

## New Moduli of Smoothness: Weighted DT Moduli Revisited and Applied

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**Abstract** We introduce new moduli of smoothness for functions  $f \in L_p[-1, 1] \cap C^{r-1}(-1, 1)$ ,  $1 \leq p \leq \infty$ ,  $r \geq 1$ , that have an  $(r-1)$ st locally absolutely continuous derivative in  $(-1, 1)$ , and such that  $\varphi^r f^{(r)}$  is in  $L_p[-1, 1]$ , where  $\varphi(x) = (1 - x^2)^{1/2}$ . These moduli are equivalent to certain weighted Ditzian–Totik (DT) moduli, but our definition is more transparent and simpler. In addition, instead of applying these weighted moduli to weighted approximation, which was the purpose of the original DT moduli, we apply these moduli to obtain Jackson-type estimates on the approximation of functions in  $L_p[-1, 1]$  (no weight), by means of algebraic polynomials. Moreover, we also prove matching inverse theorems, thus obtaining constructive characterization of various smoothness classes of functions via the degree of their approximation by algebraic polynomials.

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### 1 Motivation

The purpose of this section is to provide some motivation to the introduction of the new moduli of smoothness that we discuss in this paper.

We start with a simple example. Suppose that  $A_p^\alpha$  is the space of all functions in  $\mathbb{L}_p[-1, 1]$ ,  $1 \leq p \leq \infty$ , such that their rate of approximation by algebraic polynomials of degree  $< n$  in the  $\mathbb{L}_p$ -norm is  $O(n^{-\alpha})$ . How can we characterize this approximation space? The answer is very well known by now. There are several approaches, but the ones that became most popular in recent decades involve moduli of smoothness of Ivanov  $\tau_k(f, \delta(t, \cdot))_{q,p}$  (introduced in 1980–1981) and Ditzian–Totik  $\omega_k^\varphi(f, t)_p$  (introduced around 1984). The Ditzian–Totik (DT) modulus is defined in (2.1) by letting  $r = 0$  (see also Remark 2.3), and the Ivanov modulus (see [2, Section 16], for example) is given by

$$\tau_k(f, \delta(t, \cdot))_{q,p} = \left\| \omega_k(f, \cdot, \delta(t, \cdot))_q \right\|_p,$$

where

$$\omega_k(f, x, \delta(t, x))_q^q = \frac{1}{2\delta(t, x)} \int_{-\delta(t,x)}^{\delta(t,x)} \left| \Delta_v^k(f, x) \right|^q dv.$$

It turns out (see, e.g., [2]) that  $\omega_k^\varphi(f, t)_p \sim \tau_k(f, \Delta(t, \cdot))_{p,p}$  with  $\Delta(t, \cdot) := t\varphi(\cdot) + t^2$ , but, according to [2, p. 142], “The [Ivanov] moduli ... are a somewhat more cumbersome method to describe smoothness than ... [DT moduli], and their computation is more difficult.”

It follows from [3, Theorems 7.2.1 and 7.2.4] that, for  $0 < \alpha < k$ ,

$$f \in A_p^\alpha \iff \omega_k^\varphi(f, t)_p = O(t^\alpha), \quad t > 0. \tag{1.1}$$

A natural question now is what can be said about smoothness of the derivatives of functions from  $A_p^\alpha$ . Surely, if  $\alpha$  is large enough, then functions from  $A_p^\alpha$  have to be differentiable (or rather, almost everywhere, they coincide with functions which are differentiable). Note that (1.1) does not explicitly describe the behavior of these derivatives (but see Remark 1.1 below). While it is true that  $\omega_k^\varphi(f, t)_p \leq ct^r \omega_{k-r}^\varphi(f^{(r)}, t)_p$ , it is NOT true that, for appropriate  $\alpha$ ,  $f \in A_p^\alpha$  only if  $\omega_{k-r}^\varphi(f^{(r)}, t)_p = O(t^{\alpha-r})$ . One needs to replace  $\omega_{k-r}^\varphi(f^{(r)}, t)_p$  with an appropriated weighted modulus (as we show in Sect. 9). This is very different from the trigonometric case where the classical moduli of smoothness are used in analogous results on characterization of (trigonometric) approximation spaces.

*Remark 1.1* To be more precise, we mention that (1.1) implicitly describes the behavior of the derivatives of  $f$  since it follows from [3, Theorem 6.2.2 and Corollary 6.3.2] ([3, Corollary 6.3.2] has a couple of misprints, but this can be easily rectified) that, for  $1 \leq r < \alpha$ ,

$$\begin{aligned} \omega_k^\varphi(f, t)_p = O(t^\alpha) &\iff \omega_\varphi^{k-r}(f^{(r)}, t)_{\varphi^r, p} = O(t^{\alpha-r}) \\ &\iff \Omega_\varphi^{k-r}(f^{(r)}, t)_{\varphi^r, p} = O(t^{\alpha-r}). \end{aligned}$$

However, if  $t^\alpha$  is replaced by a more complicated function, and, correspondingly,  $A_p^\alpha$  is replaced by the space of functions  $A_p(\phi)$  whose rate of approximation is  $O(\phi(1/n))$ , then [3, Theorem 6.2.2 and Corollary 6.3.2] may no longer provide any useful information, and it becomes much harder to get an explicit description of the behavior of the derivatives of functions from  $A_p(\phi)$ . For example, if  $\phi(t) := t/(\ln(t/2))^2$ , then

$$f \in A_p(\phi) \iff \omega_2^\varphi(f, t)_p = O(\phi(t)),$$

and [3, Theorem 6.3.1 (a)] implies that  $f'$  is locally absolutely continuous and  $\Omega_\varphi(f', t)_{\varphi, p} = O(-1/\ln(t/2))$ . However, [3, Theorem 6.2.2] does not give any information about  $\omega_\varphi(f', t)_{\varphi, p}$  other than that it is bounded below by  $\Omega_\varphi(f', t)_{\varphi, p}$ . Hence, there is a need for an inverse theorem for an algebraic approximation explicitly involving derivatives of functions as in the classical trigonometric case (see [7, Theorem 6.1.3], for example). We prove such a theorem in Sect. 9 (see Theorem 9.1). In particular, it implies that

$$f \in A_p(\phi) \implies \omega_{1,1}^\varphi(f', t)_p = O(-1/\ln(t/2)).$$

In summary, there is a need for a new measure of smoothness and/or new results that would help resolve the above mentioned problems. The purpose of this paper is to introduce new moduli serving this purpose. These moduli are equivalent to certain weighted DT moduli, but, to rephrase [2, p. 142], “these weighted DT moduli are a somewhat more cumbersome method to describe smoothness than our moduli, and their computation is more difficult” (see Sect. 8 for the exact definition of these weighted DT moduli).

## 2 Introduction and Definitions

As alluded to above, we are interested in the constructive characterization of the functions in  $L_p[-1, 1]$ ,  $1 \leq p < \infty$ , and  $C[-1, 1]$  when  $p = \infty$ , with given degree of approximation by algebraic polynomials, which is analogous to the characterization of periodic functions in  $L_p[-\pi, \pi]$ , respectively,  $C[-\pi, \pi]$ , with given degree of approximation by trigonometric polynomials. Our characterization yields information on the smoothness of the derivatives of the approximated functions and is described in Sects. 8 and 9 by means of direct and inverse theorems relating certain weighted DT moduli of smoothness of a function  $f \in L_p[-1, 1]$ , respectively,  $f \in C[-1, 1]$ , to its degrees of best unweighted approximation in the space.

The first sections are devoted to introducing the abovementioned DT moduli of smoothness in a new, equivalent form, which is more transparent and simpler. We prove the equivalence via  $K$ -functionals. For  $p = \infty$ , these moduli of smoothness were introduced by the third author [6] (see also [4]), and certain direct and inverse theorems proved; however, no relations to weighted DT moduli were discussed.

Hereafter, we will have constants  $c$  that may depend only on some of the parameters involved  $(p, k, r)$  but are independent of the function and of  $t$  or  $n$ , as the case may be. The constants  $c$  may be different even if they appear in the same line.

Let  $\|\cdot\|_p := \|\cdot\|_{\mathbb{L}_p[-1,1]}$ ,  $1 \leq p \leq \infty$ , and  $\varphi(x) := \sqrt{1-x^2}$ .

For  $k \in \mathbb{N}_0$ ,  $h \geq 0$ , an interval  $J$  and  $f : J \mapsto \mathbb{R}$ , let

$$\Delta_h^k(f, x, J) := \begin{cases} \sum_{i=0}^k \binom{k}{i} (-1)^{k-i} f(x + (i - k/2)h) & \text{if } x \pm kh/2 \in J, \\ 0 & \text{otherwise,} \end{cases}$$

be the  $k$ th symmetric difference, and let  $\Delta_h^k(f, x) := \Delta_h^k(f, x, [-1, 1])$ .

**Definition 2.1** Let  $1 \leq p \leq \infty$  and  $r \in \mathbb{N}_0$ . Then, for  $r \geq 1$ , let

$$\mathbb{B}_p^r := \{ f \mid f^{(r-1)} \in AC_{loc}(-1, 1) \text{ and } \|f^{(r)}\varphi^r\|_p < +\infty\},$$

and set  $\mathbb{B}_p^0 := \mathbb{L}_p[-1, 1]$ .

(Recall that  $AC_{loc}(-1, 1)$  denotes the set of functions which are locally absolutely continuous in  $(-1, 1)$ ).

**Definition 2.2** For  $f \in \mathbb{B}_p^r$ , define

$$\omega_{k,r}^\varphi(f^{(r)}, t)_p := \sup_{0 \leq h \leq t} \left\| \mathcal{W}_{kh}^r(\cdot) \Delta_{h\varphi(\cdot)}^k(f^{(r)}, \cdot) \right\|_p, \tag{2.1}$$

where

$$\mathcal{W}_\delta(x) := ((1 - x - \delta\varphi(x)/2)(1 + x - \delta\varphi(x)/2))^{1/2}.$$

For  $\delta > 0$ , define

$$\begin{aligned} \mathfrak{D}_\delta &:= \{x \mid 1 - \delta\varphi(x)/2 \geq |x|\} \setminus \{\pm 1\} \\ &= \left\{ x \mid |x| \leq \frac{4 - \delta^2}{4 + \delta^2} \right\} = [-1 + \mu(\delta), 1 - \mu(\delta)], \end{aligned}$$

where

$$\mu(\delta) := 2\delta^2/(4 + \delta^2).$$

Observe that  $\mathfrak{D}_\delta = \emptyset$  if  $\delta > 2$ , and note that  $\Delta_{h\varphi(x)}^k(f, x)$  is defined to be identically 0 if  $x \notin \mathfrak{D}_{kh}$  and that  $\mathcal{W}_\delta$  is well defined on  $\mathfrak{D}_\delta$  (in fact, if  $\delta \leq 2$ , then  $\text{Dom}(\mathcal{W}_\delta) = \mathfrak{D}_\delta \cup \{\pm 1\}$ ).

Hence,

$$\omega_{k,r}^\varphi(f^{(r)}, t)_p = \sup_{0 < h \leq t} \left\| \mathcal{W}_{kh}^r(\cdot) \Delta_{h\varphi(\cdot)}^k(f^{(r)}, \cdot) \right\|_{\mathbb{L}_p(\mathfrak{D}_{kh})}$$

and

$$\omega_{k,r}^\varphi(f^{(r)}, t)_p = \omega_{k,r}^\varphi(f^{(r)}, 2/k)_p \quad \text{for } t \geq 2/k. \tag{2.2}$$

*Remark 2.3* When  $r = 0$ ,  $\omega_{k,0}^\varphi(f, t)_p$  reduces to the well-known  $k$ th DT modulus of smoothness  $\omega_k^\varphi(f, t)_p$  (see, e.g., [3]).

*Remark 2.4* When  $p = \infty$ ,  $\omega_{k,r}^\varphi(f, t)_\infty$  reduces to the modulus of smoothness introduced by the third author (see, e.g., [4, 6]).

Our moduli of smoothness are certain type of weighted DT moduli (see Sect. 5 for details). However, we give a more transparent and simpler definition of the moduli, which, in particular, makes their monotonicity in  $t$  self-evident. Moreover, we are not interested in weighted approximation; rather, we are interested in applying these moduli to estimates on the nonweighted approximation of  $f \in \mathbb{B}_p^r$  (see Sect. 8 for details).

If  $1 \leq p < \infty$ , our moduli are equivalent to the following averaged moduli of smoothness.

**Definition 2.5** Let  $k \in \mathbb{N}$ ,  $r \in \mathbb{N}_0$ , and  $f \in \mathbb{B}_p^r$ ,  $1 \leq p < \infty$ . Then, the averaged modulus of smoothness is defined as

$$\omega_{k,r}^{*\varphi}(f^{(r)}, t)_p := \left( \frac{1}{t} \int_0^t \int_{\mathfrak{D}_{k\tau}} |\mathcal{W}_{k\tau}^r(x) \Delta_{\tau\varphi(x)}^k(f^{(r)}, x)|^p dx d\tau \right)^{1/p}.$$

For convenience, for  $p = \infty$ , we also define

$$\omega_{k,r}^{*\varphi}(f^{(r)}, t)_\infty := \omega_{k,r}^\varphi(f^{(r)}, t)_\infty.$$

While the modulus  $\omega_{k,r}^\varphi(f^{(r)}, t)_p$  is obviously a nondecreasing function of  $t$ , the averaged modulus  $\omega_{k,r}^{*\varphi}(f^{(r)}, t)_p$  does not have to be nondecreasing. At the same time, it immediately follows from Definition 2.5 that

$$\omega_{k,r}^{*\varphi}(f^{(r)}, t_1)_p \leq (t_2/t_1)^{1/p} \omega_{k,r}^{*\varphi}(f^{(r)}, t_2)_p, \quad \text{for } 0 < t_1 \leq t_2. \tag{2.3}$$

It turns out that the abovedefined moduli are equivalent to the following  $K$ -functional.

**Definition 2.6** (*K-functional*) For  $k \in \mathbb{N}$ ,  $r \in \mathbb{N}_0$ ,  $1 \leq p \leq \infty$ , and  $f \in \mathbb{B}_p^r$ , define

$$K_{k,r}^\varphi(f^{(r)}, t^k)_p := \inf_{g \in \mathbb{B}_p^{k+r}} (\|f^{(r)} - g^{(r)}\|_p + t^k \|g^{(k+r)}\|_p).$$

The following result is valid.

**Theorem 2.7** *If  $k \in \mathbb{N}$ ,  $r \in \mathbb{N}_0$ ,  $1 \leq p \leq \infty$ , and  $f \in \mathbb{B}_p^r$ , then, for all  $0 < t \leq 2/k$ ,*

$$cK_{k,r}^\varphi(f^{(r)}, t^k)_p \leq \omega_{k,r}^{*\varphi}(f^{(r)}, t)_p \leq \omega_{k,r}^\varphi(f^{(r)}, t)_p \leq cK_{k,r}^\varphi(f^{(r)}, t^k)_p, \quad (2.4)$$

where constants  $c$  may depend only on  $k$ ,  $r$ , and  $p$ .

*Remark 2.8* Note that with an additional restriction that  $t \leq t_0$ , in the case  $r = 0$ , Theorem 2.7 becomes [3, Theorem 2.1.1] (with  $\varphi(x) = \sqrt{1 - x^2}$ ) and that  $t_0$  can be taken to be  $(2k)^{-1}$  as was shown in [1, Theorem 6.6.2].

*Remark 2.9* It follows from Theorem 2.7 that

$$\omega_{k,r}^\varphi(f^{(r)}, t)_p \sim \omega_\varphi^k(f^{(r)}, t)_{\varphi^r, p},$$

where  $\omega_\varphi^k(f^{(r)}, t)_{\varphi^r, p}$  is a weighted DT modulus defined in (5.1) (see Sect. 5 for more details).

Since it is obvious that

$$\omega_{k,r}^{*\varphi}(f^{(r)}, t)_p \leq \omega_{k,r}^\varphi(f^{(r)}, t)_p,$$

only the first and last inequalities in (2.4) need to be proved. Their proofs are given, respectively, in Sects. 6 and 4.

We conclude this section with an immediate consequence of Theorem 2.7.

**Corollary 2.10** *Let  $k \in \mathbb{N}$ ,  $r \in \mathbb{N}_0$ ,  $1 \leq p \leq \infty$ ,  $f \in \mathbb{B}_p^r$ , and  $\lambda \geq 1$ . Then, for all  $t > 0$ ,*

$$\omega_{k,r}^\varphi(f^{(r)}, \lambda t)_p \leq c\lambda^k \omega_{k,r}^\varphi(f^{(r)}, t)_p.$$

*Proof* Using Theorem 2.7, identity (2.2), and the monotonicity, in  $t$ , of both the  $K$ -functional  $K_{k,r}^\varphi(f^{(r)}, t^k)_p$ , and the modulus  $\omega_{k,r}^\varphi(f^{(r)}, t)_p$ , and denoting  $\tilde{t} := \min\{\lambda t, 2/k\}$ , we have

$$\begin{aligned} \omega_{k,r}^\varphi(f^{(r)}, \lambda t)_p &= \omega_{k,r}^\varphi(f^{(r)}, \tilde{t})_p \leq cK_{k,r}^\varphi(f^{(r)}, \tilde{t}^k)_p \leq c\lambda^k K_{k,r}^\varphi(f^{(r)}, (\tilde{t}/\lambda)^k)_p \\ &\leq c\lambda^k \omega_{k,r}^\varphi(f^{(r)}, \tilde{t}/\lambda)_p \leq c\lambda^k \omega_{k,r}^\varphi(f^{(r)}, t)_p \end{aligned}$$

for any  $t > 0$ . □

*Remark 2.11* With an additional restriction that  $t \leq t_0$ , in the case  $r = 0$ , Corollary 2.10 becomes [3, Theorem 4.1.2].

### 3 Auxiliary Results

In the following proposition, we list several useful properties of the weights  $\mathcal{W}_\delta(x)$  and the sets  $\mathfrak{D}_\delta$ ,  $\delta > 0$ , which will be used below. (Note that the statements in this proposition are vacuously true for  $\delta$ 's such that  $\mathfrak{D}_\delta$  and/or  $\mathfrak{D}_{2\delta}$  are empty.)

**Proposition 3.1** (Properties of  $\mathcal{W}_\delta(x)$  and  $\mathfrak{D}_\delta$ ).

- (i)  $\mathcal{W}_\delta(x) \leq \varphi(u)$  for  $x \in \mathfrak{D}_\delta$  and  $u \in [-|x| - \delta\varphi(x)/2, |x| + \delta\varphi(x)/2]$ .
- (ii)  $\mathcal{W}_\delta(x) \leq \varphi(x)$  for  $x \in \mathfrak{D}_\delta$ .
- (iii)  $\varphi(x) \leq 2\mathcal{W}_\delta(x)$  for  $x \in \mathfrak{D}_{2\delta}$ .
- (iv)  $\delta|\varphi'(x)| \leq 1$  for  $x \in \mathfrak{D}_\delta$ .
- (v) If  $y(x) := x + \delta_1\varphi(x)/2$  and  $|\delta_1| \leq \delta$ , then  $1/2 \leq y'(x) \leq 3/2$  for all  $x \in \mathfrak{D}_\delta$ .
- (vi) If  $\delta_1 > \delta_2$ , then  $\mathfrak{D}_{\delta_1} \subset \mathfrak{D}_{\delta_2}$ .

*Proof* For  $x \in \mathfrak{D}_\delta$  and  $u \in [-|x| - \delta\varphi(x)/2, |x| + \delta\varphi(x)/2]$ , we have

$$\varphi^2(u) - \mathcal{W}_\delta^2(x) \geq \varphi^2(|x| + \delta\varphi(x)/2) - \mathcal{W}_\delta^2(x) = (1 - |x| - \delta\varphi(x)/2)\delta\varphi(x) \geq 0,$$

which implies (i).

Choosing  $u$  to be  $x$  in (i), we get (ii).

Now,  $\delta \leq 1$  implies  $1 + |x| \leq 2(1 + |x| - \delta\varphi(x)/2)$ , and  $x \in \mathfrak{D}_{2\delta}$ , that is,  $\delta\varphi(x) \leq 1 - |x|$ , yields  $1 - |x| \leq 2(1 - |x| - \delta\varphi(x)/2)$ . Hence,

$$\varphi^2(x) = (1 - |x|)(1 + |x|) \leq 4(1 - |x| - \delta\varphi(x)/2)(1 + |x| - \delta\varphi(x)/2) = 4\mathcal{W}_\delta^2(x),$$

$x \in \mathfrak{D}_{2\delta}$ ,

and so (iii) is verified. If  $\delta > 1$ , then (iii) is vacuously true since  $\mathfrak{D}_{2\delta} = \emptyset$ .

If  $x \in \mathfrak{D}_\delta$ , then  $\delta\varphi(x)/2 \leq 1 - |x| = (1 + |x|)^{-1}\varphi^2(x)$ ; that is,  $\varphi(x) \geq \delta(1 + |x|)/2$ . Hence,

$$\delta|\varphi'(x)| = \delta \frac{|x|}{\varphi(x)} \leq \frac{2|x|}{1 + |x|} \leq 1,$$

which is (iv).

Property (v) immediately follows from (iv), and (vi) is obvious. □

The first important property of the new moduli is stated in the following lemma.

**Lemma 3.2** If  $r \in \mathbb{N}_0$ ,  $1 \leq p < \infty$ , and  $f \in \mathbb{B}_p^r$ , then

$$\lim_{t \rightarrow 0^+} \omega_{k,r}^\varphi(f^{(r)}, t)_p = 0.$$

*Proof* Let  $\epsilon > 0$ . Then, there is  $\delta > 0$  such that

$$\int_{[-1,1] \setminus \mathfrak{D}_\delta} |\varphi^r(x) f^{(r)}(x)|^p dx < \left(\frac{\epsilon}{2^{k+2}}\right)^p.$$

Set

$$g^{(r)}(x) := \begin{cases} f^{(r)}(x) & \text{if } x \in \mathfrak{D}_\delta, \\ 0 & \text{otherwise.} \end{cases}$$

Since  $g^{(r)} \in L_p[-1, 1]$ , there exists  $t_0 > 0$  such that

$$\omega_k^\varphi(g^{(r)}, t)_p < \epsilon/2, \quad 0 < t \leq t_0.$$

For each  $h > 0$ , we have

$$\begin{aligned} & \left\| \mathcal{W}_{kh}^r(\cdot) \Delta_{h\varphi(\cdot)}^k(f^{(r)}, \cdot) \right\|_p \\ & \leq \left\| \mathcal{W}_{kh}^r(\cdot) \Delta_{h\varphi(\cdot)}^k(g^{(r)}, \cdot) \right\|_p + \left\| \mathcal{W}_{kh}^r(\cdot) \Delta_{h\varphi(\cdot)}^k(f^{(r)} - g^{(r)}, \cdot) \right\|_p \\ & \leq \left\| \Delta_{h\varphi(\cdot)}^k(g^{(r)}, \cdot) \right\|_p + \left\| \mathcal{W}_{kh}^r(\cdot) \Delta_{h\varphi(\cdot)}^k(f^{(r)} - g^{(r)}, \cdot) \right\|_p \\ & =: I_1 + I_2. \end{aligned}$$

Now, if  $h \leq t_0$ , then  $I_1 < \epsilon/2$ , and

$$\begin{aligned} I_2 & \leq \sum_{i=0}^k \binom{k}{i} \left( \int_{\mathfrak{D}_{kh}} \left( \mathcal{W}_{kh}^r(x) |f^{(r)}(x + (i - k/2)h\varphi(x)) \right. \right. \\ & \quad \left. \left. - g^{(r)}(x + (i - k/2)h\varphi(x)) \right|^p dx \right)^{1/p} \\ & \leq \sum_{i=0}^k \binom{k}{i} \left( \int_{\mathfrak{D}_{kh}} \left( \varphi^r(x + (i - k/2)h\varphi(x)) |f^{(r)}(x + (i - k/2)h\varphi(x)) \right. \right. \\ & \quad \left. \left. - g^{(r)}(x + (i - k/2)h\varphi(x)) \right|^p dx \right)^{1/p} \\ & \leq 2 \sum_{i=0}^k \binom{k}{i} \left( \int_{-1}^1 \left( \varphi^r(u) |f^{(r)}(u) - g^{(r)}(u)| \right)^p du \right)^{1/p} \\ & \leq 2 \sum_{i=0}^k \binom{k}{i} \left( \int_{[-1,1] \setminus \mathfrak{D}_\delta} |\varphi^r(u) f^{(r)}(u)|^p \right)^{1/p} \leq \epsilon/2, \end{aligned}$$

where for the second inequality we used, for  $x \in \mathfrak{D}_{kh}$ , the inequality  $\mathcal{W}_{kh}(x) \leq \varphi(u(x))$ , where  $u = u(x) := x + (i - k/2)h\varphi(x)$ , and the third inequality follows because Proposition 3.1(v) implies that  $u'(x) \geq 1/2$  when  $x \in \mathfrak{D}_{kh}$ . This completes the proof.  $\square$

*Remark 3.3* Note that  $f \in \mathbb{B}_\infty^r$  only implies that  $\omega_{k,r}^\varphi(f^{(r)}, t)_\infty < \infty$  for  $t > 0$  and does NOT imply that  $\lim_{t \rightarrow 0^+} \omega_{k,r}^\varphi(f^{(r)}, t)_\infty = 0$  even if we assume that  $f^{(r)} \in$



$C(-1, 1)$ . For example, if  $f$  is such that  $f^{(r)}(x) := \varphi^{-r}(x)$ ,  $r \in \mathbb{N}$ , then  $f \in \mathbb{B}_\infty^r \cap C^r(-1, 1)$  and  $\omega_{k,r}^\varphi(f^{(r)}, t)_\infty \geq \text{const} > 0$ .

It was proved in [4] (see also [6]) that, for  $r \in \mathbb{N}$  and  $f \in C^r(-1, 1)$ ,

$$\lim_{t \rightarrow 0^+} \omega_{k,r}^\varphi(f^{(r)}, t)_\infty = 0 \text{ if and only if } \lim_{x \rightarrow \pm 1} \varphi^r(x) f^{(r)}(x) = 0.$$

In the case  $r = 0$ , it is easy to see (see also [3, p. 37]) that  $\lim_{t \rightarrow 0^+} \omega_{k,0}^\varphi(f, t)_\infty = 0$  if and only if  $f \in C[-1, 1]$ .

**Lemma 3.4** *Let  $1 \leq p \leq \infty$  and  $r \in \mathbb{N}_0$ . If  $g \in \mathbb{B}_p^{r+1}$ , then*

$$\left\| \varphi^\gamma g^{(r)} \right\|_p < \infty$$

for any  $\gamma \geq 0$  such that  $\gamma > r - 1$ .

**Corollary 3.5** *Let  $1 \leq p \leq \infty$  and  $r \in \mathbb{N}_0$ . If  $g \in \mathbb{B}_p^{r+1}$ , then  $g \in \mathbb{B}_p^r$ .*

*Proof of Lemma 3.4* Suppose that we are given  $g \in \mathbb{B}_p^{r+1}$ . Without loss of generality, we can assume that  $g^{(r)}(0) = 0$ .

First, we consider the case  $p = \infty$ . Since  $\varphi(u) \geq \varphi(x)$  for  $|u| \leq |x|$ , we have

$$\begin{aligned} \left\| \varphi^\gamma g^{(r)} \right\|_\infty &\leq \left\| \varphi^\gamma(x) \int_0^x g^{(r+1)}(u) \, du \right\|_\infty \\ &\leq \left\| \varphi^{r+1} g^{(r+1)} \right\|_\infty \left\| \varphi^\gamma(x) \int_0^x \varphi^{-r-1}(u) \, du \right\|_\infty \\ &\leq \left\| \varphi^{r+1} g^{(r+1)} \right\|_\infty \left\| \int_0^x \varphi^{\gamma-r-1}(u) \, du \right\|_\infty \\ &\leq c \left\| \varphi^{r+1} g^{(r+1)} \right\|_\infty. \end{aligned}$$

Similarly, if  $p = 1$ , then

$$\begin{aligned} \left\| \varphi^\gamma g^{(r)} \right\|_1 &= \int_{-1}^1 \varphi^\gamma(x) \left| \int_0^x g^{(r+1)}(u) \, du \right| \, dx \\ &\leq \int_{-1}^1 \varphi^{\gamma-r-1}(x) \left| \int_0^x \varphi^{r+1}(u) |g^{(r+1)}(u)| \, du \right| \, dx \\ &\leq \left\| \varphi^{r+1} g^{(r+1)} \right\|_1 \int_{-1}^1 \varphi^{\gamma-r-1}(x) \, dx \leq c \left\| \varphi^{r+1} g^{(r+1)} \right\|_1. \end{aligned}$$

Suppose now that  $1 < p < \infty$  and  $q = p/(p - 1)$  is such that  $(r + 1)q \neq 2$  (i.e., either  $r \geq 1$ , or  $r = 0$  and  $p \neq 2$ ). Using Hölder’s inequality, we have

$$\begin{aligned}
 & \left\| \varphi^\gamma g^{(r)} \right\|_p^p \\
 &= \int_{-1}^1 \varphi^{\gamma p}(x) \left| \int_0^x g^{(r+1)}(u) \, du \right|^p dx \\
 &\leq \int_{-1}^1 \varphi^{\gamma p}(x) \left| \left( \int_0^x \varphi^{-(r+1)q}(u) \, du \right)^{1/q} \left( \int_0^x |\varphi^{r+1}(u) g^{(r+1)}(u)|^p \, du \right)^{1/p} \right|^p dx \\
 &\leq \left\| \varphi^{r+1} g^{(r+1)} \right\|_p^p \int_{-1}^1 \varphi^{\gamma p}(x) \left| \int_0^x \varphi^{-(r+1)q}(u) \, du \right|^{p/q} dx \\
 &= 2 \left\| \varphi^{r+1} g^{(r+1)} \right\|_p^p \int_0^1 \varphi^{\gamma p}(x) \left( \int_0^x \varphi^{-(r+1)q}(u) \, du \right)^{p/q} dx \\
 &\leq 2^{1+\gamma p/2} \left\| \varphi^{r+1} g^{(r+1)} \right\|_p^p \int_0^1 (1-x)^{\gamma p/2} \left( \int_0^x (1-u)^{-(r+1)q/2} \, du \right)^{p/q} dx \\
 &\leq c \left\| \varphi^{r+1} g^{(r+1)} \right\|_p^p \int_0^1 (1-x)^{\gamma p/2} \max\{1, (1-x)^{-1-p(r-1)/2}\} dx \\
 &\leq c \left\| \varphi^{r+1} g^{(r+1)} \right\|_p^p \int_0^1 \max\{(1-x)^{\gamma p/2}, (1-x)^{-1+p(\gamma-r+1)/2}\} dx \\
 &\leq c \left\| \varphi^{r+1} g^{(r+1)} \right\|_p^p.
 \end{aligned}$$

Finally, if  $p = 2$  and  $r = 0$ , then

$$\begin{aligned}
 \left\| \varphi^\gamma g \right\|_2^2 &\leq 2^{1+\gamma} \left\| \varphi g' \right\|_2^2 \int_0^1 (1-x)^\gamma \int_0^x \frac{du}{(1-u)} dx \\
 &\leq c \left\| \varphi g' \right\|_2^2 \int_0^1 \int_0^x \frac{du}{(1-u)} dx \leq c \left\| \varphi g' \right\|_2^2.
 \end{aligned}$$

□

#### 4 Proof of Theorem 2.7: The Upper Estimate

The upper estimate of our modulus by the  $K$ -functional in Theorem 2.7 (i.e., the last inequality in (2.4)) immediately follows from the following lemma.

**Lemma 4.1** *If  $k \in \mathbb{N}$ ,  $r \in \mathbb{N}_0$ ,  $1 \leq p \leq \infty$ , and  $f \in \mathbb{B}_p^r$ , then*

$$\omega_{k,r}^\varphi(f^{(r)}, t)_p \leq c(k, r, p) K_{k,r}^\varphi(f^{(r)}, t^k)_p \quad \text{for all } t > 0.$$

*Proof* In view of (2.2) and the monotonicity of the  $K$ -functional in  $t$ , we may assume that  $t \leq 2/k$ . Take any  $g \in \mathbb{B}_p^{k+r}$ . Corollary 3.5 implies that  $g \in \mathbb{B}_p^r$ , whence

$$\omega_{k,r}^\varphi(f^{(r)}, t)_p \leq \omega_{k,r}^\varphi(f^{(r)} - g^{(r)}, t)_p + \omega_{k,r}^\varphi(g^{(r)}, t)_p.$$

Take  $h$  such that  $0 < h \leq t$ .

For each  $0 \leq i \leq k$ , set  $y_i(x) := x + (i - k/2)h\varphi(x)$ . Then, Proposition 3.1(v) implies that  $y'_i(x) \geq 1/2$  for  $x \in \mathfrak{D}_{kh}$ , and so we have (with obvious modifications if  $p = \infty$ )

$$\begin{aligned} & \|\varphi^r(y_i)(f^{(r)}(y_i) - g^{(r)}(y_i))\|_{L_p(\mathfrak{D}_{kh})} \\ &= \left( \int_{\mathfrak{D}_{kh}} \varphi^{rp}(y_i(x)) |f^{(r)}(y_i(x)) - g^{(r)}(y_i(x))|^p dx \right)^{1/p} \\ &\leq 2^{1/p} \left( \int_{-1}^1 \varphi^{rp}(y) |f^{(r)}(y) - g^{(r)}(y)|^p dy \right)^{1/p} \\ &= 2^{1/p} \|\varphi^r(f^{(r)} - g^{(r)})\|_p. \end{aligned}$$

Since  $\mathcal{W}_\delta(x) \leq \varphi(y)$  for all  $x \in \mathfrak{D}_\delta$ ,  $y \in [x - \delta\varphi(x)/2, x + \delta\varphi(x)/2]$ , and  $0 < \delta \leq 2$ , we get

$$\begin{aligned} \omega_{k,r}^\varphi(f^{(r)} - g^{(r)}, t)_p &\leq \sup_{0 < h \leq t} \left\| \sum_{i=0}^k \binom{k}{i} \varphi^r(y_i) |f^{(r)}(y_i) - g^{(r)}(y_i)| \right\|_{\mathbb{L}_p(\mathfrak{D}_{kh})} \\ &\leq 2^{k+1/p} \|\varphi^r(f^{(r)} - g^{(r)})\|_p. \end{aligned}$$

To estimate the second term  $\omega_{k,r}^\varphi(g^{(r)}, t)_p$ , using the identity

$$\Delta_h^k(f, x) = \int_{-h/2}^{h/2} \dots \int_{-h/2}^{h/2} f^{(k)}(x + u_1 + \dots + u_k) du_1 \dots du_k, \tag{4.1}$$

we have

$$\begin{aligned} & \omega_{k,r}^\varphi(g^{(r)}, t)_p \\ &= \sup_{0 < h \leq t} \left\| \mathcal{W}_{kh}^r \Delta_{h\varphi}^k(g^{(r)}, \cdot) \right\|_{\mathbb{L}_p(\mathfrak{D}_{kh})} \\ &= \sup_{0 < h \leq t} \left\| \mathcal{W}_{kh}^r \int_{-h\varphi/2}^{h\varphi/2} \dots \int_{-h\varphi/2}^{h\varphi/2} g^{(k+r)}(\cdot + u_1 + \dots + u_k) du_1 \dots du_k \right\|_{\mathbb{L}_p(\mathfrak{D}_{kh})}. \end{aligned}$$

By Hölder’s inequality (with  $1/p + 1/q = 1$ ), for each  $u$  satisfying  $-1 < x + u - h\varphi(x)/2 < x + u + h\varphi(x)/2 < 1$ , we have

$$\begin{aligned} \left| \int_{-h\varphi(x)/2}^{h\varphi(x)/2} g^{(k+r)}(x + u + u_k) du_k \right| &= \left| \int_{x+u-h\varphi(x)/2}^{x+u+h\varphi(x)/2} g^{(k+r)}(v) dv \right| \\ &\leq \int_{x+u-h\varphi(x)/2}^{x+u+h\varphi(x)/2} \frac{\varphi^{k+r}(v) |g^{(k+r)}(v)|}{\varphi^{k+r}(v)} dv \\ &\leq \|\varphi^{k+r} g^{(k+r)}\|_{L_p(\mathcal{A}(x,u))} \|\varphi^{-k-r}\|_{L_q(\mathcal{A}(x,u))}. \end{aligned}$$

where

$$A(x, u) := \left[ x + u - \frac{h}{2}\varphi(x), x + u + \frac{h}{2}\varphi(x) \right].$$

Thus, in order to complete the proof, it suffices to prove

$$\begin{aligned} & \int_{\mathfrak{D}_{kh}} \left( \mathcal{W}_{kh}^r(x) \int_{-h\varphi(x)/2}^{h\varphi(x)/2} \cdots \int_{-h\varphi(x)/2}^{h\varphi(x)/2} \|\varphi^{-k-r}\|_{\mathbb{L}_q(\mathcal{A}(x, u_1 + \dots + u_{k-1}))} \right. \\ & \quad \left. \times \|\varphi^{k+r} g^{(k+r)}\|_{L_p(\mathcal{A}(x, u_1 + \dots + u_{k-1}))} \mathbf{d}u_1 \cdots \mathbf{d}u_{k-1} \right)^p \mathbf{d}x \\ & \leq ch^{kp} \|g^{(k+r)} \varphi^{k+r}\|_p^p, \end{aligned} \tag{4.2}$$

noting that, in the case  $k = 1$ , this inequality is understood as

$$\begin{aligned} & \int_{\mathfrak{D}_h} \left( \mathcal{W}_h^r(x) \|\varphi^{-1-r}\|_{\mathbb{L}_q(\mathcal{A}(x, 0))} \|\varphi^{1+r} g^{(1+r)}\|_{L_p(\mathcal{A}(x, 0))} \right)^p \mathbf{d}x \\ & \leq ch^p \|g^{(1+r)} \varphi^{1+r}\|_p^p, \end{aligned} \tag{4.3}$$

and if  $p = \infty$ , then (4.2) is replaced by

$$\begin{aligned} & \sup_{x \in \mathfrak{D}_{kh}} \left( \mathcal{W}_{kh}^r(x) \int_{-h\varphi(x)/2}^{h\varphi(x)/2} \cdots \int_{-h\varphi(x)/2}^{h\varphi(x)/2} \|\varphi^{-k-r}\|_{\mathbb{L}_1(\mathcal{A}(x, u_1 + \dots + u_{k-1}))} \right. \\ & \quad \left. \times \|\varphi^{k+r} g^{(k+r)}\|_{L_\infty(\mathcal{A}(x, u_1 + \dots + u_{k-1}))} \mathbf{d}u_1 \cdots \mathbf{d}u_{k-1} \right) \\ & \leq ch^k \|g^{(k+r)} \varphi^{k+r}\|_\infty. \end{aligned} \tag{4.4}$$

To this end, we write

$$\int_{\mathfrak{D}_{kh}} = \int_{\mathfrak{D}_{2kh}} + \int_{(\mathfrak{D}_{kh} \setminus \mathfrak{D}_{2kh}) \cap [0, 1]} + \int_{(\mathfrak{D}_{kh} \setminus \mathfrak{D}_{2kh}) \cap [-1, 0]} =: I_1(p) + I_2(p) + I_3(p)$$

if  $1 \leq p < \infty$ , and

$$\sup_{\mathfrak{D}_{kh}} \leq \sup_{\mathfrak{D}_{2kh}} + \sup_{(\mathfrak{D}_{kh} \setminus \mathfrak{D}_{2kh}) \cap [0, 1]} + \sup_{(\mathfrak{D}_{kh} \setminus \mathfrak{D}_{2kh}) \cap [-1, 0]} =: I_1(\infty) + I_2(\infty) + I_3(\infty)$$

if  $p = \infty$ .

**Part I: estimate of  $I_1$**

First, we note that if  $h > 1/k$ , then  $\mathfrak{D}_{2kh} = \emptyset$ , and so no estimate of  $I_1$  is needed. Hence, in this part, we may assume that  $h \leq 1/k$ .

For  $x \in \mathfrak{D}_{2\delta}$  and  $u \in [x - \delta\varphi(x)/2, x + \delta\varphi(x)/2]$ , we have

$$\begin{aligned} \frac{1}{2}(1 - |u|) &\leq \frac{1}{2}(1 - |x| + \delta\varphi(x)/2) \leq 1 - |x| \\ &\leq 2(1 - |x| - \delta\varphi(x)/2) \leq 2(1 - |u|), \end{aligned}$$

where for the second and third inequalities we applied the fact that  $\delta\varphi(x) \leq 1 - |x|$ . Also, obviously,

$$\frac{1}{2}(1 + |u|) \leq 1 + |x| \leq 2(1 + |u|).$$

Hence, for  $x \in \mathfrak{D}_{2\delta}$  and  $u \in [x - \delta\varphi(x)/2, x + \delta\varphi(x)/2]$ ,

$$\frac{1}{2}\varphi(u) \leq \varphi(x) \leq 2\varphi(u). \tag{4.5}$$

Also, note that  $\delta\varphi(x) \leq 1 - |x|$  (i.e.,  $x \in \mathfrak{D}_{2\delta}$ ) implies

$$\delta \leq \varphi(x). \tag{4.6}$$

So, if  $x \in \mathfrak{D}_{2kh}$ , then by (4.5),

$$\begin{aligned} \mathcal{W}_{kh}^r(x) &\int_{-h\varphi(x)/2}^{h\varphi(x)/2} \cdots \int_{-h\varphi(x)/2}^{h\varphi(x)/2} \|\varphi^{-k-r}\|_{\mathbb{L}_q(\mathcal{A}(x, u_1 + \dots + u_{k-1}))} \, du_1 \dots du_{k-1} \\ &\leq \varphi^r(x)(h\varphi(x))^{k-1} \frac{2^{k+r}}{\varphi^{k+r}(x)} (h\varphi(x))^{1/q} \\ &= 2^{k+r} h^{k-1+1/q} \varphi^{1/q-1}(x) = 2^{k+r} h^{k-1/p} \varphi^{-1/p}(x), \end{aligned}$$

where we applied Proposition 3.1(ii). Note that the above estimate is also valid for  $k = 1$ .

Therefore, by (4.5) and (4.6), for  $1 \leq p < \infty$ , we have

$$\begin{aligned} I_1(p) &\leq \int_{\mathfrak{D}_{2kh}} \left( 2^{k+r} h^{k-1/p} \varphi^{-1/p}(x) \|\varphi^{k+r} g^{(k+r)}\|_{L_p([x-kh\varphi(x)/2, x+kh\varphi(x)/2])} \right)^p dx \\ &= 2^{p(k+r)} h^{kp-1} \int_{\mathfrak{D}_{2kh}} \frac{1}{\varphi(x)} \int_{x-kh\varphi(x)/2}^{x+kh\varphi(x)/2} \left| \varphi^{k+r}(u) g^{(k+r)}(u) \right|^p du dx \\ &\leq ch^{kp-1} \int_{\mathfrak{D}_{2kh}} \int_{x-kh\varphi(x)/2}^{x+kh\varphi(x)/2} \frac{1}{\varphi(u) + kh/2} \left| \varphi^{k+r}(u) g^{(k+r)}(u) \right|^p du dx \\ &= ch^{kp-1} \int_{a_1}^{a_2} \int_{b_1(u)}^{b_2(u)} \frac{1}{\varphi(u) + kh/2} \left| \varphi^{k+r}(u) g^{(k+r)}(u) \right|^p dx du \\ &= ch^{kp-1} \int_{a_1}^{a_2} \left| \varphi^{k+r}(u) g^{(k+r)}(u) \right|^p \frac{b_2(u) - b_1(u)}{\varphi(u) + kh/2} du, \end{aligned}$$

where  $-1 < a_1 < a_2 < 1$  and

$$b_2(u) - b_1(u) \leq \frac{kh\sqrt{1 - u^2 + (kh/2)^2}}{1 + (kh/2)^2} \leq kh(\varphi(u) + kh/2).$$

Hence,

$$I_1(p) \leq ch^{kp} \|g^{(k+r)} \varphi^{k+r}\|_p^p.$$

If  $p = \infty$ , then

$$\begin{aligned} I_1(\infty) &\leq \sup_{\mathfrak{D}_{2kh}} \left( 2^{k+r} h^k \|\varphi^{k+r} g^{(k+r)}\|_{L_\infty([x - kh\varphi(x)/2, x + kh\varphi(x)/2])} \right) \\ &\leq ch^k \|\varphi^{k+r} g^{(k+r)}\|_\infty. \end{aligned}$$

**Part II: estimate of  $I_2$**

In this part, we estimate  $I_2$ , the estimate of  $I_3$  being completely analogous. It is convenient to introduce the notation

$$\begin{aligned} \mathcal{F}_q(x, k, r) &:= \mathcal{W}_{kh}^r(x) \int_{-h\varphi(x)/2}^{h\varphi(x)/2} \cdots \int_{-h\varphi(x)/2}^{h\varphi(x)/2} \|\varphi^{-k-r}\|_{\mathbb{L}_q(\mathcal{A}(x, u_1 + \dots + u_{k-1}))} \mathbf{d}u_1 \dots \mathbf{d}u_{k-1}, \\ \mathcal{F}_q(x, k) &:= \mathcal{F}_q(x, k, 0), \end{aligned}$$

and

$$\mathcal{E}_{kh} := (\mathfrak{D}_{kh} \setminus \mathfrak{D}_{2kh}) \cap [0, 1].$$

The required estimates for  $I_2(p)$  and  $I_2(\infty)$  follow, respectively, from

$$\int_{\mathcal{E}_{kh}} (\mathcal{F}_q(x, k, r))^p \mathbf{d}x \leq ch^{kp}, \tag{4.7}$$

and

$$\sup_{x \in \mathcal{E}_{kh}} \mathcal{F}_1(x, k, r) \leq ch^k. \tag{4.8}$$

First, we observe that if  $v \in \mathcal{A}(x, u_1 + \dots + u_{k-1})$  and  $|u_i| \leq h\varphi(x)/2$ , then

$$x - kh\varphi(x)/2 \leq v \leq x + kh\varphi(x)/2,$$

which, by Proposition 3.1(i) implies, for  $x \in \mathfrak{D}_{kh}$ , that

$$\mathcal{W}_{kh}(x) \leq \varphi(v).$$

Hence

$$\mathcal{W}_{kh}^r(x) \|\varphi^{-k-r}\|_{\mathbb{L}_q(\mathcal{A}(x, u_1 + \dots + u_{k-1}))} \leq \|\varphi^{-k}\|_{\mathbb{L}_q(\mathcal{A}(x, u_1 + \dots + u_{k-1}))},$$

so that

$$\mathcal{F}_q(x, k, r) \leq \mathcal{F}_q(x, k), \quad x \in \mathcal{D}_{kh}.$$

Thus, (4.7) and (4.8) follow, respectively, from

$$\int_{\mathcal{E}_{kh}} (\mathcal{F}_q(x, k))^p dx \leq ch^{kp} \tag{4.9}$$

and

$$\sup_{x \in \mathcal{E}_{kh}} \mathcal{F}_1(x, k) \leq ch^k. \tag{4.10}$$

Recall that  $\mu(\delta) = 2\delta^2/(4 + \delta^2)$ , and note that  $\mathcal{E}_{kh} = (1 - \mu(2kh), 1 - \mu(kh)] \cap [0, 1]$ , i.e.,

$$\mathcal{E}_{kh} = \begin{cases} (1 - \mu(2kh), 1 - \mu(kh)] & \text{if } h \leq 1/k, \\ [0, 1 - \mu(kh)] & \text{if } 1/k < h \leq 2/k. \end{cases}$$

It will be convenient for us to separate the proof for “small” and “large”  $h$ . We first consider the case when  $h \leq 1/(\sqrt{2}k)$ .

**Part II(i):**  $h \leq 1/(\sqrt{2}k)$

It is easy to see that if  $h \leq 1/(\sqrt{2}k)$ , then

$$\frac{8k^2h^2}{9} \leq \text{meas}(\mathcal{E}_{kh}) \leq \frac{3k^2h^2}{2}$$

and, for  $x \in \mathcal{E}_{kh}$ ,

$$\frac{4k^2h^2}{9} \leq 1 - x \leq 2k^2h^2 \quad \text{and} \quad \frac{2kh}{3} \leq \varphi(x) \leq 2kh.$$

It is important to note that if  $h \leq 1/(\sqrt{2}k)$  and  $x \in \mathcal{E}_{kh}$ , then

$$x - kh\varphi(x)/2 \geq 0. \tag{4.11}$$

This implies that if  $v \in \mathcal{A}(x, u_1 + \dots + u_{k-1})$ , where  $x \in \mathcal{E}_{kh}$  and  $|u_i| \leq h\varphi(x)/2$ , then  $v \geq 0$ , and so

$$\sqrt{1 - v} \leq \varphi(v) \leq \sqrt{2(1 - v)}.$$

Now, for any  $q < \infty$ ,  $x \in \mathcal{E}_{kh}$ , and  $u \in [-(k - 1)h\varphi(x)/2, (k - 1)h\varphi(x)/2]$ , we have

$$\begin{aligned} & \|\varphi^{-k}\|_{\mathbb{L}_q(\mathcal{A}(x,u))} \\ & \leq \left( \int_{x+u-h\varphi(x)/2}^{x+u+h\varphi(x)/2} (1-v)^{-kq/2} dv \right)^{1/q} \\ & \leq c(k, q) \begin{cases} (1-x-u-h\varphi(x)/2)^{-k/2+1/q} & \text{if } kq > 2, \\ \left( \ln \frac{1-x-u+h\varphi(x)/2}{1-x-u-h\varphi(x)/2} \right)^{1/q} & \text{if } kq = 2, \\ h^{-k+2/q} & \text{if } kq < 2, \end{cases} \end{aligned} \tag{4.12}$$

and note that (4.12) is also valid if  $q = \infty$ .

We also observe that if  $k \geq 2$ , then for any  $x \in \mathcal{E}_{kh}$  and  $u \in [-(k-2)h\varphi(x)/2, (k-2)h\varphi(x)/2]$ ,

$$\int_{-h\varphi(x)/2}^{h\varphi(x)/2} \int_{-h\varphi(x)/2}^{h\varphi(x)/2} (1-x-u-u_1-u_2)^{-1} du_1 du_2 \leq ch^2. \tag{4.13}$$

Indeed, using the fact that  $1-x-u \geq h\varphi(x)$  and changing variables to  $v := -2u_1/(h\varphi(x))$  and  $w := -2u_2/(h\varphi(x))$ , we have

$$\begin{aligned} & \int_{-h\varphi(x)/2}^{h\varphi(x)/2} \int_{-h\varphi(x)/2}^{h\varphi(x)/2} (1-x-u-u_1-u_2)^{-1} du_1 du_2 \\ & \leq \int_{-h\varphi(x)/2}^{h\varphi(x)/2} \int_{-h\varphi(x)/2}^{h\varphi(x)/2} (h\varphi(x)-u_1-u_2)^{-1} du_1 du_2 \\ & = \frac{h\varphi(x)}{2} \int_{-1}^1 \int_{-1}^1 (2+v+w)^{-1} dv dw \\ & = (2 \ln 2)h\varphi(x) \leq c(k)h^2. \end{aligned}$$

Thus, (4.13) implies that for any  $k \geq 2$ ,  $\alpha > 1-k$ ,  $x \in \mathcal{E}_{kh}$ , and  $u \in [-h\varphi(x)/2, h\varphi(x)/2]$ ,

$$\int_{-h\varphi(x)/2}^{h\varphi(x)/2} \dots \int_{-h\varphi(x)/2}^{h\varphi(x)/2} (1-x-u-u_1-\dots-u_{k-1})^\alpha du_1 \dots du_{k-1} \leq ch^{2k+2\alpha-2}. \tag{4.14}$$

Now, for any  $x \in \mathcal{E}_{kh}$ , we have

$$\begin{aligned} \mathcal{F}_1(x, k) &= \int_{-h\varphi(x)/2}^{h\varphi(x)/2} \dots \int_{-h\varphi(x)/2}^{h\varphi(x)/2} \varphi^{-k}(x+u_1+\dots+u_k) du_1 \dots du_k \\ &\leq \int_{-h\varphi(x)/2}^{h\varphi(x)/2} \dots \int_{-h\varphi(x)/2}^{h\varphi(x)/2} (1-x-u_1-\dots-u_k)^{-k/2} du_1 \dots du_k \\ &\leq ch^k, \end{aligned}$$



which implies (4.10) and so completes the proof in the case  $p = \infty$ .

For  $1 \leq p < \infty$ , it is convenient to break the proof of (4.9) into several cases.

CASE 1:  $1 \leq p < \infty, k = 1$ , and  $q > 2$

Using (4.12), we have

$$\begin{aligned} \int_{\mathcal{E}_h} (\mathcal{F}_q(x, 1))^p dx &= \int_{\mathcal{E}_h} \|\varphi^{-1}\|_{\mathbb{L}_q(\mathcal{A}(x,0))}^p dx \\ &\leq c \int_{\mathcal{E}_h} (1 - x - h\varphi(x)/2)^{-p/2+p/q} dx \\ &= c \int_{1-\mu(2h)}^{1-\mu(h)} (1 - x - h\varphi(x)/2)^{p/2-1} dx \\ &\leq c \int_{1/(1+h^2)}^1 (1 - y)^{p/2-1} dy \\ &\leq ch^p. \end{aligned}$$

CASE 2:  $1 \leq p < \infty, k = 1$ , and  $q = 2$

Applying (4.12), we obtain

$$\begin{aligned} \int_{\mathcal{E}_h} (\mathcal{F}_2(x, 1))^2 dx &\leq c \int_{1-\mu(2h)}^{1-\mu(h)} \ln \left( 1 + \frac{h\varphi(x)}{1 - x - h\varphi(x)/2} \right) dx \\ &\leq c \int_{1-\mu(2h)}^{1-\mu(h)} \ln \left( 1 + \frac{2h^2}{1 - x - h\varphi(x)/2} \right) dx \\ &\leq c \int_{1/(1+h^2)}^1 \ln \left( 1 + \frac{2h^2}{1 - y} \right) dy \\ &\leq ch^2. \end{aligned}$$

CASE 3:  $1 \leq p < \infty, k = 1$ , and  $q < 2$

We apply (4.12) and get

$$\int_{\mathcal{E}_h} (\mathcal{F}_q(x, 1))^p dx \leq c \int_{\mathcal{E}_h} h^{p-2} dx \leq c \text{meas}(\mathcal{E}_h) h^{p-2} \leq ch^p.$$

CASE 4:  $1 \leq p < \infty, k \geq 2$ , and  $2/p < k$

Note that in this case,  $kh > 2$  and  $k/2 + 1/q > 1$ , and applying (4.12) and (4.14), we get

$$\begin{aligned} &\int_{\mathcal{E}_{kh}} (\mathcal{F}_q(x, k))^p dx \\ &\leq c \int_{\mathcal{E}_{kh}} \left( \int_{-h\varphi(x)/2}^{h\varphi(x)/2} \cdots \int_{-h\varphi(x)/2}^{h\varphi(x)/2} \right. \\ &\quad \left. (1 - x - u_1 - \cdots - u_{k-1} - h\varphi(x)/2)^{-k/2+1/q} du_1 \cdots du_{k-1} \right)^p dx \end{aligned}$$

$$\leq c \operatorname{meas}(\mathcal{E}_{kh})h^{(k-2+2/q)p} \leq ch^{kp}.$$

CASE 5:  $1 \leq p < \infty$  and  $2/p \geq k \geq 2$

Since  $2 \leq k \leq 2/p$  and  $p \geq 1$  can hold simultaneously only if  $k = 2$  and  $p = 1$ , using (4.12) for  $q = \infty$ , we have

$$\begin{aligned} & \int_{\mathcal{E}_{2h}} \mathcal{F}_\infty(x, 2) dx \\ &= \int_{\mathcal{E}_{2h}} \int_{-h\varphi(x)/2}^{h\varphi(x)/2} \|\varphi^{-2}\|_{\mathbb{L}_\infty(\mathcal{A}(x,u))} du dx \\ &\leq c \int_{\mathcal{E}_{2h}} \int_{-h\varphi(x)/2}^{h\varphi(x)/2} (1 - x - u - h\varphi(x)/2)^{-1} du dx \\ &= c \int_{1-\mu(4h)}^{1-\mu(2h)} \ln \left( 1 + \frac{h\varphi(x)}{1 - x - h\varphi(x)} \right) dx \\ &\leq c \int_{1-\mu(4h)}^{1-\mu(2h)} \ln \left( 1 + \frac{4h^2}{1 - x - h\varphi(x)} \right) dx \\ &\leq c \int_{1/(1+4h^2)}^1 \ln \left( 1 + \frac{4h^2}{1 - y} \right) dy \\ &\leq ch^2. \end{aligned}$$

It remains to consider the case when  $1/(\sqrt{2}k) < h \leq 2/k$ .

**Part II(ii):**  $1/(\sqrt{2}k) < h \leq 2/k$

If  $1/(\sqrt{2}k) < h \leq 2/k$ , then, for  $x \in \mathcal{E}_{kh}$ ,  $h \sim 1 - x \sim \varphi(x) \sim c(k)$ , and  $\operatorname{meas}(\mathcal{E}_{kh}) \leq c(k)$  (“ $\leq$ ” cannot be replaced with “ $\sim$ ” since  $\operatorname{meas}(\mathcal{E}_2) = 0$ ). Inequalities (4.9) and (4.10) which we need to verify become

$$\int_{\mathcal{E}_{kh}} (\mathcal{F}_q(x, k))^p dx \leq c \tag{4.15}$$

and

$$\sup_{x \in \mathcal{E}_{kh}} \mathcal{F}_1(x, k) \leq c. \tag{4.16}$$

We can prove (4.15) and (4.16) using the proof used in Part II(i) with the only difference that we can no longer use the fact that  $\varphi(v) \sim \sqrt{1 - v}$  for  $v \in \mathcal{A}(x, u_1 + \dots + u_{k-1})$  with  $x \in \mathcal{E}_{kh}$  and  $|u_i| \leq h\varphi(x)/2$ . At the same time, since we no longer need to keep track of powers of  $h$ ’s, this proof can be considerably simplified.

First, let  $F \in C[-1, 1]$  be such that  $F^{(k)}(x) = \varphi^{-k}(x)$ ,  $x \in (-1, 1)$ . Applying the identity (4.1) for any  $x \in \mathcal{E}_{kh}$ , we have

$$\begin{aligned} \mathcal{F}_1(x, k) &= \int_{-h\varphi(x)/2}^{h\varphi(x)/2} \dots \int_{-h\varphi(x)/2}^{h\varphi(x)/2} \varphi^{-k}(x + u_1 + \dots + u_k) du_1 \dots du_k \\ &= \Delta_{h\varphi(x)}^k(F, x) \leq c, \end{aligned}$$

which implies (4.16) and so completes the proof in the case  $p = \infty$ .

Now, observing that, for  $-1 < a < b < 1$ ,

$$\int_a^b \varphi^\gamma(t) dt \leq c(\gamma) \begin{cases} \varphi^{2+\gamma}(a) + \varphi^{2+\gamma}(b) & \text{if } \gamma < -2, \\ 1 & \text{if } \gamma > -2, \end{cases} \tag{4.17}$$

we conclude that

$$\|\varphi^{-k}\|_{\mathbb{L}_q[a,b]} \leq c(k, q) \begin{cases} \varphi^{-k+2/q}(a) + \varphi^{-k+2/q}(b) & \text{if } kq > 2, \\ 1 & \text{if } kq < 2. \end{cases}$$

Therefore, in particular,

$$\begin{aligned} \mathcal{F}_q(x, 1) &= \|\varphi^{-1}\|_{\mathbb{L}_q[x-h\varphi(x)/2, x+h\varphi(x)/2]} \\ &\leq c(k, q) \begin{cases} \varphi^{-1+2/q}(x - h\varphi(x)/2) + \varphi^{-1+2/q}(x + h\varphi(x)/2) & \text{if } q > 2, \\ 1 & \text{if } q < 2. \end{cases} \end{aligned}$$

Hence, (4.15) is verified if  $k = 1$  and  $q < 2$  ( $p > 2$ ), and for  $q > 2$  ( $1 \leq p < 2$ ), we have

$$\begin{aligned} \int_{\mathcal{E}_h} (\mathcal{F}_q(x, 1))^p dx &\leq c \int_{\mathcal{D}_h} \left( \varphi^{p-2}(x - h\varphi(x)/2) + \varphi^{p-2}(x + h\varphi(x)/2) \right) dx \\ &\leq c \int_{\mathcal{D}_h} \varphi^{p-2}(x + h\varphi(x)/2) dx \\ &\leq c \int_{-1}^1 \varphi^{p-2}(v) dv \leq c. \end{aligned}$$

Finally, if  $k = 1$  and  $p = q = 2$ , then we observe that, for a centrally symmetric set  $S \subset \mathbb{R}^2$ , we have  $\iint_S f(-\bar{x}) d\bar{x} = \iint_S f(\bar{x}) d\bar{x}$ . Hence,

$$\begin{aligned} \int_{\mathcal{E}_h} (\mathcal{F}_2(x, 1))^2 dx &= \int_{\mathcal{E}_h} \int_{x-h\varphi(x)/2}^{x+h\varphi(x)/2} \frac{1}{1-t^2} dt dx \\ &\leq c \int_{\mathcal{D}_h} \int_{x-h\varphi(x)/2}^{x+h\varphi(x)/2} \frac{1}{1-t} dt dx \\ &\leq c - c \int_{\mathcal{D}_h} \ln(1 - x - h\varphi(x)/2) dx \\ &\leq c - c \int_{-1}^1 \ln(1 - v) dv \leq c, \end{aligned}$$

and so (4.15) is verified for all  $1 \leq p < \infty$  if  $k = 1$ .

If  $1 \leq p < \infty$  and  $k \geq 2$ , then  $kq > 2$ , and so

$$\begin{aligned} \mathcal{F}_q(x, k) &\leq c \int_{-h\varphi(x)/2}^{h\varphi(x)/2} \dots \int_{-h\varphi(x)/2}^{h\varphi(x)/2} \left( \varphi^{-k+2/q}(x + u_1 + \dots + u_{k-1} - h\varphi(x)/2) \right. \\ &\quad \left. + \varphi^{-k+2/q}(x + u_1 + \dots + u_{k-1} + h\varphi(x)/2) \right) du_1 \dots du_{k-1}. \end{aligned}$$

Again, let  $F \in C[-1, 1]$  be such that  $F^{(k-1)}(x) = \varphi^{-k+2/q}(x)$  (this is possible provided  $k + 2/q > 2$ ). Applying the identity (4.1), we have

$$\mathcal{F}_q(x, k) \leq c\Delta_{h\varphi(x)}^{k-1}(F, x - h\varphi(x)/2) + c\Delta_{h\varphi(x)}^{k-1}(F, x + h\varphi(x)/2) \leq c.$$

This implies (4.15) in all remaining cases except for  $k = 2$  and  $p = 1$  ( $q = \infty$ ). Finally, twice using the fact that  $\{(x, u) \mid x \in \mathcal{D}_{2h}, |u| \leq h\varphi(x)/2\}$  is centrally symmetric, we have

$$\begin{aligned} &\int_{\mathcal{E}_{2h}} \mathcal{F}_\infty(x, 2) dx \\ &\leq c \int_{\mathcal{E}_{2h}} \int_{-h\varphi(x)/2}^{h\varphi(x)/2} \left( \varphi^{-2}(x + u - h\varphi(x)/2) + \varphi^{-2}(x + u + h\varphi(x)/2) \right) dudx \\ &\leq c \int_{\mathcal{D}_{2h}} \int_{-h\varphi(x)/2}^{h\varphi(x)/2} \varphi^{-2}(x + u + h\varphi(x)/2) dudx \\ &\leq c \int_{\mathcal{D}_{2h}} \int_{-h\varphi(x)/2}^{h\varphi(x)/2} (1 - x - u - h\varphi(x)/2)^{-1} dudx \\ &= c \int_{\mathcal{D}_{2h}} (\ln(1 - x) - \ln(1 - x - h\varphi(x))) dx \\ &\leq c \int_{-1}^1 |\ln(1 - x)| dx \leq c. \end{aligned}$$

This completes the proof of the lemma. □

### 5 Weighted DT Moduli

The following weighted DT moduli are defined in [3, p. 218] (with  $D = (0, 1)$ ):

$$\begin{aligned} \omega_\psi^k(f, t)_{w,p} &:= \sup_{0 < h \leq t} \left\| w \Delta_{h\psi}^k f \right\|_{\mathbb{L}_p[t_0^*, 1-t_1^*]} \\ &\quad + \sup_{0 < h \leq t_0^*} \left\| w \overrightarrow{\Delta}_h^k f \right\|_{\mathbb{L}_p[0, 12t_0^*]} + \sup_{0 < h \leq t_1^*} \left\| w \overleftarrow{\Delta}_h^k f \right\|_{\mathbb{L}_p[1-12t_1^*, 1]}, \end{aligned}$$

where if  $\psi(x) = \sqrt{x(1-x)}$ , then  $t_0^* = t_1^* = k^2 t^2$ .

It was shown in [3, Theorem 6.1.1] that, under certain restrictions on  $\psi$  and  $w$ ,  $\omega_{\psi}^k(f, t)_{w,p}$  is equivalent to the following weighted  $K$ -functional  $K_{k,\psi}(f, t^k)_{w,p}$ :

$$K_{k,\psi}(f, t^k)_{w,p} := \inf_{g^{(k-1)} \in \text{AC}_{\text{loc}}} \left( \|(f - g)w\|_{\mathbb{L}_p(D)} + t^k \|w\psi^k g^{(k)}\|_{\mathbb{L}_p(D)} \right).$$

In particular, with obvious modifications for  $(-1, 1)$  instead of  $D = (0, 1)$ ,  $\psi := \varphi$  and  $w := \varphi^r$ , we have

$$\begin{aligned} \omega_{\varphi}^k(f, t)_{\varphi^r,p} &= \sup_{0 < h \leq t} \left\| \varphi^r \Delta_{h\varphi}^k f \right\|_{\mathbb{L}_p[-1+t^*, 1-t^*]} \\ &+ \sup_{0 < h \leq t^*} \left\| \varphi^r \overrightarrow{\Delta}_{h\varphi}^k f \right\|_{\mathbb{L}_p[-1, -1+At^*]} + \sup_{0 < h \leq t^*} \left\| \varphi^r \overleftarrow{\Delta}_{h\varphi}^k f \right\|_{\mathbb{L}_p[1-At^*, 1]}, \end{aligned} \tag{5.1}$$

where  $t^* := 2k^2t^2$  and  $A$  is an absolute constant (for example,  $A = 12$  as in [3]), and note that it is readily seen that the  $K$ -functional defined in Definition 2.6 satisfies

$$K_{k,r}^{\varphi}(f, t^k)_p = K_{k,\varphi}(f, t^k)_{\varphi^r,p} = \inf_{g^{(k-1)} \in \text{AC}_{\text{loc}}} \left( \|(f - g)\varphi^r\|_p + t^k \|\varphi^{k+r} g^{(k)}\|_p \right).$$

It follows from [3, Theorem 6.1.1] that

$$M^{-1} \omega_{\varphi}^k(f, t)_{\varphi^r,p} \leq K_{k,r}^{\varphi}(f, t^k)_p \leq M \omega_{\varphi}^k(f, t)_{\varphi^r,p} \tag{5.2}$$

for some  $M > 1$  and  $0 < t \leq t_0$ .

A similar quantity to the following averaged modulus was considered in [3, (6.1.9)] (recall that  $t^* := 2k^2t^2$ ):

$$\begin{aligned} \omega_{\varphi}^{*k}(f, t)_{w,p} &= \left( \frac{1}{t} \int_0^t \int_{-1+t^*}^{1-t^*} |w(x) \Delta_{\tau\varphi(x)}^k(f, x)|^p \, dx \, d\tau \right)^{1/p} \\ &+ \left( \frac{1}{t^*} \int_0^{t^*} \int_{-1}^{-1+At^*} |w(x) \overrightarrow{\Delta}_u^k(f, x)|^p \, dx \, du \right)^{1/p} \\ &+ \left( \frac{1}{t^*} \int_0^{t^*} \int_{1-At^*}^1 |w(x) \overleftarrow{\Delta}_u^k(f, x)|^p \, dx \, du \right)^{1/p}, \end{aligned} \tag{5.3}$$

where  $1 \leq p < \infty$ .

Also, from the statement in [3, p. 57], we conclude that, for sufficiently small  $t > 0$ ,

$$K_{k,r}^{\varphi}(f, t^k)_p \leq M_1 \omega_{\varphi}^{*k}(f, t)_{\varphi^r,p}. \tag{5.4}$$

### 6 Proof of Theorem 2.7: The Lower Estimate

We will apply (5.4) (for  $1 \leq p < \infty$ ) and the second inequality in (5.2) (for  $p = \infty$ ) in order to complete the proof of the lower estimate in Theorem 2.7.

**Lemma 6.1** *Let  $k \in \mathbb{N}$ ,  $r \in \mathbb{N}_0$ ,  $1 \leq p < \infty$ , and  $f \in \mathbb{B}'_p$ . Then,*

$$\omega_{\varphi}^{*k}(f^{(r)}, t)_{\varphi^r, p} \leq c(k, r)\omega_{k, r}^{*\varphi}(f^{(r)}, c(k)t)_p, \quad 0 < t \leq c(k).$$

*Proof* We estimate each of the three terms in the definition (5.3) separately.

First, recall that  $t^* := 2k^2t^2$  and note that  $[-1 + t^*, 1 - t^*] \subset \mathfrak{D}_{2kt}$ , and so using Proposition 3.1(iii), we have

$$\begin{aligned} & \frac{1}{t} \int_0^t \int_{-1+t^*}^{1-t^*} |\varphi^r(x) \Delta_{\tau\varphi(x)}^k(f^{(r)}, x)|^p dx d\tau \\ & \leq \frac{2^{rp}}{t} \int_0^t \int_{-1+t^*}^{1-t^*} |\mathcal{W}_{k\tau}^r(x) \Delta_{\tau\varphi(x)}^k(f^{(r)}, x)|^p dx d\tau \\ & \leq \frac{2^{rp}}{t} \int_0^t \int_{\mathfrak{D}_{2kt}} |\mathcal{W}_{k\tau}^r(x) \Delta_{\tau\varphi(x)}^k(f^{(r)}, x)|^p dx d\tau \\ & \leq 2^{rp} \omega_{k, r}^{*\varphi}(f^{(r)}, t)_p^p. \end{aligned} \tag{6.1}$$

We now estimate the second term (dealing with the function near  $-1$ ), the third term being similar.

If  $t$  is sufficiently small (for example,  $t \leq (2k\sqrt{A+k/2})^{-1}$  will do), then

$$\begin{aligned} & \frac{1}{t^*} \int_0^{t^*} \int_{-1}^{-1+At^*} |\varphi^r(x) \overrightarrow{\Delta}_u^k(f^{(r)}, x)|^p dx du \\ & = \frac{1}{t^*} \int_0^{t^*} \int_{-1}^{-1+At^*} |\varphi^r(x) \Delta_u^k(f^{(r)}, x + ku/2)|^p dx du \\ & \leq \frac{1}{t^*} \int_0^{t^*} \int_{-1+ku/2}^{-1+(A+k/2)t^*} |\varphi^r(y - ku/2) \Delta_u^k(f^{(r)}, y)|^p dy du \\ & \leq \frac{1}{t^*} \int_{-1}^{-1+(A+k/2)t^*} \int_0^{2(y+1)/k} |\varphi^r(y - ku/2) \Delta_u^k(f^{(r)}, y)|^p du dy \\ & = \frac{1}{t^*} \int_{-1}^{-1+(A+k/2)t^*} \int_0^{2(y+1)/(k\varphi(y))} \varphi(y) |\varphi^r(y - kh\varphi(y)/2) \Delta_{h\varphi(y)}^k(f^{(r)}, y)|^p dh dy \\ & \leq c \frac{1}{t^*} \int_{-1}^{-1+(A+k/2)t^*} \int_0^{2(y+1)/(k\varphi(y))} \varphi(y) |\mathcal{W}_{kh}^r(y) \Delta_{h\varphi(y)}^k(f^{(r)}, y)|^p dh dy \\ & \leq c \frac{1}{\sqrt{t^*}} \int_{-1}^{-1+(A+k/2)t^*} \int_0^{2(y+1)/(k\varphi(y))} |\mathcal{W}_{kh}^r(y) \Delta_{h\varphi(y)}^k(f^{(r)}, y)|^p dh dy \\ & \leq c \frac{1}{\sqrt{t^*}} \int_0^{c\sqrt{t^*}} \int_{\mathfrak{D}_{kh} \cap [-1, -1+(A+k/2)t^*]} |\mathcal{W}_{kh}^r(y) \Delta_{h\varphi(y)}^k(f^{(r)}, y)|^p dy dh \end{aligned}$$

$$\leq c\omega_{k,r}^{*\varphi}(f^{(r)}, c(k)t)_p^p,$$

where, for the third inequality, we used the fact that  $\varphi(y - kh\varphi(y)/2) \leq \sqrt{2}\mathcal{W}_{kh}(y)$  if  $0 \leq h \leq 2(y + 1)/(k\varphi(y))$  and  $y \leq -1/2$ .  $\square$

**Lemma 6.2** *Let  $k \in \mathbb{N}$ ,  $r \in \mathbb{N}_0$ , and  $f \in \mathbb{B}'_\infty$ . Then,*

$$\omega_\varphi^k(f^{(r)}, t)_{\varphi^r, \infty} \leq c(k, r)\omega_{k,r}^\varphi(f^{(r)}, c(k)t)_\infty, \quad 0 < t \leq c(k).$$

*Proof* The proof is very similar to that of Lemma 6.1. First, recalling that  $t^* := 2k^2t^2$ , noting that  $[-1 + t^*, 1 - t^*] \subset \mathfrak{D}_{2kt}$ , and using Proposition 3.1(iii), we have

$$\sup_{0 < h \leq t} \left\| \varphi^r(\cdot) \Delta_{h\varphi(\cdot)}^k(f^{(r)}, \cdot) \right\|_{\mathbb{L}_\infty[-1+t^*, 1-t^*]} \leq 2^r \omega_{k,r}^\varphi(f^{(r)}, t)_\infty.$$

If  $0 < h \leq t^*$ , then

$$\begin{aligned} & \left\| \varphi^r(\cdot) \overrightarrow{\Delta}_h^k(f^{(r)}, \cdot) \right\|_{\mathbb{L}_\infty[-1, -1+At^*]} \\ &= \sup_{x \in [-1, -1+At^*]} \left| \varphi^r(x) \Delta_h^k(f^{(r)}, x + kh/2) \right| \\ &\leq \sup_{y \in [-1+kh/2, -1+(A+k/2)t^*]} \left| \varphi^r(y - kh/2) \Delta_h^k(f^{(r)}, y) \right|. \end{aligned}$$

Hence, if  $t$  is sufficiently small ( $t \leq (2k\sqrt{A + k/2})^{-1}$  will do), then

$$\begin{aligned} & \sup_{0 < h \leq t^*} \left\| \varphi^r(x) \overrightarrow{\Delta}_h^k(f^{(r)}, x) \right\|_{\mathbb{L}_\infty[-1, -1+At^*]} \\ &\leq \sup_{y \in [-1, -1+(A+k/2)t^*]} \sup_{0 < h \leq 2(y+1)/k} \left| \varphi^r(y - kh/2) \Delta_h^k(f^{(r)}, y) \right| \\ &= \sup_{y \in [-1, -1+(A+k/2)t^*]} \sup_{0 < h \leq 2(y+1)/(k\varphi(y))} \left| \varphi^r(y - kh\varphi(y)/2) \Delta_{h\varphi(y)}^k(f^{(r)}, y) \right| \\ &\leq 2^{r/2} \sup_{y \in [-1, -1+(A+k/2)t^*]} \sup_{0 < h \leq 2(y+1)/(k\varphi(y))} \left| \mathcal{W}_{kh}^r(y) \Delta_{h\varphi(y)}^k(f^{(r)}, y) \right| \\ &\leq 2^{r/2} \sup_{0 < h \leq c(k)t} \sup_{y \in \mathfrak{D}_{kh} \cap [-1, -1+(A+k/2)t^*]} \left| \mathcal{W}_{kh}^r(y) \Delta_{h\varphi(y)}^k(f^{(r)}, y) \right| \\ &\leq 2^{r/2} \omega_{k,r}^\varphi(f^{(r)}, c(k)t)_\infty. \end{aligned}$$

The estimate of  $\sup_{0 < h \leq t^*} \left\| \varphi^r(\cdot) \overleftarrow{\Delta}_h^k(f^{(r)}, \cdot) \right\|_{\mathbb{L}_\infty[1-At^*, 1]}$  is similar.  $\square$

We are now ready to complete the proof of the lower estimate in Theorem 2.7. First, estimates (5.2) and (5.4) together with Lemmas 6.1 and 6.2 imply that, for  $f \in \mathbb{B}'_p$ ,  $1 \leq p \leq \infty$ ,

$$K_{k,r}^\varphi(f^{(r)}, t^k)_p \leq c\omega_{k,r}^{*\varphi}(f^{(r)}, c_1t)_p, \quad 0 < t \leq c_2, \tag{6.2}$$

where  $c_1 = c_1(k)$  and  $c_2 = c_2(k)$  are some positive constants, which we now consider fixed.

Now, suppose that  $0 < t \leq 2/k$ , and let  $\mu := \max\{1, c_1, 2/(kc_2)\}$ . Then, since  $t/\mu \leq c_2$ , taking into account (2.3), we have

$$\begin{aligned} K_{k,r}^\varphi(f^{(r)}, t^k)_p &\leq \mu^k K_{k,r}^\varphi(f^{(r)}, (t/\mu)^k)_p \leq c\omega_{k,r}^{*\varphi}(f^{(r)}, c_1t/\mu)_p \\ &\leq c(\mu/c_1)^{1/p} \omega_{k,r}^{*\varphi}(f^{(r)}, t)_p, \end{aligned}$$

which completes the proof of the lower estimate in Theorem 2.7.

### 7 Hierarchy Between Moduli

The following theorem illustrates the hierarchy between the moduli of smoothness.

**Theorem 7.1** *If  $f \in \mathbb{B}_p^{r+1}$ ,  $1 \leq p \leq \infty$ ,  $r \in \mathbb{N}_0$ , and  $k \geq 2$ , then*

$$\omega_{k,r}^\varphi(f^{(r)}, t)_p \leq ct\omega_{k-1,r+1}^\varphi(f^{(r+1)}, t)_p.$$

*Proof* By virtue of [3, (6.2.9)], we have

$$\omega_\varphi^k(f^{(r)}, t)_{\varphi^r, p} \leq c \int_0^t (\Omega_\varphi^k(f^{(r)}, \tau)_{\varphi^r, p} / \tau) d\tau,$$

where  $\Omega_\varphi^k$  was defined in [3, (8.1.2)] as follows:

$$\Omega_\varphi^k(f, t)_{\varphi^r, p} := \sup_{0 < h \leq t} \left\| \varphi^r \Delta_h^k f \right\|_{\mathbb{L}_p[-1+2k^2h^2, -1+2k^2h^2]}.$$

Also, by [3, (6.3.2)], we obtain

$$\Omega_\varphi^k(f^{(r)}, t)_{\varphi^r, p} \leq ct\Omega_\varphi^{k-1}(f^{(r+1)}, t)_{\varphi^{r+1}, p}.$$

Hence,

$$\begin{aligned} \omega_\varphi^k(f^{(r)}, t)_{\varphi^r, p} &\leq c \int_0^t \Omega_\varphi^{k-1}(f^{(r+1)}, \tau)_{\varphi^{r+1}, p} d\tau \\ &\leq ct\Omega_\varphi^{k-1}(f^{(r+1)}, t)_{\varphi^{r+1}, p} \leq ct\omega_\varphi^{k-1}(f^{(r+1)}, t)_{\varphi^{r+1}, p}, \end{aligned}$$

where for the second inequality, we used the monotonicity of  $\Omega_\varphi^{k-1}(f^{(r+1)}, t)_{\varphi^{r+1}, p}$ , and for the third, we applied [3, (6.2.9)].

In view of the equivalence between our and weighted DT moduli (see Remark 2.9), our proof is complete. □



We also have the other usual hierarchy which follows by [3, Theorem 6.1.4].

**Theorem 7.2** *If  $f \in \mathbb{B}_p^r$ ,  $1 \leq p \leq \infty$ ,  $r \in \mathbb{N}_0$ , and  $k \geq 2$ , then*

$$\omega_{k,r}^\varphi(f^{(r)}, t)_p \leq c\omega_{k-1,r}^\varphi(f^{(r)}, t)_p.$$

### 8 Polynomial Approximation: Direct Results

This section is devoted to the approximation of functions  $f \in L_p[-1, 1]$ ,  $1 \leq p \leq \infty$ , by polynomials of degree  $< n$ . Let  $\mathcal{P}_n$  be the set of polynomials of degree  $< n$ , and denote by

$$E_n(f)_p = \inf_{p_n \in \mathcal{P}_n} \|f - p_n\|_p$$

the degree of approximation of  $f \in L_p[-1, 1]$  by elements of  $\mathcal{P}_n$ .

An immediate application of Theorem 7.1, together with [3, Theorem 7.2.1], is the following.

**Theorem 8.1** *If  $f \in \mathbb{B}_p^r$ ,  $1 \leq p \leq \infty$ , then*

$$E_n(f)_p \leq \frac{c}{n^r} \omega_{k,r}^\varphi(f^{(r)}, 1/n)_p, \quad n \geq k + r. \tag{8.1}$$

*Proof* It follows from [3, Theorem 7.2.1] that

$$E_n(f)_p \leq c\omega_{k+r}^\varphi(f, 1/n)_p, \quad n \geq k + r.$$

Since  $f \in \mathbb{B}_p^r$ , we apply Theorem 7.1  $r$  times and (8.1) follows. □

An immediate consequence of Theorem 8.1 is the following direct estimate.

**Corollary 8.2** *If  $f \in \mathbb{B}_p^r$ ,  $r \in \mathbb{N}_0$ ,  $1 \leq p \leq \infty$ , and if for some  $k \in \mathbb{N}$  and  $\alpha > r$ ,  $\omega_{k,r}^\varphi(f^{(r)}, t)_p = O(t^{\alpha-r})$ , then*

$$E_n(f)_p \leq cn^{-\alpha}, \quad n \geq k + r. \tag{8.2}$$

It is interesting to compare (8.1) with estimates of how well  $P_n^{(r)}$  approximates  $f^{(r)}$ . Our result here is the following.

**Theorem 8.3** *Let  $f \in \mathbb{B}_p^r$ ,  $r \in \mathbb{N}$ ,  $1 \leq p \leq \infty$ , and let  $P_n$  denotes the polynomial of best approximation of  $f$  in  $L_p[-1, 1]$ , of degree  $< n$ . If  $\int_0^1 (\omega_{k,r}^\varphi(f^{(r)}, \tau)_p / \tau) d\tau < \infty$  for some  $k \in \mathbb{N}$ , then*

$$\|(f^{(r)} - P_n^{(r)})\varphi^r\|_p \leq c \int_0^{1/n} (\omega_{k,r}^\varphi(f^{(r)}, \tau)_p / \tau) d\tau.$$

*Proof* Using Potapov’s estimate (see, e.g., [3, (7.2.7)])

$$\|\varphi^{\nu} P_n^{(\nu)}\|_p \leq c(p, \nu)n^{\nu} \|P_n\|_p, \tag{8.3}$$

we have

$$\begin{aligned} \|(f^{(r)} - P_n^{(r)})\varphi^r\|_p &\leq \sum_{j=1}^{\infty} \|(P_{2^j n}^{(r)} - P_{2^{j-1} n}^{(r)})\varphi^r\|_p \\ &\leq c \sum_{j=1}^{\infty} 2^{jr} n^r \|P_{2^j n} - P_{2^{j-1} n}\|_p \\ &\leq c \sum_{j=1}^{\infty} 2^{jr} n^r (\|P_{2^j n} - f\|_p + \|f - P_{2^{j-1} n}\|_p) \\ &\leq c \sum_{j=1}^{\infty} \omega_{k,r}^{\varphi}(f^{(r)}, 1/(2^j n))_p \\ &\leq c \sum_{j=1}^{\infty} \int_{1/(2^j n)}^{1/(2^{j-1} n)} (\omega_{k,r}^{\varphi}(f^{(r)}, \tau)_p / \tau) d\tau \\ &\leq c \int_0^{1/n} (\omega_{k,r}^{\varphi}(f^{(r)}, \tau)_p / \tau) d\tau, \end{aligned}$$

where for the first inequality, we used the fact that  $\|(f^{(r)} - P_{2^j n}^{(r)})\varphi^r\|_p \rightarrow 0$  (see details in the proof of Theorem 9.1 below), and for the fourth inequality, we used Theorem 8.1 and Corollary 2.10. □

### 9 Polynomial Approximation: Inverse Theorems

Denote by  $\Phi$  the set of nondecreasing functions  $\phi : [0, 1] \rightarrow [0, \infty)$  satisfying  $\phi(0+) = 0$ . Recalling that  $E_n(f)_p$  is the degree of approximation of  $f$  by polynomials of degree  $< n$ , we have the following inverse theorem.

**Theorem 9.1** *Let  $1 \leq p \leq \infty$ ,  $k \in \mathbb{N}$ ,  $r \in \mathbb{N}_0$ ,  $N \in \mathbb{N}$ , and  $\phi \in \Phi$  such that*

$$\int_0^1 \frac{r\phi(u)}{u^{r+1}} du < +\infty$$

(i.e., if  $r = 0$ , this condition is not needed). If

$$E_n(f)_p \leq \phi\left(\frac{1}{n}\right) \text{ for all } n \geq N,$$

then one of the representatives of  $f$  has a locally absolutely continuous derivative  $f^{(r-1)}$ ,  $f \in \mathbb{B}_p^r$ , and

$$\omega_{k,r}^\varphi(f^{(r)}, t)_p \leq c \int_0^t \frac{r\phi(u)}{u^{r+1}} du + ct^k \int_t^1 \frac{\phi(u)}{u^{k+r+1}} du + c(N)t^k E_{k+r}(f)_p, \quad t \in [0, 1/2].$$

If, in addition,  $N \leq k + r$ , then

$$\omega_{k,r}^\varphi(f^{(r)}, t)_p \leq c \int_0^t \frac{r\phi(u)}{u^{r+1}} du + ct^k \int_t^1 \frac{\phi(u)}{u^{k+r+1}} du, \quad t \in [0, 1/2].$$

*Remark 9.2* For  $p = \infty$ , this theorem was proved in [5].

*Remark 9.3* A classical restatement of Theorem 9.1 is Theorem 9.1' below. We prefer the current (integral) version since we find it rather more convenient to use.

*Proof* Since the theorem was proved in [5] for  $p = \infty$ , we may assume that  $1 \leq p < \infty$ .

We first give the proof for the case  $r \geq 1$ . Without any loss of generality, assume that  $N \geq k + r$ . Set  $m_j := N2^j$  and  $\phi_j := \phi(m_j^{-1})$ . We represent  $f$  as the telescopic series (converging to  $f$  in  $\mathbb{L}_p$ )

$$f = P_{k+r} + (P_N - P_{k+r}) + \sum_{j=0}^\infty (P_{m_{j+1}} - P_{m_j}) =: P_{k+r} + Q + \sum_{j=0}^\infty Q_j, \quad (9.1)$$

where  $P_n \in \mathbb{P}_n$  are the polynomials of best approximation of  $f$ ; that is,  $\|f - P_n\|_p = E_n(f)_p$ . Hence, the polynomials  $Q_j$  are of degree  $< m_{j+1}$  and satisfy  $\|Q_j\|_p \leq \phi_{j+1} + \phi_j \leq 2\phi_j$ .

As in the proof of Lemma 4.1, for each  $0 \leq i \leq k$ , set  $y_i(x) := x + (i - k/2)h\varphi(x)$  and recall that  $\mathcal{W}_\delta(x) \leq \varphi(y)$  for all  $x \in \mathfrak{D}_\delta$ ,  $y \in [x - \delta\varphi(x)/2, x + \delta\varphi(x)/2]$ , and  $0 < \delta \leq 2$ .

Then, using Proposition 3.1(v) and (8.3), we have

$$\begin{aligned} \|\mathcal{W}_{kh}^r \Delta_{h\varphi}^k(Q_j^{(r)}, \cdot)\|_p &\leq \left\| \sum_{i=0}^k \binom{k}{i} \varphi^r(y_i) \left| Q_j^{(r)}(y_i) \right| \right\|_{\mathbb{L}_p(\mathfrak{D}_{kh})} \leq 2^{k+1/p} \|\varphi^r Q_j^{(r)}\|_p \\ &\leq cm_{j+1}^r \|Q_j\|_p \leq cm_j^r \phi_j. \end{aligned}$$

Therefore, if we define  $J := \min\{j : 1/m_j \leq t\}$ , then we have

$$\begin{aligned} \omega_{k,r}^\varphi \left( \sum_{j=J+1}^\infty Q_j^{(r)}, t \right)_p &\leq c \sum_{j=J+1}^\infty m_j^r \phi_j \leq c \sum_{j=J+1}^\infty \int_{m_j^{-1}}^{m_{j-1}^{-1}} \frac{\phi_j}{u^{r+1}} du \\ &\leq c \sum_{j=J+1}^\infty \int_{m_j^{-1}}^{m_{j-1}^{-1}} \frac{\phi(u)}{u^{r+1}} du \leq c \int_0^{m_J^{-1}} \frac{\phi(u)}{u^{r+1}} du \\ &\leq c \int_0^t \frac{\phi(u)}{u^{r+1}} du. \end{aligned} \tag{9.2}$$

We also note that, in a similar fashion, (8.3) implies that

$$\sum_{j=0}^\infty \|\varphi^\nu Q_j^{(\nu)}\|_p \leq c \sum_{j=0}^\infty m_j^\nu \phi_j \leq c \int_0^1 \frac{\phi(u)}{u^{\nu+1}} du < \infty$$

for all  $1 \leq \nu \leq r$ . This implies that  $\{\varphi^\nu P_m^{(\nu)}\}$  is a Cauchy sequence in  $\mathbb{L}_p$  converging to some function  $\varphi^\nu f_\nu \in \mathbb{L}_p$ , and there is a subsequence of this sequence which converges pointwise almost everywhere to  $\varphi^\nu f_\nu$ . Moreover, we conclude that there exists a sequence  $\{n_l\} \subset \mathbb{N}$  such that, for each  $1 \leq \nu \leq r$ ,  $\{P_{n_l}^{(\nu)}\}$  converges pointwise almost everywhere to  $f_\nu$  and

$$\lim_{l \rightarrow \infty} \|\varphi^\nu (f_\nu - P_{n_l}^{(\nu)})\|_p = 0. \tag{9.3}$$

Now, considering  $S := [-1 + \varepsilon, 1 - \varepsilon]$ ,  $\varepsilon > 0$ , writing  $f_0 := f$ , and using the argument [7, section 6.1.3], we write, for  $x_0$  which is one of the points of convergence for all  $1 \leq \nu \leq r$ ,

$$\begin{aligned} f_{\nu-1}(x) - f_{\nu-1}(x_0) - \int_{x_0}^x f_\nu(t) dt \\ = f_{\nu-1}(x) - P_{n_l}^{(\nu-1)}(x) - \left( f_{\nu-1}(x_0) - P_{n_l}^{(\nu-1)}(x_0) \right) - \int_{x_0}^x \left( f_\nu(t) - P_{n_l}^{(\nu)}(t) \right) dt \end{aligned}$$

and conclude that

$$f_{\nu-1}(x) - f_{\nu-1}(x_0) = \int_{x_0}^x f_\nu(t) dt$$

for almost all  $x \in S$  and all  $1 \leq \nu \leq r$ . Hence, recalling that  $f_0 = f$ , it follows that almost everywhere,  $f(x)$  is identical with a function possessing an absolutely continuous derivative of order  $(r - 1)$  and  $f^{(r)} \in \mathbb{L}_p(S)$ . Hence, differentiation of (9.1) is justified. Also, (9.3) implies that  $f \in \mathbb{B}_p^r$ .

We now continue with our estimates, and using (8.3) with  $\nu = r + k$ , we have

$$\|\varphi^{r+k} Q_j^{(r+k)}\|_p \leq cm_{j+1}^{r+k} \|Q_j\|_p \leq cm_j^{r+k} \phi_j.$$

Hence, for  $0 \leq j \leq J$ , taking into account that  $1/m_j > t/2$  and setting  $m_{-1} := N/2$ , we have

$$\begin{aligned} & \omega_{k,r}^\varphi \left( \sum_{j=0}^J Q_j^{(r)}, t \right)_p \\ & \leq ct^k \sum_{j=0}^J \|\varphi^{r+k} Q_j^{(r+k)}\|_p \leq ct^k \sum_{j=0}^J m_j^{r+k} \phi_j \leq ct^k \sum_{j=0}^J \int_{m_j^{-1}}^{m_{j-1}^{-1}} \frac{\phi_j}{u^{k+r+1}} du \\ & \leq ct^k \sum_{j=0}^J \int_{m_j^{-1}}^{m_{j-1}^{-1}} \frac{\phi(u)}{u^{k+r+1}} du \leq ct^k \int_{m_{-1}^{-1}}^{2/N} \frac{\phi(u)}{u^{k+r+1}} du \leq ct^k \int_{t/2}^1 \frac{\phi(u)}{u^{k+r+1}} du \\ & \leq ct^k \int_t^1 \frac{\phi(u)}{u^{k+r+1}} du. \end{aligned} \tag{9.4}$$

Finally, we have the estimate

$$\omega_{k,r}^\varphi(Q^{(r)}, t)_p \leq ct^k \|\varphi^{k+r} Q^{(k+r)}\|_p \leq ct^k N^{r+k} \|Q\|_p \leq ct^k N^{r+k} E_{k+r}(f)_p. \tag{9.5}$$

Note that if  $N = k + r$ , then  $Q \equiv 0$ , so that the left-hand side of (9.5) vanishes and no estimate is needed.

Now, the observation that  $\Delta_{h\varphi(x)}^k(P_{k+r}^{(r)}, x) = 0$ , combined with (9.2), (9.4), and (9.5), completes the proof of the theorem for  $r \geq 1$ .

For  $r = 0$ , we write

$$f = P_k + Q + \sum_{j=0}^J Q_j + (f - P_{m_{J+1}}),$$

where  $Q := P_N - P_k$  and  $Q_j := P_{m_{j+1}} - P_{m_j}$  (see (9.1)), and complete the proof as above, just applying (9.4), (9.5), and

$$\|f - P_{m_{J+1}}\|_p = E_{m_{J+1}}(f)_p \leq \phi_{m_{J+1}}$$

(i.e., the same type of estimate as for  $\|Q_{m_{J+1}}\|_p$ ). □

Choosing

$$\phi(u) := \begin{cases} E_n(f)_p & \text{if } 1/n \leq u < 1/(n+1), \ n \geq N-1, \\ E_N(f)_p & \text{if } 1/N \leq u \leq 1, \end{cases}$$

in Theorem 9.1, we immediately get the following result, which, in fact, is equivalent to Theorem 9.1.

**Theorem 9.1'** *Let  $1 \leq p \leq \infty, k \in \mathbb{N}, r \in \mathbb{N}_0, N \in \mathbb{N}$ . If*

$$\sum_{n=1}^{\infty} rn^{r-1} E_n(f)_p < +\infty$$

(i.e., if  $r = 0$ , this condition is not needed), then one of the representatives of  $f$  has a locally absolutely continuous derivative  $f^{(r-1)}, f \in \mathbb{B}_p^r$ , and

$$\begin{aligned} \omega_{k,r}^{\varphi}(f^{(r)}, t)_p &\leq c \sum_{n>\max\{N, 1/t\}} rn^{r-1} E_n(f)_p \\ &\quad + ct^k \sum_{N \leq n \leq \max\{1/t, N\}} n^{k+r-1} E_n(f)_p \\ &\quad + c(N)t^k E_{k+r}(f)_p, \quad t \in [0, 1/2]. \end{aligned}$$

If, in addition,  $N \leq k + r$ , then

$$\begin{aligned} \omega_{k,r}^{\varphi}(f^{(r)}, t)_p &\leq c \sum_{n>\max\{N, 1/t\}} rn^{r-1} E_n(f)_p \\ &\quad + ct^k \sum_{N \leq n \leq \max\{1/t, N\}} n^{k+r-1} E_n(f)_p, \quad t \in [0, 1/2]. \end{aligned}$$

Another immediate corollary of Theorem 9.1 with  $\phi(t) := t^\alpha$  and  $N = k + r$  is the following result which is an inverse of (8.2).

**Corollary 9.4** *Let  $r \in \mathbb{N}_0, k \in \mathbb{N}$ , and  $r < \alpha < r + k$ , and let  $f \in L_p[-1, 1], 1 \leq p \leq \infty$ . If*

$$E_n(f)_p \leq n^{-\alpha}, \quad n \geq N, \tag{9.6}$$

for some  $N \geq k + r$ , then  $f \in \mathbb{B}_p^r$  and

$$\omega_{k,r}^{\varphi}(f^{(r)}, t)_p \leq c(\alpha, k, r)t^{\alpha-r} + c(N, k, r)t^k E_{k+r}(f)_p, \quad t > 0.$$

In particular, if  $N = k + r$ , then (9.6) implies that  $f \in \mathbb{B}_p^r$  and

$$\omega_{k,r}^{\varphi}(f^{(r)}, t)_p \leq c(\alpha, k, r)t^{\alpha-r}, \quad t > 0.$$

Corollaries 8.2 and 9.4 imply the following constructive characterization result.

**Corollary 9.5** *Let  $r \in \mathbb{N}_0, k \in \mathbb{N}, r < \alpha < r + k$ , and let  $f \in L_p[-1, 1], 1 \leq p \leq \infty$ . Then  $E_n(f)_p \leq cn^{-\alpha}$ , for all  $n \geq k + r$ , if and only if  $f \in \mathbb{B}_p^r$  and  $\omega_{k,r}^{\varphi}(f^{(r)}, t)_p \leq ct^{\alpha-r}, t > 0$ .*

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