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ARE THE DEGREES OF BEST (CO)CONVEX AND UNCONSTRAINED POLYNOMIAL APPROXIMATION THE SAME?*

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Dedicated to Jóska Szabados on his 70*th birthday*

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Abstract. Let $\mathbb{C}[-1,1]$ be the space of continuous functions on $[-1,1]$, and denote by Δ^2 the set of convex functions $f \in \mathbb{C}[-1,1]$. Also, let $E_n(f)$ and $E_n^{(2)}(f)$ denote the degrees of best unconstrained and convex approximation of $f \in \Delta^2$ by algebraic polynomials of degree $< n$, respectively. Clearly, $E_n(f) \leqq E_n^{(2)}(f)$, and Lorentz and Zeller proved that the inverse inequality $E_n^{(2)}(f) \leqq cE_n(f)$ is invalid even with the constant $c = c(f)$ which depends on the function $f \in \Delta^2$.

In this paper we prove, for every $\alpha > 0$ and function $f \in \Delta^2$, that

$$\sup\left\{ n^\alpha E_n^{(2)}(f) : n \in \mathbb{N} \right\} \leqq c(\alpha) \sup\left\{ n^\alpha E_n(f) : n \in \mathbb{N} \right\},$$

where $c(\alpha)$ is a constant depending only on α. Validity of similar results for the class of piecewise convex functions having s convexity changes inside $(-1,1)$ is also investigated. It turns out that there are substantial differences between the cases $s \leqq 1$ and $s \geqq 2$.

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1. Introduction and main results

Let $\mathbb{C}[-1,1]$ be the space of continuous functions on $[-1,1]$ equipped with the uniform norm $\|\cdot\|$, and denote by Δ^2 the set of all convex functions $f \in \mathbb{C}[-1,1]$. If $\mathbb{P}_n$ is the space of algebraic polynomials of degree $< n$, then

$$E_n(f) = \inf\left\{ \|f - P_n\| : P_n \in \mathbb{P}_n \right\}$$

and

$$E_n^{(2)}(f) = \inf\left\{ \|f - P_n\| : P_n \in \mathbb{P}_n \cap \Delta^2 \right\},$$

denote the degrees of best unconstrained and convex approximation of a function f by polynomials from $\mathbb{P}_n$, respectively.

Clearly, $E_n(f) \leqq E_n^{(2)}(f)$, and Lorentz and Zeller [8] proved that the inverse inequality $E_n^{(2)}(f) \leqq cE_n(f)$ is not true in general even with the constant $c = c(f)$ which depends on the function $f \in \Delta^2$.

Despite all this, the following result is valid.

Theorem 1.1. *For every $\alpha > 0$ and $f \in \Delta^2$ we have*

$$\sup\left\{ n^\alpha E_n^{(2)}(f) : n \in \mathbb{N} \right\} \leqq c(\alpha) \sup\left\{ n^\alpha E_n(f) : n \in \mathbb{N} \right\}, \tag{1.1}$$

where $c(\alpha)$ is a constant that depends only on α.

Clearly (1.1) is meaningful only when the right-hand side is $< \infty$.

A natural question now is whether similar results are valid for piecewise convex functions, i.e., functions which change their convexity $s < \infty$ times in the interval $(-1,1)$. Surprisingly, we discovered that the situations are rather different in the cases $s \leqq 1$ and $s \geqq 2$.

In order to give precise statements we need some additional definitions. Let $\mathbb{Y}_s$, $s \in \mathbb{N}$, be the set of all collections $Y_s := \{y_i\}_{i=1}^s$ of points y_i, such that $-1 < y_s < \cdots < y_1 < 1$. For $Y_s \in \mathbb{Y}_s$ denote by $\Delta^2(Y_s)$ the set of all piecewise convex functions $f \in \mathbb{C}[-1,1]$, that change convexity at the points Y_s, and are convex on $[y_1, 1]$. In other words, a continuous function f is in $\Delta^2(Y_s)$ if and only if it is convex (concave) on $[y_{i+1}, y_i]$, for all even (odd) indices $0 \leqq i \leqq s$, where $y_0 := 1$ and $y_{s+1} := -1$. Note that if f is twice continuously differentiable in $(-1,1)$, then $f \in \Delta^2(Y_s)$ if and only if

$$f''(x)\Pi(x;Y_s) \geqq 0, \quad x \in (-1,1), \quad \text{where} \quad \Pi(x;Y_s) := \prod_{i=1}^s (x - y_i).$$

Denote by

$$E_n^{(2)}(f, Y_s) = \inf\left\{ \|f - P_n\| : P_n \in \mathbb{P}_n \cap \Delta^2(Y_s) \right\},$$

the degree of best coconvex approximation of a function $f \in \Delta^2(Y_s)$. For the sake of uniformity, we also write $\mathbb{Y}_0 := \{\emptyset\}$, $\Delta^2(\mathbb{Y}_0) := \Delta^2$, $E_n^{(2)}(f, \mathbb{Y}_0) := E_n^{(2)}(f)$, etc.

Throughout this paper, we denote by c absolute positive constants which may vary from one occurrence to another even when they appear in the same line. Similarly, $c(\dots)$ and $N(\dots)$ denote positive real constants and natural numbers, respectively, which depend only on the parameters/sets/functions in the parentheses. For example, $N(\alpha, Y_s)$ denotes a natural number which depends only on α and Y_s and does not depend on anything else.

In the case $s = 1$ (i.e., when the function changes convexity only once), we have

THEOREM 1.2. *For every $\alpha > 0$, $\alpha \neq 4$, $Y_1 \in \mathbb{Y}_1$ and $f \in \Delta^2(Y_1)$, we have*

$$\sup\left\{n^\alpha E_n^{(2)}(f, Y_1) : n \in \mathbb{N}\right\} \leqq c(\alpha) \sup\left\{n^\alpha E_n(f) : n \in \mathbb{N}\right\}. \tag{1.2}$$

The case $s = 1$, $\alpha = 4$, turns out to be completely different and resembles what is happening for $s \geqq 2$ (see below). For this case, on the positive side, we have

THEOREM 1.3. *There is an absolute constant c, such that for every $Y_1 \in \mathbb{Y}_1$ and a function $f \in \Delta^2(Y_1)$ the inequality*

$$\sup\left\{n^4 E_n^{(2)}(f, Y_1) : n > (1 - y_1^2)^{-1/2}\right\} \leqq c \sup\left\{n^4 E_n(f) : n \in \mathbb{N}\right\}, \tag{1.3}$$

holds.

Let

$$\varphi(x) := \sqrt{1 - x^2}.$$

Since constant functions are in $\Delta^2(Y_1)$ for every $Y_1 \in \mathbb{Y}_1$, we have

$$n^4 E_n^{(2)}(f, Y_1) \leqq n^4 E_1^{(2)}(f, Y_1) = n^4 E_1(f) \leqq \varphi(y_1)^{-4} \sup\left\{n^4 E_n(f) : n \in \mathbb{N}\right\},$$

for all $1 \leqq n \leqq \varphi(y_1)^{-1}$. Therefore, the following statement immediately follows from Theorem 1.3.

COROLLARY 1.4. *For every $Y_1 \in \mathbb{Y}_1$ and a function $f \in \Delta^2(Y_1)$ the inequality*

$$\sup\left\{n^4 E_n^{(2)}(f, Y_1) : n \in \mathbb{N}\right\} \leqq c(Y_1) \sup\left\{n^4 E_n(f) : n \in \mathbb{N}\right\},$$

holds.

Theorem 1.3 and Corollary 1.4 cannot be improved since, on the negative side, we have

THEOREM 1.5. *For every* $Y_1 \in \mathbb{Y}_1$ *there exists a function* $f \in \Delta^2(Y_1)$*, satisfying*

$$\sup\left\{ n^4 E_n(f) : n \in \mathbb{N} \right\} = 1,$$

such that for each $m \in \mathbb{N}$*, we have*

$$m^4 E_m^{(2)}(f, Y_1) \geqq \left(c \ln \frac{m}{1 + m^2 \varphi(y_1)} - 1 \right), \tag{1.4}$$

and

$$\sup\left\{ n^4 E_n^{(2)}(f, Y_1) : n \in \mathbb{N} \right\} \geqq c \left| \ln \varphi(y_1) \right|. \tag{1.5}$$

As we have already indicated above, the situation when $s \geqq 2$ is quite different than the general case for $s = 1$.

THEOREM 1.6. *Let* $s \geqq 2$*. For every* $\alpha > 0$ *and each* $Y_s \in \mathbb{Y}_s$ *there exists* $N(\alpha, Y_s)$ *with the property that if* $f \in \Delta^2(Y_s)$*, then*

$$\sup\left\{ n^\alpha E_n^{(2)}(f, Y_s) : n \geqq N(\alpha, Y_s) \right\} \leqq c(\alpha, s) \sup\left\{ n^\alpha E_n(f) : n \in \mathbb{N} \right\}. \tag{1.6}$$

A negative result related to Theorem 1.6 (i.e., the fact that, for no $\alpha > 0$ and $s \geqq 2$, can N in (1.6) be made independent of Y_s) is an immediate consequence of the following theorem.

THEOREM 1.7. *Let* $s \geqq 2$*. Then for every* $\alpha > 0$ *and* $m \in \mathbb{N}$*, there exist a collection* $Y_s \in \mathbb{Y}_s$ *and a function* $f \in \Delta^2(Y_s)$*, such that*

$$m^\alpha E_m^{(2)}(f, Y_s) \geqq c(\alpha, s) m^{\alpha_*} \sup\left\{ n^\alpha E_n(f) : n \in \mathbb{N} \right\}, \tag{1.7}$$

where $\alpha_* = \alpha + 1 - \lceil \alpha \rceil$ *and* $\lceil \alpha \rceil$ *is the smallest integer not smaller than* α*.*

2. Auxiliary results

2.1. Ditzian–Totik weighted moduli of smoothness and related function classes. Recall that $\varphi(x) = \sqrt{1 - x^2}$, and let $\mathbb{B}^r$, $r \in \mathbb{N}$, be the space of all functions $f \in \mathbb{C}[-1, 1]$ with locally absolutely continuous $(r-1)$st derivative in $(-1, 1)$ such that $\left\| \varphi^r f^{(r)} \right\|_\infty < \infty$, where for $g \in \mathbb{L}_\infty(-1, 1)$, we write $\|g\|_\infty := \operatorname{ess\,sup}_{x \in [-1,1]} \left| g(x) \right|$. It is well known, that if $f \in \mathbb{B}^r$, then

$$n^r E_n(f) \leqq c(r) \left\| \varphi^r f^{(r)} \right\|_\infty, \quad n \geqq r. \tag{2.1}$$

Also let $\mathbb{C}_\varphi^r$, be the space of functions $f \in \mathbb{C}^r(-1,1) \cap \mathbb{C}[-1,1]$ such that

$$\lim_{x\to\pm 1} \varphi(x)^r f^{(r)}(x) = 0,$$

and $\mathbb{C}_\varphi^0 := \mathbb{C}[-1,1]$. Then the weighted Ditzian–Totik modulus of smoothness of rth derivative of a function $f \in \mathbb{C}_\varphi^r$, is defined by

$$\omega_{k,r}^{\varphi}\big(f^{(r)},t\big) := \sup_{0<h\leqq t} \sup_{x} \varphi(x)^{r/2}\big(\varphi(x) - kh\big(1+|x|\big)/2\big)^{r/2}\big|\Delta_{h\varphi(x)}^k\big(f^{(r)},x\big)\big|,$$

where the inner supremum is taken over all x, such that $\varphi(x) > kh\big(1+|x|\big)/2$, and

$$\Delta_\delta^k(g,x) := \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} g(x - k\delta/2 + i\delta)$$

is the kth symmetric difference of a function g with a step δ.

If $r = 0$, then

$$\omega_k^{\varphi}(f,t) := \omega_{k,0}^{\varphi}(f,t)$$

is the (usual) Ditzian–Totik modulus of smoothness. Finally, we denote by $\|f\|_{C[a,b]}$ the sup-norm of $f \in \mathbb{C}[a,b]$ (note that $\|f\|_{C[-1,1]} = \|f\|$). Recall that the ordinary kth modulus of smoothness of $f \in \mathbb{C}[a,b]$ is

$$\omega_k\big(f,t,[a,b]\big) := \sup_{0<h\leqq t} \big\| \Delta_h^k(f,\cdot)\big\|_{\mathbb{C}[a+kh/2,b-kh/2]},$$

and denote $\omega_k(f,t) := \omega_k\big(f,t,[-1,1]\big)$.

Clearly

$$\mathbb{C}_\varphi^r \subset \mathbb{B}^r, \tag{2.2}$$

while it is known (see, *e.g.*, [2, Ch. 3.10]) that if $f \in \mathbb{B}^r$, then $f \in \mathbb{C}_\varphi^l$ for all $0 \leqq l < r$, and

$$\omega_{r-l,l}^{\varphi}\big(f^{(l)},t\big) \leqq ct^{r-l}\big\|\varphi^r f^{(r)}\big\|_\infty, \qquad t > 0. \tag{2.3}$$

Note that if $f \in \mathbb{C}_\varphi^r$, then the following inequality holds for all $0 \leqq l \leqq r$ and $k \geqq 1$ (see [2, Ch. 3.10]):

$$\omega_{k+r-l,l}^{\varphi}\big(f^{(l)},t\big) \leqq c\, t^{r-l} \omega_{k,r}^{\varphi}\big(f^{(r)},t\big), \qquad t > 0. \tag{2.4}$$

2.2. Auxiliary lemmas. The following results are so-called inverse theorems. They characterize the smoothness (i.e., describe the class) of functions that have the order of polynomial approximation $n^{-\alpha}$.

Theorem 2.1 [2, Ch. 7.2]. *Let $r \in \mathbb{N}_0$, $k \in \mathbb{N}$ and $\alpha > 0$, be such that $r < \alpha < k+r$, and let $f \in \mathbb{C}[-1,1]$. If*

$$n^\alpha E_n(f) \leqq 1, \quad \text{for all } n \geqq k+r,$$

then $f \in \mathbb{C}^r_\varphi$ and

$$\omega^\varphi_{k,r}\big(f^{(r)}, t\big) \leqq c(\alpha, k, r) t^{\alpha - r}.$$

Theorem 2.2. *Let $2r < \alpha < 2k+2r$, and $f \in \mathbb{C}[-1,1]$. If*

$$n^\alpha E_n(f) \leqq 1, \qquad n \geqq k+r,$$

then $f \in \mathbb{C}^r[-1,1]$ and

$$\omega_k\big(f^{(r)}, t\big) \leqq c(\alpha, k, r)\big)\, t^{\alpha/2 - r}. \tag{2.5}$$

Proof. Set $\rho_n(x) := \max\big\{n^{-2}, n^{-1}\varphi(x)\big\}$. Since $n^{-2} \leqq \rho_n(x)$, there is a sequence of polynomials $P_n \in \mathbb{P}_n$, such that

$$\big|f(x) - P_n(x)\big| \leqq c\rho_n^{\alpha/2}(x)$$

for all $x \in [-1,1]$ and $n \geqq k+r$. By the classical inverse theorem [2, Theorem 7.1.2] this implies that $f \in \mathbb{C}^r[-1,1]$ and (2.5) is satisfied. $\square$

Let $x_j := \cos(j\pi/n)$, $0 \leqq j \leqq n$, be the Chebyshev knots, and denote $I_j := [x_j, x_{j-1}]$ and $|I_j| := x_{j-1} - x_j$, $1 \leqq j \leqq n$. Denote by $\Sigma_{k,n}$ the collection of all continuous piecewise polynomials of degree $< k$, on the Chebyshev partition $\{x_j\}_{j=0}^n$. Also, denote by $\Sigma_{k,n}(Y_s)$ the subset of $\Sigma_{k,n}$ consisting of those continuous piecewise polynomials S that do not have any knots "too close" to the points $y_i \in Y_s$ of convexity change. More precisely, if $Y_s \in \mathbb{Y}_s$ and j_i, $1 \leqq i \leqq s$, are chosen so that $y_i \in [x_{j_i}, x_{j_i - 1})$, then S is in $\Sigma_{k,n}(Y_s)$ if and only if $S \in \Sigma_{k,n}$ and, for every $1 \leqq i \leqq s$, the restriction of S to (x_{j_i+1}, x_{j_i-2}) is a polynomial, where $x_{n+l} := -1$, $x_{-l} := 1$, $l \in \mathbb{N}$.

We will need the following well known relation for $f \in \mathbb{C}^r_\varphi$ (see, e.g., [4], (3.4)),

$$|I_j|^r \omega_k\big(f^{(r)}, |I_j|, I_j\big) \leqq c(k,r) n^{-r} \omega_{k,r}\big(f^{(r)}, 1/n\big), \quad 2 \leqq j \leqq n-1.$$

Then Besov's inequality (see, e.g., [2, (3.5.2)]) implies

Lemma 2.3. *For each k, r and $f \in C^r_\varphi$,*

$$|I_j|^r \big\|f^{(r)}\big\|_{C(I_j)} \leqq c(k,r)\big(\|f\|_{C(I_j)} + n^{-r}\omega^\varphi_{k,r}\big(f^{(r)}, 1/n\big)\big), \quad 2 \leqq j \leqq n-1. \tag{2.6}$$

The following lemma allows us to reduce the proofs of the direct estimates of (co)convex polynomial approximation to those for spline approximation which are usually much simpler.

LEMMA 2.4 [7, Theorem 3]. *For every* $m \geqq 1$ *and* $s \geqq 0$ *there are constants* $c = c(m,s)$ *and* $c_* = c_*(m,s)$, *such that if* $n \geqq 1$, $Y_s \in \mathbb{Y}_s$, *and* $S \in \Sigma_{m,n}(Y_s) \cap \Delta^2(Y_s)$, *then*

$$E^{(2)}_{c_* n}(S, Y_s) \leqq c\omega^{\varphi}_m(S, 1/n). \tag{2.7}$$

The following two lemmas immediately follow from [6, Proof of Theorem 2.5].

LEMMA 2.5. *For every* $f \in \Delta^2 \cap \mathbb{C}^2_\varphi$, $k \in \mathbb{N}$ *and* $n \geqq 5$, *there is* $S \in \Sigma_{k+2,n} \cap \Delta^2$, *such that*

$$\|f - S\|_{\mathbb{C}[x_{n-2}, x_2]} \leqq c(k) n^{-2} \omega^{\varphi}_{k,2}(f'', 1/n).$$

LEMMA 2.6. *Given* $Y_s \in \mathbb{Y}_s$, $s \geqq 1$, *and* $k \in \mathbb{N}$, *there exists* $N = N(k, Y_s)$ *such that, for every* $f \in \Delta^2(Y_s) \cap \mathbb{C}^3_\varphi$ *and* $n \geqq N$, *there is* $S \in \Sigma_{k+3,n}(Y_s) \cap \Delta^2(Y_s)$, *such that*

$$\|f - S\|_{\mathbb{C}[x_{n-2}, x_2]} \leqq c(k,s) n^{-3} \omega^{\varphi}_{k,3}\big(f^{(3)}, 1/n\big).$$

Moreover, if $s = 1$, *then* $N(k, Y_1) = 5$.

The following lemma is a consequence of [6, Proof of Theorem 2.7 and 2.8].

LEMMA 2.7. *Given* $Y_1 = \{y_1\} \in \mathbb{Y}_1$, *for every* $f \in \Delta^2(Y_1) \cap \mathbb{C}^2_\varphi$ *and* $n \geqq 5$, *there is* $S \in \Sigma_{5,n}(Y_1) \cap \Delta^2(Y_1)$, *such that*

$$\|f - S\|_{\mathbb{C}[x_{n-2}, x_2]} \leqq cn^{-2} \omega^{\varphi}_{3,2}(f'', 1/n).$$

The next auxiliary lemma is needed for the proof of the positive results.

LEMMA 2.8. I. *Let* $f \in \Delta^2$. *Then, for* $f \in \mathbb{C}^2_\varphi \cap \mathbb{C}^1[-1,1]$, *we have*

$$E^{(2)}_n(f) \leqq c(k) n^{-2} \omega^{\varphi}_{k,2}(f'', 1/n) + c(k) n^{-2} \omega_2(f', 1/n^2), \quad n \geqq 3. \tag{2.8}$$

Moreover, if $f \in \mathbb{C}^2_\varphi \cap \mathbb{C}^2[-1,1]$, *and* $k, l \in \mathbb{N}$, *then, for* $n \geqq l + 2$, *we have*

$$E^{(2)}_n(f) \leqq c(k,l) n^{-2} \omega^{\varphi}_{k,2}(f'', 1/n) + c(k,l) n^{-4} \omega_l(f'', 1/n^2). \tag{2.9}$$

II. *Let* $f \in \Delta^2(Y_1)$. *Then*

$$E^{(2)}_n(f, Y_1) \leqq c\omega^{\varphi}_3(f, 1/n) + c\omega_2(f, 1/n^2), \quad n \geqq 2. \tag{2.10}$$

If, in addition, $f \in \mathbb{C}^2_\varphi \cap \mathbb{C}^1[-1,1]$, *then*

$$E_n^{(2)}(f, Y_1) \leqq cn^{-2}\omega^\varphi_{3,2}(f'', 1/n) + cn^{-2}\omega_1(f', 1/n^2), \quad n \geqq 2, \tag{2.11}$$

and

$$E_n^{(2)}(f, Y_1) \leqq cn^{-2}\omega^\varphi_{3,2}(f'', 1/n) + cn^{-2}\omega_2(f', 1/n^2), \quad n\varphi(y_1) > 1. \tag{2.12}$$

Moreover, if we actually have $f \in \mathbb{C}^3_\varphi \cap \mathbb{C}^2[-1,1]$, *then for any* $k \in \mathbb{N}$,

$$E_n^{(2)}(f, Y_1) \leqq c(k)n^{-3}\omega^\varphi_{k,3}\big(f^{(3)}, 1/n\big) + c(k)n^{-4}\omega_2(f'', 1/n^2), \quad n \geqq 4. \tag{2.13}$$

III. *If* $f \in \Delta^2(Y_s) \cap \mathbb{C}^3_\varphi \cap \mathbb{C}^2[-1,1]$, $s \in \mathbb{N}$, *and* $k, l \in \mathbb{N}$, *then there exists* $N(Y_s, k, l)$ *such that, for all* $n \geqq N(Y_s, k, l)$,

$$E_n^{(2)}(f, Y_s) \leqq c(k,l,s)n^{-3}\omega^\varphi_{k,3}\big(f^{(3)}, 1/n\big) + c(k,l,s)n^{-4}\omega_l(f'', 1/n^2). \tag{2.14}$$

PROOF. Suppose that we have proved (2.8)–(2.14) for the approximation by piecewise polynomials instead of by polynomials. More precisely, suppose that (2.8)–(2.14) are valid if we replace $E_n^{(2)}(f, Y_s)$ on the left-hand sides of these inequalities by

$$\sigma^{(2)}_{k_1,n}(f, Y_s) := \inf\big\{\, \|f - S\| : \, S \in \Sigma_{k_1,n}(Y_s) \cap \Delta^2(Y_s)\big\}$$

with sufficiently large k_1. Observing that (2.7) implies

$$E_n^{(2)}(S, Y_s) \leqq c\omega^\varphi_{k_1}(S, 1/n), \quad n \geqq c_*, \tag{2.15}$$

and taking $S \in \Sigma_{k_1,n}(Y_s) \cap \Delta^2(Y_s)$ so that $\|f - S\| \leqq 2\sigma^{(2)}_{k_1,n}(f, Y_s)$, we conclude that

$$E_n^{(2)}(f, Y_s) \leqq E_n^{(2)}(S, Y_s) + 2\sigma^{(2)}_{k_1,n}(f, Y_s) \leqq c\omega^\varphi_{k_1}(f, 1/n) + c\sigma^{(2)}_{k_1,n}(f, Y_s),$$

for all $n \geqq c_*$. Since, by (2.4), $\omega^\varphi_{k_1}(f, 1/n) \leqq cn^{-r}\omega^\varphi_{k_1-r,r}\big(f^{(r)}, 1/n\big)$, if k_1 is chosen to be sufficiently large ($k_1 = k + 4$ will do), then this implies (2.8)–(2.14) with an additional restriction that $n \geqq c_*$. This verifies (2.14). For $n < c_*$, all other inequalities follow from the Whitney-type estimates with n equal to the corresponding lower bounds. Namely, for $f \in \Delta^2$, inequalities (2.8) and (2.9) follow, respectively, from

$$E_3^{(2)}(f) \leqq c\omega_2(f', 1), \quad \text{if } \ f \in \mathbb{C}^1[-1,1], \tag{2.16}$$

and

$$E^{(2)}_{l+2}(f) \leqq c\omega_l(f'', 1), \quad \text{if } f \in \mathbb{C}^2[-1,1]. \tag{2.17}$$

For $f \in \Delta^2(Y_1)$, (2.10) and (2.11) are consequences of

$$E^{(2)}_2(f, Y_1) \leqq c\omega_2(f, 1). \tag{2.18}$$

Taking into account that, if $1/\varphi(y_1) < n < c_*$, then $\varphi(y_1) \sim c$, inequality (2.12) follows from

$$E^{(2)}_2(f, Y_1) \leqq 4\varphi(y_1)^{-2}\omega_2(f', 1), \quad \text{if } f \in \mathbb{C}^1[-1,1]. \tag{2.19}$$

(We would like to emphasize that the fact that $E^{(2)}_2$ and not $E^{(2)}_3$ appears on the left-hand side of (2.19) is not a misprint.) Finally, if $f \in \mathbb{C}^2[-1,1]$, then

$$E^{(2)}_4(f, Y_1) \leqq c\omega_2(f'', 1) \tag{2.20}$$

yields (2.13). Inequalities (2.16)–(2.20) are simple corollaries of the Whitney inequality, and we omit details of their proofs.

Therefore, it is sufficient to verify (2.8)–(2.14) for the approximation by piecewise polynomials instead of by polynomials. Except for the proof of (2.10), one may take for the interval $[x_{n-2}, x_2]$, the piecewise polynomials S, guaranteed by Lemmas 2.5-2.7. Thus, we only need to define approximating piecewise polynomials near the endpoints of $[-1, 1]$ so that the shape is preserved and appropriate estimates hold. The proof of (2.10) will be slightly different since we do not have a reference for a direct analogue of Lemmas 2.5-2.7 suitable for its proof (see below).

Due to symmetry, we only consider the right endpoint. By doubling or quadrupling n if necessary we may assume that there are no points y_i in $[x_3, 1]$ in the case $s \geqq 2$ (and, of course, if $s = 0$), and that either $y_1 < x_3$ or $y_1 \in [x_1, 1]$ in the case $s = 1$. Hence, it is sufficient to construct polynomials p of small degrees such that

$$p(x_2) = f(x_2), \quad p'(x_2) = f'(x_2), \tag{2.21}$$

$$p''(x) \geqq 0, \quad x \in [x_2, 1], \tag{2.22}$$

if $s \neq 1$, or $s = 1$ and $y_1 \notin [x_3, 1]$, and

$$(x - y_1)p''(x) \geqq 0, \quad x \in [x_2, 1], \tag{2.23}$$

if $s=1$ and $y_1 \in [x_1,1]$, and appropriate estimates for $\|f-p\|_{\mathbb{C}[x_2,1]}$ hold. We then extend S (which is close to f on $[x_{n-2},x_2]$) to $(x_2,1]$ by defining

$$S|_{(x_2,1]}(x) := p(x) + S(x_2) + S'(x_2-)(x-x_2) - f(x_2) - f'(x_2)(x-x_2),$$

$x \in (x_2,1]$. It is clear by the definition that S satisfies the required coconvexity, and (2.6) implies, for $x \in [x_2,1]$ and $r=2$ or 3 (depending on which of the inequalities (2.8)–(2.14) we are proving),

$$\begin{aligned} \bigl|S'(x_2-) - f'(x_2)\bigr|\,|x-x_2| &\leqq \|S'-f'\|_{C(I_3)}|x-x_2| \\ \leqq c(k_1)|I_3|^{-1}|x-x_2|\bigl(\|S-f\|_{C(I_3)} &+ n^{-1}\omega^{\varphi}_{k_1-1,1}(f',1/n)\bigr) \\ \leqq c(k_1)\|S-f\|_{C[x_{n-2},x_2]} &+ c(k_1)n^{-r}\omega^{\varphi}_{k_1-r,r}\bigl(f^{(r)},1/n\bigr), \end{aligned}$$

so that we may conclude that $\|f-S\|_{C[x_2,1]}$ is appropriately small.

Thus, our aim is the construction of $p(x)$. To this end, if $f \in \mathbb{C}^2[-1,1]$, then let $p_* \in \mathbb{P}_l$ be the polynomial of best approximation of f'' on $[x_2,1]$, and define

$$\begin{aligned} p(x) &:= f(x_2) + (x-x_2)f'(x_2) \\ &+ \frac{1}{2}(x-x_2)^2\|f''-p_*\|_{\mathbb{C}[x_2,1]} + \int_{x_2}^{x}(x-t)p_*(t)\,dt. \end{aligned}$$

If f'' is non-negative on $[x_2,1]$, then (2.21)–(2.22) are satisfied and, by Whitney's inequality,

$$\|f-p\|_{\mathbb{C}[x_2,1]} \leqq c(1-x_2)^2\|f''-p_*\|_{\mathbb{C}[x_2,1]} \leqq c(l)n^{-4}\omega_l(f'',1/n^2), \tag{2.24}$$

which completes the proof of (2.9) and (2.14).

If $s=1$ and $y_1 \in [x_1,1]$, then we define

$$p(x) := f(x_2) + (x-x_2)f'(x_2) + \int_{x_2}^{x}(x-t)L(t;f'',x_2,y_1)\,dt,$$

where $L(x;g,a,b)$ denotes the linear polynomial interpolating g at the points a and b.

Then p satisfies (2.21) and (2.23), and

$$\|f-p\|_{\mathbb{C}[x_2,1]} \leqq cn^{-4}\omega_2(f'',1/n^2), \tag{2.25}$$

which completes the proof of (2.13).

If $f \in \mathbb{C}^1[-1,1]$ and f is convex on $[x_2,1]$, then we define

$$p(x) := f(x_2) + \int_{x_2}^{x} L(t; f', x_2, 1)\, dt,$$

which verifies (2.8) and (2.12) (the estimates of the rate of approximation are obtained using Whitney's inequality similarly to (2.24) and (2.25)). Additionally, in the case $s=1$, if $y_1 \in [x_1,1]$, then we choose $p(x) := f(x_2) + (x-x_2)f'(x_2)$ which completes the proof of (2.11).

Finally, for the proof of (2.10) we note that, in the case $x_{n-3} < y_1 < x_3$, it was proved in [4] that $E_n^{(2)}(f, Y_1) \leqq c\omega_3^{\varphi}(f, 1/n)$ which implies (2.10). In the case $y_1 \in [x_1, 1]$, denoting by $Q(x; g, a, b, c)$ the quadratic polynomial interpolating g at the points a, b and c, we define

$$S(x) := \begin{cases} Q(x; f, x_n, x_{n-1}, x_{n-2}), & x \in [-1, x_{n-1}), \\ \min\left\{ Q(x; f, x_{j+1}, x_j, x_{j-1}), Q(x; f, x_j, x_{j-1}, x_{j-2}) \right\}, & \\ & x \in [x_j, x_{j-1}),\ 3 \leqq j \leqq n-1, \\ L(x; f, x_2, x_1), & x \in [x_2, 1]. \end{cases}$$

It is not difficult to check that $S \in \Sigma_{3,n}(Y_1) \cap \Delta^2(Y_1)$ and

$$\|f - S\| \leqq c\omega_3^{\varphi}(f, 1/n) + c\omega_2(f, 1/n^2).$$

This completes the proof of (2.10) and, hence, of the lemma. $\square$

3. Proofs of the negative results

Proof of Theorem 1.5. Noting that (1.4) is trivial if $|y_1| < 3/4$ (the right-hand side of the inequality is negative) and using symmetry, without loss of generality we may assume that $y_1 \leqq -3/4$. Put $\varepsilon := 1 + y_1$ and define

$$F(x) := \begin{cases} \dfrac{1}{x+1} - \dfrac{1}{2\sqrt{\varepsilon}}, & \text{if } -1 < x < -1 + 4\sqrt{\varepsilon}, \\ -\dfrac{1}{x+1}, & \text{if } -1 + 4\sqrt{\varepsilon} \leqq x \leqq 1, \end{cases}$$

and set

$$f(x) := \frac{1}{2}\int_{-1+\varepsilon}^{x} (x-t)^2 F(t)\, dt, \quad -1 \leqq x \leqq 1.$$

Since $f^{(3)} = F$, it follows that f'' is increasing on $\bigl(-1, -1+2\sqrt{\varepsilon}\bigr)$ and decreasing on $\bigl(-1+2\sqrt{\varepsilon}, 1\bigr)$. Also $f''(y_1) = 0$ and $f''(1) = \int_{-1+\varepsilon}^{1} F(t)\,dt = 3\ln 2 - 2 + \sqrt{\varepsilon}/2 > 0$. Thus, we conclude that $f''(x) < 0$ for $-1 < x < y_1$, and $f''(x) > 0$ for $y_1 < x \leqq 1$, which establishes that $f \in \Delta^2(Y_1)$.

Note that $\bigl|f^{(4)}(x)\bigr| = \bigl|F'(x)\bigr| = (1+x)^{-2}$, $x \neq -1$. Hence $\bigl\|\varphi^4 f^{(4)}\bigr\| = 4$, so that for $n \geqq 4$, (2.1) yields

$$n^4 E_n(f) \leqq c_1, \tag{3.1}$$

for some absolute constant c_1. The cases $1 \leqq n < 4$ in (3.1) follow immediately from the case $n = 1$ (i.e., approximation by a constant function), perhaps at the expense of somewhat increasing c_1. Indeed, since $f'(y_1) = 0$, the fact that $f \in \Delta^2(Y_1)$ implies that f is nondecreasing on $[-1,1]$, and taking into account that $\bigl|F(x)\bigr| \leqq (1+x)^{-1}$, $-1 < x \leqq 1$, we have

$$\begin{aligned}
E_1(f) = \frac{1}{2}\bigl(f(1) - f(-1)\bigr) &= \frac{1}{4}\int_{y_1}^{1} (1-t)^2 F(t)\,dt + \frac{1}{4}\int_{-1}^{y_1} (1+t)^2 F(t)\,dt \\
&= \frac{1}{4}\int_{-1}^{1} (1+t)^2 F(t)\,dt - \int_{y_1}^{1} (1+t)F(t)\,dt + \int_{y_1}^{1} F(t)\,dt \\
&\leqq \frac{1}{4}\int_{-1}^{1} (1+t)\,dt + \int_{y_1}^{1} dt + 3\ln 2 - 2 + \frac{\sqrt{\varepsilon}}{2} \leqq 3.
\end{aligned}$$

Hence, (3.1) is valid for $1 \leqq n \leqq 3$ if we assume that $c_1 \geqq 3^5$. This completes the proof of (3.1) for all $n \in \mathbb{N}$.

Now, let

$$T_m(x) := \frac{1}{2}\int_{-1+\varepsilon}^{-1+m^{-2}} (x-t)^2 F(t)\,dt$$

(note that T_m is a quadratic polynomial), and set $g(x) := f(x) - T_m(x)$. Then,

$$T_m'' \equiv \begin{cases} -\ln\left(m^2\varepsilon\right) - 1/\bigl(2m^2\sqrt{\varepsilon}\bigr) + \sqrt{\varepsilon}/2, & \text{if } m^{-2} < 4\sqrt{\varepsilon}, \\ 2\ln m + 4\ln 2 - 2 + \sqrt{\varepsilon}/2, & \text{if } m^{-2} \geqq 4\sqrt{\varepsilon}, \end{cases}$$

which in turn implies

$$T_m'' \geqq \ln\frac{m^2}{1+m^4\varepsilon} - 2, \quad \text{for all } m \in \mathbb{N}.$$

Since $g(x) = \frac{1}{2}\int_{-1+m^{-2}}^{x}(x-t)^2F(t)\,dt$, recalling that $\left|F(x)\right| \leqq (1+x)^{-1}$, we obtain for $-1 \leqq x < -1+m^{-2}$,

$$\left|g(x)\right| \leqq \frac{1}{2}\int_{x}^{-1+m^{-2}} \frac{(t-x)^2}{1+t}\,dt \leqq \frac{1}{2}\int_{-1}^{-1+m^{-2}} \frac{(t+1)^2}{1+t}\,dt = \frac{1}{4m^4},$$

while for $-1+m^{-2} \leqq x \leqq 1$, we have

$$\left|g(x)\right| \leqq \frac{1}{2}\int_{-1+m^{-2}}^{x} \frac{(x-t)^2}{1+t}\,dt \leqq \frac{m^2(x+1)^2}{2}\int_{-1+m^{-2}}^{x} dt \leqq m^2(x+1)^3/2.$$

Hence,

$$\left|g(x)\right| \leqq \frac{1+m^6(x+1)^3}{m^4}, \quad -1 \leqq x \leqq 1.$$

Now, let $P_m \in \Delta^2(Y_1)$ be an arbitrary polynomial of degree $< m$. Since $P_m''(-1) \leqq 0$, it follows that

$$T_m''(-1) - P_m''(-1) \geqq T_m''(-1) \geqq \ln\frac{m^2}{1+m^4\varepsilon} - 2. \tag{3.2}$$

On the other hand, it follows from the well known Dzyadyk type inequality (see, e.g., [5, Lemma 5.2]) that for any polynomial $Q_m \in \mathbb{P}_m$,

$$\left|Q_m''(-1)\right| \leqq c_2 m^4 \left\|\frac{Q_m}{1+m^6(1+\cdot)^3}\right\|,$$

where $c_2 \geqq 1$ is an absolute constant. In particular,

$$\left|T''(-1) - P_m''(-1)\right| \leqq c_2 m^4 \left\|\frac{P_m - T}{1+m^6(1+\cdot)^3}\right\|. \tag{3.3}$$

Now, denoting $A_m := m^4\|f-P_m\|$ we have

$$\begin{aligned}\left|T_m(x) - P_m(x)\right| &\leqq \|f-P_m\| + \left|g(x)\right| \leqq m^{-4}A_m + m^{-4}\big(1+m^6(x+1)^3\big)\\ &\leqq (A_m+1)m^{-4}\big(1+m^6(x+1)^3\big),\end{aligned}$$

and (3.3) implies that

$$\left|T_m''(-1) - P_m''(-1)\right| \leqq c_2(A_m+1).$$

Therefore, by (3.2)

$$A_m + 1 \geqq \frac{1}{c_2}\big(T_m''(-1) - P_m''(-1)\big)$$

$$\geqq \frac{1}{c_2}\left(\ln\frac{m^2}{1+m^4\varepsilon} - 2\right) \geqq \frac{1}{c_2}\ln\frac{m^2}{1+m^4\varepsilon} - 2,$$

whence

$$\frac{m^4 E_m(f, Y_1)}{\sup\left\{n^4 E_n(f) : n \in \mathbb{N}\right\}} \geqq \frac{1}{c_1c_2}\ln\frac{m^2}{1+m^4\varepsilon} - \frac{3}{c_1}$$

$$\geqq \frac{1}{c_1c_2}\ln\frac{m^2}{1+m^4(1-y_1^2)} - 1,$$

which implies (1.4).

Finally, for small values of $1-y_1^2$, (1.5) follows from (1.4) and, for values bounded away from 0, (1.5) is a consequence of an obvious inequality

$$\sup\left\{n^4 E_n^{(2)}(f, Y_1) : n \in \mathbb{N}\right\} \geqq \sup\left\{n^4 E_n(f) : n \in \mathbb{N}\right\}.$$

The proof of Theorem 1.5 is now complete. □

Proof of Theorem 1.7. Given α and m, denote $r := \lceil \alpha \rceil$ and $h := (6m)^{-1}$. For $Y_s = \{y_i\}_{i=1}^s$ such that $-1 < y_s < \cdots < y_3 \leqq -1/2$, $y_2 := -h$ and $y_1 := h$, let

$$f_m(x) := \int_0^x (x-t) f_m''(t)\, dt,$$

where

$$f_m''(t) := \begin{cases} -(h^2-t^2)^r, & \text{if } |t| \leqq h, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $f_m \in \mathbb{C}^r \cap \Delta^2(Y_s)$, and by the proof of Theorem 2.2 in [6],

$$m^r E_m^{(2)}(f_m, Y_s) \geqq c(r) m \big\| f_m^{(r)} \big\|.$$

Hence the function $f(x) := \big\| f_m^{(r)} \big\|^{-1} f_m(x)$ satisfies

$$m^\alpha E_m^{(2)}(f, Y_s) \geqq c(r) m^{\alpha+1-r} = c(r) m^{\alpha_*}. \tag{3.4}$$

On the other hand, since $\left\|f^{(r)}\right\| = 1$, Jackson's inequality implies

$$n^r E_n(f) \leqq c(r), \tag{3.5}$$

for all $n \geqq r$, while straightforward computations yield $\|f'\| \leqq c(r)h^{r-1}$, so that

$$\|f\| \leqq c(r)h^{r-1} \leqq c(r),$$

which implies (3.5) for $n < r$. Hence

$$n^\alpha E_n(f) \leqq n^r E_n(f) \leqq c(r), \quad n \in \mathbb{N}. \tag{3.6}$$

Theorem 1.7 follows by combining (3.4) and (3.6). $\square$

4. Proofs of the positive results

For $\alpha > 0$, it is convenient to denote

$$r^* := r^*(\alpha) := \lceil \alpha \rceil - 1 \quad \text{and} \quad k^* := k^*(\alpha) := \begin{cases} 1, & \text{if } \alpha \notin \mathbb{N}, \\ 2, & \text{if } \alpha \in \mathbb{N}. \end{cases}$$

Note that $r^* < \alpha < k^* + r^*$, and so by Theorem 2.1, $n^\alpha E_n(f) \leqq 1$, $n \geqq k^* + r^*$, implies that $f \in \mathbb{C}_\varphi^{r^*}$ and

$$\omega_{k^*,r^*}^{\varphi}\left(f^{(r^*)}, t\right) \leqq c(\alpha) t^{\alpha - r^*}. \tag{4.1}$$

4.1. The case $s = 0$: proof of Theorem 1.1. If the right-hand side of (1.1) is infinite, then there is nothing to prove. Thus, without loss of generality, we may assume that

$$\sup\left\{n^\alpha E_n(f) : n \in \mathbb{N}\right\} \leqq 1. \tag{4.2}$$

Since constant functions are in Δ^2, it is sufficient to prove inequality (1.1) for $n \geqq N$, where N is an absolute constant. Indeed, for $1 \leqq n < N$, we have

$$E_n^{(2)}(f) \leqq E_1^{(2)}(f) = E_1(f) \leqq 1 \leqq N^\alpha n^{-\alpha}.$$

For $\alpha \notin [3,5]$, Theorem 1.1 immediately follows from (4.1) and the following lemma (see [3, Theorem 2] and [5, lines 2–5 in the table on p. 5]).

LEMMA 4.1. *Let $k = 1, 2$ and $r \in \mathbb{N}_0$ be such that*

$$(k, r) \notin \{(1,3), (1,4), (2,2), (2,3), (2,4)\}.$$

If $f \in \mathbb{C}^r_\varphi \cap \Delta^2$, then

$$E_n^{(2)}(f) \leqq c(r) n^{-r} \omega^\varphi_{k,r}\big(f^{(r)}, 1/n\big), \quad n \geqq k + r.$$

For $\alpha \in (4, 5]$, Theorem 1.1 follows from (4.1) and [3, Theorem 6] (a proof similar to the one in the case $\alpha \in [3, 4]$ below, works as well).

Now, if inequality (4.2) is satisfied for $\alpha \in [3, 4]$, then Theorem 2.1 (with $r = 2$ and $k = 3$) and Theorem 2.2 (with $r = 1$ and $k = 2$) imply that $f \in \mathbb{C}^2_\varphi \cap \mathbb{C}^1[-1, 1]$,

$$\omega^\varphi_{3,2}(f'', t) \leqq c(\alpha) t^{\alpha - 2} \quad \text{and} \quad \omega_2(f', t) \leqq c(\alpha) t^{\alpha/2 - 1}.$$

Hence, (2.8) (with $k = 3$) yields $E_n^{(2)}(f) \leqq c(\alpha) n^{-\alpha}$ for $n \geqq 3$ as needed.

4.2. The case $s = 1$ and $\alpha \neq 4$: proof of Theorem 1.2. As in the case $s = 0$, we may assume that (4.2) holds, and since constant functions are in $\Delta^2(Y_1)$ it is sufficient to prove inequality (1.2) for $n \geqq N$, where $N = N(\alpha)$.

For $\alpha > 7$ and $\alpha < 2$, Theorem 1.2 readily follows from (4.1) and the following theorem (see [6, Corollaries 2.10 and 2.12]).

THEOREM 4.2. *Let $s = 1$ and $Y_1 \in \mathbb{Y}_1$. If $k = 1, 2$ and $r \geqq 7$, or $(k, r) \in \{(1,1), (1,0), (2,0)\}$, and $f \in \mathbb{C}^r_\varphi \cap \Delta^2(Y_1)$, then*

$$E_n^{(2)}(f, Y_1) \leqq c(r) n^{-r} \omega^\varphi_{k,r}\big(f^{(r)}, 1/n\big), \quad n \geqq k + r.$$

If $4 < \alpha \leqq 7$, then Theorem 2.1 (with $r = 3$ and $k = 5$) and Theorem 2.2 (with $r = 2$ and $k = 2$) imply that $f \in \mathbb{C}^3_\varphi \cap \mathbb{C}^2[-1, 1]$,

$$\omega^\varphi_{5,3}\big(f^{(3)}, t\big) \leqq c(\alpha) t^{\alpha - 3} \quad \text{and} \quad \omega_2(f'', t) \leqq c(\alpha) t^{\alpha/2 - 2}.$$

Hence, (2.13) (with $k = 5$) immediately yields $E_n^{(2)}(f, Y_1) \leqq c(\alpha) n^{-\alpha}$ for $n \geqq 4$ which completes the proof of Theorem 1.2 for $4 < \alpha \leqq 7$.

If $2 < \alpha < 4$, then Theorem 2.1 (with $r = 2$ and $k = 3$) and Theorem 2.2 (with $r = 1$ and $k = 1$) imply that $f \in \mathbb{C}^2_\varphi \cap \mathbb{C}^1[-1, 1]$,

$$\omega^\varphi_{3,2}(f'', t) \leqq c(\alpha) t^{\alpha - 2} \quad \text{and} \quad \omega_1(f', t) \leqq c(\alpha) t^{\alpha/2 - 1},$$

and (2.11) completes the proof for this range of α.

Finally, if $\alpha = 2$, then Theorem 2.1 (with $r = 0$ and $k = 3$) and Theorem 2.2 (with $r = 0$ and $k = 2$) imply that

$$\omega_3^\varphi(f,t) \leqq c(\alpha)t^2 \quad \text{and} \quad \omega_2(f,t) \leqq ct,$$

and estimate (2.10) now completes the proof Theorem 1.2.

4.3. The case $s = 1$ and $\alpha = 4$: proof of Theorem 1.3. The inequality $\sup\{n^4 E_n(f) : n \in \mathbb{N}\} \leqq 1$, Theorem 2.1 (with $r = 2$ and $k = 3$) and Theorem 2.2 (with $r = 1$ and $k = 2$) imply that $f \in \mathbb{C}_\varphi^2 \cap \mathbb{C}^1[-1,1]$,

$$\omega_{3,2}^\varphi(f'',t) \leqq ct^2 \quad \text{and} \quad \omega_2(f',t) \leqq ct.$$

Thus, it follows from (2.12) that $E_n^{(2)}(f,Y_1) \leqq cn^{-4}$, for $n\varphi(y_1) > 1$, and the proof of Theorem 1.3 is complete.

4.4. The case $s \geqq 2$: proof of Theorem 1.6. Once again we may assume that (4.2) is satisfied. For $\alpha \neq 5$, Theorem 1.6 immediately follows from (4.1) and the following result (see [6, Corollary 2.9]).

THEOREM 4.3. *Let $s \geqq 2$, $k = 1,2$ and $r \in \mathbb{N}_0$ be such that $(k,r) \neq (2,4)$, and $Y_s \in \mathbb{Y}_s$. If $f \in \mathbb{C}_\varphi^r \cap \Delta^2(Y_s)$, then*

$$E_n^{(2)}(f,Y_s) \leqq c(r,s)n^{-r}\omega_{k,r}^\varphi\big(f^{(r)},1/n\big), \quad n \geqq N(r,Y_s).$$

In the case $\alpha = 5$, Theorem 2.1 (with $r = 3$ and $k = 3$) and Theorem 2.2 (with $r = 2$ and $k = 1$) imply that $f \in \mathbb{C}_\varphi^3 \cap \mathbb{C}^2[-1,1]$,

$$\omega_{3,3}^\varphi\big(f^{(3)},t\big) \leqq ct^2 \quad \text{and} \quad \omega_1(f'',t) \leqq ct^{1/2}.$$

It follows from (2.14) (with $k = 3$ and $m = 1$) that $E_n^{(2)}(f,Y_s) \leqq cn^{-5}$, for $n > N(Y_s)$, which completes the proof of Theorem 1.6.

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