On Some Properties of Moduli of Smoothness with Jacobi Weights

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Dedicated to the memory of our friend, colleague, and collaborator
Yingkang Hu (July 6, 1949–March 11, 2016)

Abstract We discuss some properties of the moduli of smoothness with Jacobi weights that we have recently introduced and that are defined as

$$\omega_{k,r}^{\varphi}(f^{(r)},t)_{\alpha,\beta,p} := \sup_{0\le h\le t} \left\| \mathcal{W}_{kh}^{r/2+\alpha,r/2+\beta}(\cdot)\Delta_{h\varphi(\cdot)}^{k}(f^{(r)},\cdot)\right\|_{p},$$

where $\varphi(x)=\sqrt{1-x^2}$, $\Delta_h^k(f,x)$ is the kth symmetric difference of f on $[-1,1]$,

$$\mathcal{W}_{\delta}^{\xi,\zeta}(x) := (1-x-\delta\varphi(x)/2)^{\xi}(1+x-\delta\varphi(x)/2)^{\zeta},$$

and $\alpha,\beta > -1/p$ if $0<p<\infty$, and $\alpha,\beta\ge 0$ if $p=\infty$.

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We show, among other things, that for all $m, n \in \mathbb{N}$, $0 < p \le \infty$, polynomials P_n of degree $< n$ and sufficiently small t,

$$\omega^{\varphi}_{m,0}(P_n, t)_{\alpha,\beta,p} \sim t\omega^{\varphi}_{m-1,1}(P_n', t)_{\alpha,\beta,p} \sim \cdots \sim t^{m-1}\omega^{\varphi}_{1,m-1}(P_n^{(m-1)}, t)_{\alpha,\beta,p}$$
$$\sim t^m \left\| w_{\alpha,\beta}\varphi^m P_n^{(m)} \right\|_p ,$$

where $w_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$ is the usual Jacobi weight.

In the spirit of Yingkang Hu's work, we apply this to characterize the behavior of the polynomials of best approximation of a function in a Jacobi weighted L_p space, $0 < p \le \infty$. Finally we discuss sharp Marchaud and Jackson type inequalities in the case $1 < p < \infty$.

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1 Introduction

Recall that the Jacobi weights are defined as $w_{\alpha,\beta}(x) := (1-x)^\alpha(1+x)^\beta$, where parameters α and β are usually assumed to be such that $w_{\alpha,\beta} \in L_p[-1,1]$, i.e.,

$$\alpha, \beta \in J_p := \begin{cases} (-1/p, \infty), & \text{if } 0 < p < \infty, \\ [0, \infty), & \text{if } p = \infty. \end{cases}$$

We denote by $\mathbb{P}_n$ the set of all algebraic polynomials of degree $\le n-1$, and $L_p^{\alpha,\beta}(I) := \left\{ f \mid \|w_{\alpha,\beta} f\|_{L_p(I)} < \infty \right\}$, where $I \subseteq [-1,1]$. For convenience, if $I = [-1,1]$, then we omit I from the notation. For example, $\|\cdot\|_p := \|\cdot\|_{L_p[-1,1]}$, $L_p^{\alpha,\beta} := L_p^{\alpha,\beta}[-1,1]$, etc.

Following [5] we denote $\mathbb{B}^0_p(w_{\alpha,\beta}) := L_p^{\alpha,\beta}$, and

$$\mathbb{B}^r_p(w_{\alpha,\beta}) := \left\{ f \mid f^{(r-1)} \in AC_{loc} \quad \text{and} \quad \varphi^r f^{(r)} \in L_p^{\alpha,\beta} \right\}, \quad r \ge 1,$$

where AC_{loc} denotes the set of functions which are locally absolutely continuous in $(-1,1)$, and $\varphi(x) := \sqrt{1-x^2}$. Also (see [5]), for $k, r \in \mathbb{N}$ and $f \in \mathbb{B}^r_p(w_{\alpha,\beta})$, let

$$\omega^{\varphi}_{k,r}(f^{(r)}, t)_{\alpha,\beta,p} := \sup_{0 \le h \le t} \left\| \mathcal{W}^{r/2+\alpha, r/2+\beta}_{kh}(\cdot)\Delta^k_{h\varphi(\cdot)}(f^{(r)}, \cdot) \right\|_p \tag{1.1}$$
$$= \sup_{0 < h \le t} \left\| \mathcal{W}^{r/2+\alpha, r/2+\beta}_{kh}(\cdot)\Delta^k_{h\varphi(\cdot)}(f^{(r)}, \cdot) \right\|_{L_p(\mathfrak{D}_{kh})},$$

where

$$\Delta_h^k(f,x;S) := \begin{cases} \sum_{i=0}^k \binom{k}{i}(-1)^{k-i} f(x-\frac{kh}{2}+ih), & \text{if } [x-\frac{kh}{2}, x+\frac{kh}{2}] \subseteq S\,, \\ 0, & \text{otherwise} \end{cases}$$

is the kth symmetric difference, $\Delta_h^k(f,x) := \Delta_h^k(f,x;[-1,1])$,

$$\mathcal{W}_\delta^{\xi,\zeta}(x) := (1-x-\delta\varphi(x)/2)^\xi(1+x-\delta\varphi(x)/2)^\zeta,$$

and

$$\mathfrak{D}_\delta := [-1+\mu(\delta), 1-\mu(\delta)], \quad \mu(\delta) := 2\delta^2/(4+\delta^2)$$

(note that $\Delta_{h\varphi(x)}^k(f,x) = 0$ if $x \notin \mathfrak{D}_{kh}$).

We define the main part weighted modulus of smoothness as

$$\Omega_{k,r}^\varphi(f^{(r)}, A, t)_{\alpha,\beta,p} := \sup_{0\le h\le t} \left\| w_{\alpha,\beta}(\cdot)\varphi^r(\cdot)\Delta_{h\varphi(\cdot)}^k(f^{(r)}, \cdot; \mathcal{I}_{A,h})\right\|_{L_p(\mathcal{I}_{A,h})}, \tag{1.2}$$

where $\mathcal{I}_{A,h} := [-1+Ah^2, 1-Ah^2]$ and $A > 0$.

We also denote

$$\Psi_{k,r}^\varphi(f^{(r)}, t)_{\alpha,\beta,p} := \sup_{0\le h\le t} \left\| w_{\alpha,\beta}(\cdot)\varphi^r(\cdot)\Delta_{h\varphi(\cdot)}^k(f^{(r)}, \cdot)\right\|_p, \tag{1.3}$$

i.e., $\Psi_{k,r}^\varphi$ is "the main part modulus $\Omega_{k,r}^\varphi$ with $A = 0$." However, we want to emphasize that while $\Omega_{k,r}^\varphi(f^{(r)}, A, t)_{\alpha,\beta,p}$ with $A > 0$ and $\omega_{k,r}^\varphi(f^{(r)}, t)_{\alpha,\beta,p}$ are bounded for all $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$ (see [5, Lemma 2.4]), the modulus $\Psi_{k,r}^\varphi(f^{(r)}, t)_{\alpha,\beta,p}$ may be infinite for such functions (for example, this is the case for f such that $f^{(r)}(x) = (1-x)^{-\gamma}$ with $1/p \le \gamma < \alpha + r/2 + 1/p$).

Remark 1.1 We note that the main part modulus is sometimes defined with the difference inside the norm not restricted to $\mathcal{I}_{A,h}$, i.e.,

$$\widetilde{\Omega}_{k,r}^\varphi(f^{(r)}, A, t)_{\alpha,\beta,p} := \sup_{0\le h\le t} \left\| w_{\alpha,\beta}(\cdot)\varphi^r(\cdot)\Delta_{h\varphi(\cdot)}^k(f^{(r)}, \cdot)\right\|_{L_p(\mathcal{I}_{A,h})}. \tag{1.4}$$

Clearly, $\Omega_{k,r}^\varphi(f^{(r)}, A, t)_{\alpha,\beta,p} \le \widetilde{\Omega}_{k,r}^\varphi(f^{(r)}, A, t)_{\alpha,\beta,p}$. Moreover, we have an estimate in the opposite direction as well if we replace A with a larger constant A'. For example, $\widetilde{\Omega}_{k,r}^\varphi(f^{(r)}, A', t)_{\alpha,\beta,p} \le \Omega_{k,r}^\varphi(f^{(r)}, A, t)_{\alpha,\beta,p}$, where $A' = 2\max\{A, k^2\}$ (see (2.9)). At the same time, if A is so small that $\mathfrak{D}_{kh} \subset \mathcal{I}_{A,h}$ (for example, if $A \le k^2/4$), then $\widetilde{\Omega}_{k,r}^\varphi(f^{(r)}, A, t)_{\alpha,\beta,p} = \Psi_{k,r}^\varphi(f^{(r)}, t)_{\alpha,\beta,p}$. Hence, all our results in this paper are valid with the modulus (1.2) replaced by (1.4) with an additional assumption that A is sufficiently large (assuming that $A \ge 2k^2$ will do).

Throughout this paper, we use the notation

$$q := \min\{1, p\},$$

and ϱ stands for some sufficiently small positive constant depending only on α, β, k, and q, and independent of n, to be prescribed in the proof of Theorem 2.1.

2 The Main Result

The following theorem is our main result.

Theorem 2.1 *Let $k, n \in \mathbb{N}$, $r \in \mathbb{N}_0$, $A > 0$, $0 < p \le \infty$, $\alpha + r/2, \beta + r/2 \in J_p$, and let $0 < t \le \varrho n^{-1}$, where ϱ is some positive constant that depends only on α, β, k, and q. Then, for any $P_n \in \mathbb{P}_n$,*

$$\omega^{\varphi}_{k,r}(P_n^{(r)}, t)_{\alpha,\beta,p} \sim \Psi^{\varphi}_{k,r}(P_n^{(r)}, t)_{\alpha,\beta,p} \sim \Omega^{\varphi}_{k,r}(P_n^{(r)}, A, t)_{\alpha,\beta,p} \tag{2.1}$$
$$\sim t^k \left\| w_{\alpha,\beta}\varphi^{k+r} P_n^{(k+r)} \right\|_p ,$$

where the equivalence constants depend only on k, r, α, β, A, and q.

The following is an immediate corollary of Theorem 2.1 by virtue of the fact that if $\alpha, \beta \in J_p$, then $\alpha + r/2, \beta + r/2 \in J_p$ for all $r \ge 0$.

Corollary 2.2 *Let $m, n \in \mathbb{N}$, $A > 0$, $0 < p \le \infty$, $\alpha, \beta \in J_p$, and let $0 < t \le \varrho n^{-1}$. Then, for any $P_n \in \mathbb{P}_n$, and any $k \in \mathbb{N}$ and $r \in \mathbb{N}_0$ such that $k + r = m$,*

$$t^{-k}\omega^{\varphi}_{k,r}(P_n^{(r)}, t)_{\alpha,\beta,p} \sim t^{-k}\Psi^{\varphi}_{k,r}(P_n^{(r)}, t)_{\alpha,\beta,p} \sim t^{-k}\Omega^{\varphi}_{k,r}(P_n^{(r)}, A, t)_{\alpha,\beta,p}$$
$$\sim \left\| w_{\alpha,\beta}\varphi^{m} P_n^{(m)} \right\|_p ,$$

where the equivalence constants depend only on m, α, β, A, and q.

It was shown in [5, Corollary 1.9] that, for $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $r/2 + \alpha \ge 0$, $r/2 + \beta \ge 0$, $1 \le p \le \infty$, $f \in \mathbb{B}^r_p(w_{\alpha,\beta})$, $\lambda \ge 1$, and all $t > 0$,

$$\omega^{\varphi}_{k,r}(f^{(r)}, \lambda t)_{\alpha,\beta,p} \le c\lambda^k \omega^{\varphi}_{k,r}(f^{(r)}, t)_{\alpha,\beta,p}.$$

Hence, in the case $1 \le p \le \infty$, we can strengthen Corollary 2.2 for the moduli $\omega^{\varphi}_{k,r}$. Namely, the following result is valid.

Corollary 2.3 *Let $m, n \in \mathbb{N}$, $1 \le p \le \infty$, $\alpha, \beta \in J_p$, $\Lambda > 0$, and let $0 < t \le \Lambda n^{-1}$. Then, for any $P_n \in \mathbb{P}_n$, and any $k \in \mathbb{N}$ and $r \in \mathbb{N}_0$ such that $k + r = m$,*

$$t^{-k}\omega^{\varphi}_{k,r}(P_n^{(r)}, t)_{\alpha,\beta,p} \sim \left\| w_{\alpha,\beta}\varphi^{m} P_n^{(m)} \right\|_p ,$$

where the equivalence constants depend only on m, α, β, and Λ.

Remark 2.4 In the case $1 \le p \le \infty$, several equivalences in Theorem 2.1 and Corollary 2.2 follow from [4, Theorems 4 and 5], since, as was shown in [5, (1.8)], for $1 \le p \le \infty$,

$$\omega_{k,r}^{\varphi}(f^{(r)}, t)_{\alpha,\beta,p} \sim \omega_{\varphi}^{k}(f^{(r)}, t)_{w_{\alpha,\beta}\varphi^r,p}, \quad 0 < t \le t_0, \tag{2.2}$$

where $\omega_{\varphi}^{k}(g, t)_{w,p}$ is the three-part weighted Ditzian–Totik modulus of smoothness (see, e.g., [5, (5.1)] for its definition).

Note that it is still an open problem if (2.2) is valid if $0 < p < 1$.

Proof of Theorem 2.1 The main idea of the proof is not much different from that of [4, Theorems 3–5].

First, we note that it suffices to prove Theorem 2.1 in the case $r = 0$. Indeed, suppose we proved that, for $k, n \in \mathbb{N}$, $A > 0$, $0 < t \le \varrho n^{-1}$, $0 < p \le \infty$, $\alpha, \beta \in J_p$, and any polynomial $Q_n \in \mathbb{P}_n$,

$$\begin{aligned}\omega_{k,0}^{\varphi}(Q_n, t)_{\alpha,\beta,p} \sim \Psi_{k,0}^{\varphi}(Q_n, t)_{\alpha,\beta,p} \sim \Omega_{k,0}^{\varphi}(Q_n, A, t)_{\alpha,\beta,p} \\ \sim t^k \left\| w_{\alpha,\beta}\varphi^k Q_n^{(k)} \right\|_p .\end{aligned} \tag{2.3}$$

Then, if P_n is an arbitrary polynomial from $\mathbb{P}_n$, and r is an arbitrary natural number, assuming that $n > r$ (otherwise, $P_n^{(r)} \equiv 0$ and there is nothing to prove) and denoting $Q := P_n^{(r)} \in \mathbb{P}_{n-r}$, we have

$$\omega_{k,r}^{\varphi}(P_n^{(r)}, t)_{\alpha,\beta,p} = \omega_{k,0}^{\varphi}(Q, t)_{\alpha+r/2,\beta+r/2,p},$$

$$\Psi_{k,r}^{\varphi}(P_n^{(r)}, t)_{\alpha,\beta,p} = \Psi_{k,0}^{\varphi}(Q, t)_{\alpha+r/2,\beta+r/2,p},$$

$$\Omega_{k,r}^{\varphi}(P_n^{(r)}, t)_{\alpha,\beta,p} = \Omega_{k,0}^{\varphi}(Q, A, t)_{\alpha+r/2,\beta+r/2,p},$$

and

$$\left\| w_{\alpha,\beta}\varphi^{k+r} P_n^{(k+r)} \right\|_p = \left\| \omega_{\alpha+r/2,\beta+r/2}\varphi^k Q^{(k)} \right\|_p ,$$

and so (2.1) follows from (2.3) with α and β replaced by $\alpha + r/2$ and $\beta + r/2$, respectively.

Now, note that it immediately follows from the definition that

$$\omega_{k,0}^{\varphi}(g, t)_{\alpha,\beta,p} \le \Psi_{k,0}^{\varphi}(g, t)_{\alpha,\beta,p}.$$

Also, for $A > 0$,

$$\Omega_{k,0}^{\varphi}(g, A, t)_{\alpha,\beta,p} \le c\omega_{k,0}^{\varphi}(g, t)_{\alpha,\beta,p},$$

since $w_{\alpha,\beta}(x) \le c\mathcal{W}_{kh}^{\alpha,\beta}(x)$ for x such that $x \pm kh\varphi(x)/2 \in \mathcal{I}_{A,h}$.

Hence, in order to prove (2.3), it suffices to show that

$$\Psi_{k,0}^{\varphi}(Q_n, t)_{\alpha,\beta,p} \le ct^k \left\| w_{\alpha,\beta}\varphi^k Q_n^{(k)} \right\|_p \tag{2.4}$$

and

$$t^k \left\| w_{\alpha,\beta}\varphi^k Q_n^{(k)} \right\|_p \le c\Omega_{k,0}^{\varphi}(Q_n, A, t)_{\alpha,\beta,p}. \tag{2.5}$$

Recall the following Bernstein–Dzyadyk-type inequality that follows from [4, (2.24)]: if $0 < p \le \infty$, $\alpha, \beta \in J_p$, and $P_n \in \mathbb{P}_n$, then

$$\left\| w_{\alpha,\beta}\varphi^s P_n' \right\|_p \le cns \left\| w_{\alpha,\beta}\varphi^{s-1} P_n \right\|_p, \quad 1 \le s \le n-1,$$

where c depends only on α, β and q, and is independent of n and s.

This implies that, for any $Q_n \in \mathbb{P}_n$ and $k, j \in \mathbb{N}$,

$$\left\| w_{\alpha,\beta}\varphi^{k+j} Q_n^{(k+j)} \right\|_p \le (c_0 n)^j \frac{(k+j)!}{k!} \left\| w_{\alpha,\beta}\varphi^k Q_n^{(k)} \right\|_p, \quad 1 \le k+j \le n-1. \tag{2.6}$$

We now use the following identity (see [4, (2.4)]):

for any $Q_n \in \mathbb{P}_n$ and $k \in \mathbb{N}$, we have

$$\Delta_{h\varphi(x)}^k(Q_n, x) = \sum_{i=0}^{K} \frac{1}{(2i)!}\varphi^{k+2i}(x) Q_n^{(k+2i)}(x) h^{k+2i} \xi_{k+2i}^{2i}, \tag{2.7}$$

where $K := \lfloor (n-1-k)/2 \rfloor$, and $\xi_j \in (-k/2, k/2)$ depends only on k and j.

Applying (2.6), we obtain, for $0 \le i \le K$ and $0 < h \le t \le \varrho n^{-1}$,

$$\begin{aligned}
\left\| \frac{1}{(2i)!} w_{\alpha,\beta}\varphi^{k+2i} Q_n^{(k+2i)} \right\|_p h^{2i} |\xi_{k+2i}|^{2i} &\le (c_0 \varrho k/2)^{2i} \frac{(k+2i)!}{(2i)!k!} \left\| w_{\alpha,\beta}\varphi^k Q_n^{(k)} \right\|_p \\
&\le [c_0 \varrho k(k+1)/2]^{2i} \left\| w_{\alpha,\beta}\varphi^k Q_n^{(k)} \right\|_p \\
&\le B^{2i} \left\| w_{\alpha,\beta}\varphi^k Q_n^{(k)} \right\|_p,
\end{aligned}$$

where we used the estimate $(k+2i)!/((2i)!k!) \le (k+1)^{2i}$, and where ϱ is taken so small that the last estimate holds with $B := (1/3)^{1/(2q)}$. Note that $\sum_{i=1}^{\infty} B^{2iq} = 1/2$.

Hence, it follows from (2.7) that

$$\begin{aligned}\left\| w_{\alpha,\beta}\Delta_{h\varphi}^k(Q_n,\cdot)\right\|_p^q &\le h^{kq}\sum_{i=0}^{K}\left\|\frac{1}{(2i)!}w_{\alpha,\beta}\varphi^{k+2i}Q_n^{(k+2i)}\right\|_p^q h^{2iq}|\xi|_{k+2i}^{2iq}\\ &\le h^{kq}\left\|w_{\alpha,\beta}\varphi^k Q_n^{(k)}\right\|_p^q\left(1+\sum_{i=1}^{K}B^{2iq}\right)\\ &\le 3/2\cdot h^{kq}\left\|w_{\alpha,\beta}\varphi^k Q_n^{(k)}\right\|_p^q .\end{aligned}$$

This immediately implies

$$\Psi_{k,0}^{\varphi}(Q_n,t)_{\alpha,\beta,p} \le (3/2)^{1/q}t^k\left\|w_{\alpha,\beta}\varphi^k Q_n^{(k)}\right\|_p ,$$

and so (2.4) is proved.

Recall now the following Remez-type inequality (see, e.g., [4, (2.22)]):

If $0 < p \le \infty$, $\alpha, \beta \in J_p$, $a \ge 0$, $n \in \mathbb{N}$ is such that $n > \sqrt{a}$, and $P_n \in \mathbb{P}_n$, then

$$\left\|w_{\alpha,\beta}P_n\right\|_p \le c\left\|w_{\alpha,\beta}P_n\right\|_{L_p[-1+an^{-2},1-an^{-2}]}, \tag{2.8}$$

where c depends only on α, β, a, and q.

Note that

$$\begin{aligned}\Omega_{k,0}^{\varphi}(Q_n,A,t)_{\alpha,\beta,p} &= \sup_{0\le h\le t}\left\|w_{\alpha,\beta}(\cdot)\Delta_{h\varphi(\cdot)}^k(Q_n,\cdot;\mathcal{I}_{A,h})\right\|_{L_p(\mathcal{I}_{A,h})}\\ &= \sup_{0\le h\le t}\left\|w_{\alpha,\beta}(\cdot)\Delta_{h\varphi(\cdot)}^k(Q_n,\cdot)\right\|_{L_p(\mathcal{S}_{k,A,h})},\end{aligned}$$

where the set $\mathcal{S}_{k,A,h}$ is an interval containing all x so that $x \pm kh\varphi(x)/2 \in \mathcal{I}_{A,h}$. Observe that

$$\mathcal{S}_{k,A,h} \supset \mathcal{I}_{A',h},$$

where $A' := 2\max\{A, k^2\}$, and so

$$\Omega_{k,0}^{\varphi}(Q_n,A,t)_{\alpha,\beta,p} \ge \sup_{0\le h\le t}\left\|w_{\alpha,\beta}(\cdot)\Delta_{h\varphi(\cdot)}^k(Q_n,\cdot)\right\|_{L_p(\mathcal{I}_{A',h})}. \tag{2.9}$$

Now it follows from (2.7) that $\Delta^k_{h\varphi(x)}(Q_n, x)$ is a polynomial from $\mathbb{P}_n$ if k is even, and it is a polynomial from $\mathbb{P}_{n-1}$ multiplied by φ if k is odd.

Hence, (2.8) implies that, for $h \le 1/(\sqrt{2A'}n)$,

$$\left\| w_{\alpha,\beta}\Delta^k_{h\varphi}(Q_n,\cdot)\right\|_{L_p(\mathcal{I}_{A',h})} \ge \left\| w_{\alpha,\beta}\Delta^k_{h\varphi}(Q_n,\cdot)\right\|_{L_p[-1+n^{-2}/2,1-n^{-2}/2]} \tag{2.10}$$

$$\ge c\left\| w_{\alpha,\beta}\Delta^k_{h\varphi}(Q_n,\cdot)\right\|_p .$$

It now follows from (2.7) that

$$\Delta^k_{h\varphi(x)}(Q_n, x) - \varphi^k(x)Q_n^{(k)}(x)h^k = \sum_{i=1}^{K}\frac{1}{(2i)!}\varphi^{k+2i}(x)Q_n^{(k+2i)}(x)h^{k+2i}\xi^{2i}_{k+2i},$$

and so, as above,

$$\left\| w_{\alpha,\beta}\left(\Delta^k_{h\varphi}(Q_n,\cdot) - \varphi^k Q_n^{(k)}h^k\right)\right\|_p^q \le 1/2\cdot h^{kq}\left\| w_{\alpha,\beta}\varphi^k Q_n^{(k)}\right\|_p^q .$$

Therefore,

$$\left\| w_{\alpha,\beta}\Delta^k_{h\varphi}(Q_n,\cdot)\right\|_p^q \ge 1/2\cdot h^{kq}\left\| w_{\alpha,\beta}\varphi^k Q_n^{(k)}\right\|_p^q ,$$

which combined with (2.9) and (2.10) implies (2.5). □

3 The Polynomials of Best Approximation

For $f \in L_p^{\alpha,\beta}$, let $P_n^* = P_n^*(f) \in \mathbb{P}_n$ and $E_n(f)_{w_{\alpha,\beta},p}$ be a polynomial and the degree of its best weighted approximation, respectively, i.e.,

$$E_n(f)_{w_{\alpha,\beta},p} := \inf_{p_n\in\mathbb{P}_n}\|w_{\alpha,\beta}(f-p_n)\|_p = \|w_{\alpha,\beta}(f-P_n^*)\|_p .$$

Recall (see [5, Lemma 2.4] and [6, Theorem 1.4]) that if $\alpha \ge 0$ and $\beta \ge 0$, then, for any $k \in \mathbb{N}$, $0 < p \le \infty$ and $f \in L_p^{\alpha,\beta}$,

$$\omega^{\varphi}_{k,0}(f,t)_{\alpha,\beta,p} \le c\left\| w_{\alpha,\beta}f\right\|_p, \quad t > 0, \tag{3.1}$$

with c depending only on k, α, β, and q. Also, for any $0 < \vartheta \le 1$,

$$E_n(f)_{w_{\alpha,\beta},p} \le c\omega^{\varphi}_{k,0}(f,\vartheta n^{-1})_{\alpha,\beta,p}, \quad n \ge k, \tag{3.2}$$

where c depends on ϑ as well as k, α, β, and q.

Theorem 3.1 *Let $k \in \mathbb{N}$, $\alpha, \beta \ge 0$, $0 < p \le \infty$, and $f \in L_p^{\alpha,\beta}$. Then, for any $n \in \mathbb{N}$,*

$$n^{-k}\|w_{\alpha,\beta}\varphi^k P_n^{*(k)}\|_p \le c\omega_{k,0}^{\varphi}(P_n^*, t)_{\alpha,\beta,p} \le c\omega_{k,0}^{\varphi}(f, t)_{\alpha,\beta,p}, \quad t \ge \varrho n^{-1}, \tag{3.3}$$

where constants c depend only on k, α, β, and q.

Conversely, for $0 < t \le \varrho/k$ and $n := \lfloor \varrho/t \rfloor$,

$$\begin{aligned} \omega_{k,0}^{\varphi}(f, t)_{\alpha,\beta,p} &\le c\left(\sum_{j=0}^{\infty} \omega_{k,0}^{\varphi}(P_{2^j n}^*, \varrho 2^{-j} n^{-1})_{\alpha,\beta,p}^q\right)^{1/q} \\ &\le c\left(\sum_{j=0}^{\infty} 2^{-jkq} n^{-kq}\|w_{\alpha,\beta}\varphi^k P_{2^j n}^{*(k)}\|_p^q\right)^{1/q}, \end{aligned} \tag{3.4}$$

where c depends only on k, α, β, and q.

Corollary 3.2 *Let $k \in \mathbb{N}$, $\alpha, \beta \ge 0$, $0 < p \le \infty$, $f \in L_p^{\alpha,\beta}$, and $\gamma > 0$. Then,*

$$\|w_{\alpha,\beta}\varphi^k P_n^{*(k)}\|_p = O(n^{k-\gamma}) \quad \textit{iff} \quad \omega_{k,0}^{\varphi}(f, t)_{\alpha,\beta,p} = O(t^{\gamma}). \tag{3.5}$$

Proof of Theorem 3.1 In order to prove (3.3), one may assume that $n \ge k$. By Theorem 2.1 we have

$$n^{-k}\|w_{\alpha,\beta}\varphi^k P_n^{*(k)}\|_p \le c\varrho^{-k}\omega_{k,0}^{\varphi}(P_n^*, \varrho n^{-1})_{\alpha,\beta,p} \le c\omega_{k,0}^{\varphi}(P_n^*, t)_{\alpha,\beta,p}.$$

At the same time, by (3.1) and (3.2) with $\vartheta = \varrho$,

$$\begin{aligned} \omega_{k,0}^{\varphi}(P_n^*, t)_{\alpha,\beta,p}^q &\le \omega_{k,0}^{\varphi}(f - P_n^*, t)_{\alpha,\beta,p}^q + \omega_{k,0}^{\varphi}(f, t)_{\alpha,\beta,p}^q \\ &\le c\|w_{\alpha,\beta}(f - P_n^*)\|_p^q + \omega_{k,0}^{\varphi}(f, t)_{\alpha,\beta,p}^q \\ &\le c\omega_{k,0}^{\varphi}(f, \varrho n^{-1})_{\alpha,\beta,p}^q + \omega_{k,0}^{\varphi}(f, t)_{\alpha,\beta,p}^q \\ &\le c\omega_{k,0}^{\varphi}(f, t)_{\alpha,\beta,p}^q, \end{aligned}$$

and (3.3) follows.

In order to prove (3.4) we follow [4]. Assume that $0 < t \le \varrho/k$ and note that $n = \lfloor \varrho/t \rfloor \ge k$. Let $\hat{P}_n \in \mathbb{P}_n$ be a polynomial of best weighted approximation of P_{2n}^*, i.e.,

$$I_n := \left\|w_{\alpha,\beta}(P_{2n}^* - \hat{P}_n)\right\|_p = E_n(P_{2n}^*)_{w_{\alpha,\beta},p}.$$

Then, (3.2) with $\vartheta = \varrho/2$ implies that

$$I_n \le c\omega_{k,0}^{\varphi}(P_{2n}^*, \varrho(2n)^{-1})_{\alpha,\beta,p},$$

while

$$I_n^q \ge \|w_{\alpha,\beta}(f - \hat{P}_n)\|_p^q - \left\|w_{\alpha,\beta}(f - P_{2n}^*)\right\|_p^q \ge E_n(f)_{w_{\alpha,\beta},p}^q - E_{2n}(f)_{w_{\alpha,\beta},p}^q.$$

Combining the above inequalities we obtain

$$\begin{aligned} E_n(f)_{w_{\alpha,\beta},p}^q &= \sum_{j=0}^{\infty}\left(E_{2^j n}(f)_{w_{\alpha,\beta},p}^q - E_{2^{j+1}n}(f)_{w_{\alpha,\beta},p}^q\right) \le \sum_{j=0}^{\infty} I_{2^j n}^q \\ &\le c\sum_{j=1}^{\infty} \omega_{k,0}^{\varphi}(P_{2^j n}^*, \varrho 2^{-j} n^{-1})_{\alpha,\beta,p}^q. \end{aligned}$$

Hence,

$$\begin{aligned} \omega_{k,0}^{\varphi}(f,t)_{\alpha,\beta,p}^q &\le c\omega_{k,0}^{\varphi}(f - P_n^*, t)_{\alpha,\beta,p}^q + c\omega_{k,0}^{\varphi}(P_n^*, t)_{\alpha,\beta,p}^q \\ &\le cE_n(f)_{w_{\alpha,\beta},p}^q + c\omega_{k,0}^{\varphi}(P_n^*, \varrho n^{-1})_{\alpha,\beta,p}^q \\ &\le c\sum_{j=0}^{\infty} \omega_{k,0}^{\varphi}(P_{2^j n}^*, \varrho 2^{-j} n^{-1})_{\alpha,\beta,p}^q \\ &\le c\sum_{j=0}^{\infty} 2^{-jkq} n^{-kq} \|w_{\alpha,\beta}\varphi^k P_{2^j n}^{*(k)}\|_p^q, \end{aligned}$$

where, for the last inequality, we used Theorem 2.1. This completes the proof of (3.4). □

4 Further Properties of the Moduli

Following [5, Definition 1.4], for $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, and $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$, $1 \le p \le \infty$, we define the weighted K-functional as follows:

$$\begin{aligned} &K_{k,r}^{\varphi}(f^{(r)}, t^k)_{\alpha,\beta,p} \\ &\quad := \inf_{g \in \mathbb{B}_p^{k+r}(w_{\alpha,\beta})} \left\{ \left\| w_{\alpha,\beta}\varphi^r (f^{(r)} - g^{(r)}) \right\|_p + t^k \left\| w_{\alpha,\beta}\varphi^{k+r} g^{(k+r)} \right\|_p \right\}. \end{aligned}$$

We note that

$$K_{k,\varphi}(f,t^k)_{w_{\alpha,\beta},p} = K^{\varphi}_{k,0}(f,t^k)_{\alpha,\beta,p},$$

where $K_{k,\varphi}(f,t^k)_{w,p}$ is the weighted K-functional that was defined in [3, p. 55 (6.1.1)] as

$$K_{k,\varphi}(f,t^k)_{w,p} := \inf_{g\in\mathbb{B}^k_p(w)}\{\|w(f-g)\|_p + t^k\|w\varphi^k g^{(k)}\|_p\}.$$

The following lemma immediately follows from [5, Corollary 1.7].

Lemma 4.1 *If* $k\in\mathbb{N}$*,* $r\in\mathbb{N}_0$*,* $r/2+\alpha\ge 0$*,* $r/2+\beta\ge 0$*,* $1\le p\le\infty$*, and* $f\in\mathbb{B}^r_p(w_{\alpha,\beta})$*, then, for all* $0<t\le 2/k$*,*

$$K^{\varphi}_{k,r}(f^{(r)},t^k)_{\alpha,\beta,p} \le c\omega^{\varphi}_{k,r}(f^{(r)},t)_{\alpha,\beta,p} \le cK^{\varphi}_{k,r}(f^{(r)},t^k)_{\alpha,\beta,p}.$$

Hence,

$$\omega^{\varphi}_{k,r}(f^{(r)},t)_{\alpha,\beta,p} \sim K^{\varphi}_{k,r}(f^{(r)},t^k)_{\alpha,\beta,p} = K_{k,\varphi}(f^{(r)},t^k)_{w_{\alpha+r/2,\beta+r/2},p}, \tag{4.1}$$

provided that all conditions in Lemma 4.1 are satisfied.

The following sharp Marchaud inequality was proved in [1] for $f\in L^{\alpha,\beta}_p$, $1<p<\infty$.

Theorem 4.2 ([1, Theorem 7.5]) *For* $m\in\mathbb{N}$*,* $1<p<\infty$*, and* $\alpha,\beta\in J_p$*, we have*

$$K_{m,\varphi}(f,t^m)_{w_{\alpha,\beta},p} \le Ct^m\left(\int_t^1 \frac{K_{m+1,\varphi}(f,u^{m+1})^{s_*}_{w_{\alpha,\beta},p}}{u^{ms_*+1}}\,du + E_m(f)^{s_*}_{w_{\alpha,\beta},p}\right)^{1/s_*}$$

and

$$K_{m,\varphi}(f,t^m)_{w_{\alpha,\beta},p} \le Ct^m\left(\sum_{n<1/t} n^{s_*m-1}E_n(f)^{s_*}_{w_{\alpha,\beta},p}\right)^{1/s_*},$$

where $s_*=\min\{2,p\}$.

In view of (4.1), the following result holds.

Corollary 4.3 *For $1 < p < \infty$, $r \in \mathbb{N}_0$, $m \in \mathbb{N}$, $r/2+\alpha \ge 0$, $r/2+\beta \ge 0$, and $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$, we have*

$$\omega_{m,r}^{\varphi}(f^{(r)}, t)_{\alpha,\beta,p} \le Ct^m \left(\int_t^1 \frac{\omega_{m+1,r}^{\varphi}(f^{(r)}, u)_{\alpha,\beta,p}^{s_*}}{u^{ms_*+1}}\, du + E_m(f^{(r)})_{w_{\alpha,\beta}\varphi^r,p}^{s_*} \right)^{1/s_*}$$

and

$$\omega_{m,r}^{\varphi}(f^{(r)}, t)_{\alpha,\beta,p} \le Ct^m \left(\sum_{n<1/t} n^{s_* m-1} E_n(f^{(r)})_{w_{\alpha,\beta}\varphi^r,p}^{s_*} \right)^{1/s_*},$$

where $s_* = \min\{2, p\}$.

The following sharp Jackson inequality was proved in [2].

Theorem 4.4 ([2, Theorem 6.2]) *For $1 < p < \infty$, $\alpha, \beta \in J_p$, and $m \in \mathbb{N}$, we have*

$$2^{-nm} \left(\sum_{j=j_0}^{n} 2^{mjs^*} E_{2^j}(f)_{w_{\alpha,\beta},p}^{s^*} \right)^{1/s^*} \le CK_{m,\varphi}(f, 2^{-nm})_{w_{\alpha,\beta},p}$$

and

$$2^{-nm} \left(\sum_{j=j_0}^{n} 2^{mjs^*} K_{m+1,\varphi}(f, 2^{-j(m+1)})_{w_{\alpha,\beta},p}^{s^*} \right)^{1/s^*} \le CK_{m,\varphi}(f, 2^{-nm})_{w_{\alpha,\beta},p},$$

where $2^{j_0} \ge m$ and $s^* = \max\{p, 2\}$.

Again, by virtue of (4.1), we have

Corollary 4.5 *For $1 < p < \infty$, $r \in \mathbb{N}_0$, $m \in \mathbb{N}$, $r/2+\alpha \ge 0$, $r/2+\beta \ge 0$, and $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$, we have*

$$2^{-nm} \left(\sum_{j=j_0}^{n} 2^{mjs^*} E_{2^j}(f^{(r)})_{w_{\alpha,\beta}\varphi^r,p}^{s^*} \right)^{1/s^*} \le C\omega_{m,r}^{\varphi}(f^{(r)}, 2^{-n})_{\alpha,\beta,p}$$

and

$$2^{-nm} \left(\sum_{j=j_0}^{n} 2^{mjs^*} \omega_{m+1,r}^{\varphi}(f^{(r)}, 2^{-j})_{\alpha,\beta,p}^{s^*} \right)^{1/s^*} \le C\omega_{m,r}^{\varphi}(f^{(r)}, 2^{-n})_{\alpha,\beta,p},$$

where $2^{j_0} \ge m$ and $s^* = \max\{p, 2\}$.

Corollary 4.6 *For* $1 < p < \infty$, $r \in \mathbb{N}_0$, $m \in \mathbb{N}$, $r/2 + \alpha \geq 0$, $r/2 + \beta \geq 0$, *and* $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$, *we have*

$$t^m \left(\int_t^{1/m} \frac{\omega_{m+1,r}^{\varphi}(f^{(r)}, u)_{\alpha,\beta,p}^{s^*}}{u^{ms^*+1}} \, du \right)^{1/s^*} \leq C\omega_{m,r}^{\varphi}(f^{(r)}, t)_{\alpha,\beta,p}, \quad 0 < t \leq 1/m,$$

where $s^* = \max\{p, 2\}$.

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