c_2\left(d_n/|J_\nu|\right)^{k+1/p}\right) < 0\,,
\end{aligned}$$

for sufficiently large n, which is a contradiction. □

4. Construction of nearly non-decreasing quadratic spline

We combine the ideas of DeVore, Hu, and Leviatan [1], with a construction by Leviatan and Shevchuk [5]. The following lemma is similar to [5, Lemma 1].

Lemma 4.1. *Let $\zeta_0 < \zeta_1 < \zeta_2 < \zeta_3$, and let $f \in \mathbb{L}_p[\zeta_0,\zeta_3]$, $0 < p \le \infty$, be non-decreasing on $[\zeta_0,\zeta_3]$. Assume that the quadratic polynomials Q_0 and Q_1 are such that, for $l = 0,1$, Q_l interpolates f at ζ_i, $i = l, l+1, l+2$, and satisfies*

$$\|f-Q_l\|_{\mathbb{L}_p[\zeta_1,\zeta_2]} \le E. \tag{4.1}$$

Then, there exists a quadratic polynomial q which is non-decreasing in $[\zeta_1,\zeta_2]$, interpolates f at ζ_1 and ζ_2, and such that

$$\|f-q\|_{\mathbb{L}_p[\zeta_1,\zeta_2]} \le 2^{1/p}E.$$

Proof. If either Q_0 or Q_1 is non-decreasing on $[\zeta_1,\zeta_2]$, then we take it for q and the assertion follows from (4.1). Otherwise, necessarily Q_0 is concave and Q_1 is convex, and since both interpolate f at ζ_1 and ζ_2, if we let L be the (non-decreasing) linear Lagrange polynomial interpolating f at ζ_1 and ζ_2, then it follows that

$$Q_1(x) \le L(x) \le Q_0(x), \quad x \in [\zeta_1,\zeta_2].$$

For $0 < p < \infty$, we have by virtue of (4.1),

$$\begin{aligned}\|f - L\|^p_{\mathbb{L}_p[\zeta_1,\zeta_2]} &\le \int_{\zeta_1}^{\zeta_2} \max\{|f(x) - Q_0(x)|^p, |f(x) - Q_1(x)|^p\}\, dx \\ &\le \int_{\zeta_1}^{\zeta_2} |f(x) - Q_1(x)|^p\, dx + \int_{\zeta_1}^{\zeta_2} |f(x) - Q_0(x)|^p\, dx \le 2E^p.\end{aligned}$$

Thus, we take $q := L$ and the proof is complete. □

Theorem 4.2. *Let $f \in \mathbb{L}_p[-1,1]$, $0 < p \le \infty$, be non-decreasing. Then there exists $s \in \mathcal{S}_2(\mathbf{z}_n) \cap \mathbb{C}[-1,1] \cap \mathcal{M}^1[z_3, z_{n-2}]$, such that*

$$\|f - s\|_{\mathbb{L}_p(J_j)} \le c\omega_3(f, z_{j+5} - z_{j-4}, [z_{j-4}, z_{j+5}])_p\,, \quad 0 \le j \le n-1. \tag{4.2}$$

Proof. First, for each $1 \le j \le n-2$, we use Lemmas 4.1 and 2.6 with $r = 2$, $\zeta_l = \xi^{(2)}_{j-1+l}$, $l = 0,1,2,3$, and

$$E := \max\left\{\omega_3(f, |\bar J_{j-1}|, \bar J_{j-1})_p, \omega_3(f, |\bar J_j|, \bar J_j)_p\right\} \le c\omega_3(f, z_{j+3} - z_{j-2}, [z_{j-2}, z_{j+3}])_p\,,$$

and obtain a non-decreasing $q_j \in \Pi_2$ interpolating f at $\xi^{(2)}_j$ and $\xi^{(2)}_{j+1}$, and satisfying

$$\|f - q_j\|_{\mathbb{L}_p[\xi^{(2)}_j, \xi^{(2)}_{j+1}]} \le c\omega_3(f, z_{j+3} - z_{j-2}, [z_{j-2}, z_{j+3}])_p\,. \tag{4.3}$$

We now define $\tilde s\big|_{[\xi^{(2)}_j, \xi^{(2)}_{j+1}]} := q_j$, $1 \le j \le n-2$. Thus, $\tilde s$ is a non-decreasing continuous quadratic spline which is defined on $[\xi^{(2)}_1, \xi^{(2)}_{n-1}]$ and is close to f. However, the knots of $\tilde s$ are not at $\mathbf{z}_n$, and so we need one additional step in our construction.

Let $\tilde Q_j$, $3 \le j \le n-2$, be the quadratic polynomial interpolating $\tilde s$ at z_i, $i = j-1, j, j+1$. Then, Lemma 2.7 with $r = 2$, knots $\{\xi^{(2)}_j\}_1^{n-1}$ instead of $\mathbf{z}_n$, $I_{\mu,\nu} = [\xi^{(2)}_{j-2}, \xi^{(2)}_{j+1}]$, and interpolation points z_{j-1}, z_j and z_{j+1}, implies

$$\|\tilde s - \tilde Q_j\|_{\mathbb{L}_p[\xi^{(2)}_{j-2}, \xi^{(2)}_{j+1}]} \le c\omega_3(\tilde s, \xi^{(2)}_{j+1} - \xi^{(2)}_{j-2}, [\xi^{(2)}_{j-2}, \xi^{(2)}_{j+1}])_p =: \tilde E_j.$$

For each $3 \le j \le n-3$, we now apply Lemma 4.1 with $\zeta_l = z_{j-1+l}$, $l = 0,1,2,3$, to conclude that there is a quadratic polynomial p_j which is non-decreasing on J_j, interpolates $\tilde s$ at z_j and z_{j+1}, and

$$\|p_j - \tilde s\|_{\mathbb{L}_p(J_j)} \le c\max\{\tilde E_j, \tilde E_{j+1}\} \le c\omega_3(\tilde s, z_{j+3} - z_{j-2}, [z_{j-2}, z_{j+3}])_p\,. \tag{4.4}$$

Now, we denote $s\big|_{J_j} := p_j$, $3 \le j \le n-3$, and extend s to $[-1,1]$ by setting $s\big|_{[z_{n-2},1]} := p_{n-3}$, and $s\big|_{[-1,z_3]} := p_3$. Obviously, the extension may not be non-decreasing in $[-1,1]$, but s is non-decreasing in $[z_3, z_{n-2}]$.

It remains to prove (4.2). For $3 \le j \le n-3$, using inequalities (4.3) and (4.4), we have

$$\begin{aligned}\|f - s\|_{\mathbb{L}_p(J_j)} &\le c\|f - \tilde s\|_{\mathbb{L}_p(J_j)} + c\|\tilde s - s\|_{\mathbb{L}_p(J_j)} \\ &\le c\|f - \tilde s\|_{\mathbb{L}_p(J_j)} + c\omega_3(\tilde s, z_{j+3} - z_{j-2}, [z_{j-2}, z_{j+3}])_p \\ &\le c\|f - \tilde s\|_{\mathbb{L}_p[z_{j-2}, z_{j+3}]} + c\omega_3(f, z_{j+3} - z_{j-2}, [z_{j-2}, z_{j+3}])_p \\ &\le c\omega_3(f, z_{j+5} - z_{j-4}, [z_{j-4}, z_{j+5}])_p\,.\end{aligned}$$

Finally, Lemma 2.3 immediately implies that (4.2) is valid for $j = 0, 1, 2, n-2, n-1$ as well. □

5. Smoothing lemma

In this section, we show how nearly non-decreasing splines constructed in Section 4 (which were only continuous) can be "smoothed" to become continuously differentiable.

We introduce, for each $1 \le j \le n-1$, the auxiliary functions

$$h_j(x) := \begin{cases} \dfrac{1}{2} \cdot \dfrac{z_{j+1} - z_j}{z_{j+1} - z_{j-1}} \cdot \dfrac{(x - z_{j-1})^2}{z_j - z_{j-1}}, & x \in [z_{j-1}, z_j], \\ \dfrac{1}{2} \cdot \dfrac{z_j - z_{j-1}}{z_{j+1} - z_{j-1}} \cdot \dfrac{(x - z_{j+1})^2}{z_{j+1} - z_j}, & x \in (z_j, z_{j+1}], \\ 0, & x \notin [z_{j-1}, z_{j+1}], \end{cases}$$

and

$$\phi_j(x) := \begin{cases} \dfrac{x - z_{j-1}}{z_j - z_{j-1}}, & x \in [z_{j-1}, z_j], \\ \dfrac{x - z_{j+1}}{z_j - z_{j+1}}, & x \in (z_j, z_{j+1}], \\ 0, & x \notin [z_{j-1}, z_{j+1}]. \end{cases}$$

Note that h_j and ϕ_j are continuous functions supported on $[z_{j-1}, z_{j+1}]$.

The proof of the following lemma is straightforward and will be omitted.

Lemma 5.1. *Each $s \in \mathcal{S}_1(\mathbf{z}_n)$ has the following representation:*

$$s(x) = \sum_{i=1}^{n-1} \alpha_i h_i'(x) + \sum_{i=0}^{n} \beta_i \phi_i(x), \quad x \in [-1,1] \setminus \mathbf{z}_n, \tag{5.1}$$

where $\alpha_i := s(z_i-) - s(z_i+)$, $1 \le i \le n-1$, and

$$\beta_i := \frac{z_{i+1} - z_i}{z_{i+1} - z_{i-1}} \cdot s(z_i+) + \frac{z_i - z_{i-1}}{z_{i+1} - z_{i-1}} \cdot s(z_i-), \quad 0 \le i \le n\,.$$

Now, we are ready to prove

Lemma 5.2. *Let $s \in \mathcal{S}_2(\mathbf{z}_n) \cap \mathbb{C}[-1,1] \cap \mathcal{M}^1[z_\mu, z_\nu]$, where $0 \le \mu < \nu \le n$. Then there is $\tilde{s} \in \mathcal{S}_2(\mathbf{z}_n) \cap \mathbb{C}^1[-1,1] \cap \mathcal{M}^1[z_{\mu+1}, z_{\nu-1}]$ satisfying*

$$\|s - \tilde{s}\|_{\mathbb{L}_p(J_j)} \le c\omega_3(s, z_{j+2} - z_{j-1}, [z_{j-1}, z_{j+2}])_p, \quad 0 \le j \le n-1,$$

where c depends on r, p, and the scale $\vartheta(\mathbf{z}_n)$.

Proof. Since $s' \in \mathcal{S}_1(\mathbf{z}_n)$ it follows by Lemma 5.1 that $s'(x) = \sum_{i=1}^{n-1} \alpha_i h_i'(x) + \sum_{i=0}^{n} \beta_i \phi_i(x)$, $x \notin \mathbf{z}_n$. We now define

$$\tilde{s}(x) := s(-1) + \sum_{i=0}^{n} \beta_i \int_{-1}^{x} \phi_i(t)\, dt, \quad x \in [-1,1].$$

Then clearly $\tilde{s} \in \mathcal{S}_2(\mathbf{z}_n) \cap \mathbb{C}^1[-1,1]$ and, for $x \in J_j$, $\tilde{s}'(x) = \sum_{i=0}^{n} \beta_i \phi_i(x) = \beta_j \phi_j(x) + \beta_{j+1} \phi_{j+1}(x)$. Since $\beta_i \ge 0$ for all $\mu + 1 \le i \le \nu - 1$ (because $s'(z_i\pm) \ge 0$ for these i), we conclude that $\tilde{s}'(x) \ge 0$ for $x \in J_j$, $\mu + 1 \le j \le \nu - 2$, and so $\tilde{s} \in \mathcal{M}^1[z_{\mu+1}, z_{\nu-1}]$.

Finally, for each $0 \le j \le n-1$, taking into account that $\|h_i\|_{\mathbb{L}_p[z_{i-1},z_{i+1}]} \le c|J_i|^{1+1/p}$, and using Lemma 2.4 we have

$$\begin{aligned}
\|s-\tilde{s}\|_{\mathbb{L}_p(J_j)} &= \left\|\sum_{i=1}^{n-1} \alpha_i h_i\right\|_{\mathbb{L}_p(J_j)} = \|\alpha_j h_j + \alpha_{j+1} h_{j+1}\|_{\mathbb{L}_p(J_j)} \\
&\le c|\alpha_j|\,\|h_j\|_{\mathbb{L}_p(J_j)} + c|\alpha_{j+1}|\,\|h_{j+1}\|_{\mathbb{L}_p(J_j)} \\
&\le c|J_j|^{-1-1/p}\omega_3(s, z_{j+1}-z_{j-1}, [z_{j-1}, z_{j+1}])_p \cdot |J_j|^{1+1/p} \\
&\quad + c|J_{j+1}|^{-1-1/p}\omega_3(s, z_{j+2}-z_j, [z_j, z_{j+2}])_p \cdot |J_{j+1}|^{1+1/p} \\
&\le c\omega_3(s, z_{j+2}-z_{j-1}, [z_{j-1}, z_{j+2}])_p\,.
\end{aligned}$$

□

6. Proof of Theorem 1.1

Let $f \in \mathbb{L}_p[-1,1]$, $0 < p < \infty$, be a non-decreasing function on $[-1,1]$. Theorem 4.2 and Lemma 5.2 imply that there exists $\tilde{s} \in \mathcal{S}_2(\mathbf{z}_n) \cap \mathbb{C}^1[-1,1] \cap \mathcal{M}^1[z_4, z_{n-3}]$ such that

$$\begin{aligned}
\|f-\tilde{s}\|_{\mathbb{L}_p(J_j)} &\le c\,\|f-s\|_{\mathbb{L}_p(J_j)} + c\,\|s-\tilde{s}\|_{\mathbb{L}_p(J_j)} \\
&\le c\,\|f-s\|_{\mathbb{L}_p(J_j)} + c\omega_3(s, z_{j+2}-z_{j-1}, [z_{j-1}, z_{j+2}])_p \\
&\le c\,\|f-s\|_{\mathbb{L}_p[z_{j-1},z_{j+2}]} + c\omega_3(f, z_{j+2}-z_{j-1}, [z_{j-1}, z_{j+2}])_p \\
&\le c\omega_3(f, z_{j+5}-z_{j-4}, [z_{j-4}, z_{j+5}])_p\,, \quad 0 \le j \le n-1\,.
\end{aligned}$$

Finally, since the scales of the uniform and Chebyshev partitions are bounded, the inequality

$$\|f-\tilde{s}\|_p^p = \sum_{j=0}^{n-1} \|f-\tilde{s}\|_{\mathbb{L}_p(J_j)}^p \le c\sum_{j=0}^{n-1} \omega_3(f, z_{j+5}-z_{j-4}, [z_{j-4}, z_{j+5}])_p^p$$

and the estimates (see, *e.g.*, [3, 2]) $\sum_{j=0}^{n-1} \omega_k(f, |J_j^*|, J_j^*)_p^p \le c\omega_k(f, 1/n)_p^p$ (if $\mathbf{z}_n = \mathbf{u}_n$), and $\sum_{j=0}^{n-1} \omega_k(f, |J_j^*|, J_j^*)_p^p \le c\omega_k^\varphi(f, 1/n)_p^p$ (if $\mathbf{z}_n = \mathbf{t}_n$), where $J_j \subset J_j^*$ and $|J_j| \sim |J_j^*|$, complete the proof of Theorem 1.1 for $0 < p < \infty$. In the case $p = \infty$, the proof is similar and, in fact, simpler (also see [5]).

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