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Constrained Approximation with Jacobi Weights

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Abstract. In this paper, we prove that for $\ell = 1$ or 2 the rate of best ℓ-monotone polynomial approximation in the L_p norm ($1 \le p \le \infty$) weighted by the Jacobi weight $w_{\alpha,\beta}(x) := (1+x)^\alpha(1-x)^\beta$ with $\alpha, \beta > -1/p$ if $p < \infty$, or $\alpha, \beta \ge 0$ if $p = \infty$, is bounded by an appropriate $(\ell+1)$-st modulus of smoothness with the same weight, and that this rate cannot be bounded by the $(\ell+2)$-nd modulus. Related results on constrained weighted spline approximation and applications of our estimates are also given.

1 Introduction and Main Results

One of the central topics in Approximation Theory is the investigation of the connection between the rate with which a function can be approximated and the smoothness of this function. The goal is to prove direct and matching inverse estimates in terms of the right measure of smoothness. In other words, one strives to obtain results of the following type: "a function can be approximated with a given order if and only if it belongs to a certain smoothness class". In order to describe these smoothness classes, one usually needs to introduce appropriate moduli of smoothness that correspond to the way approximation orders are measured. For example, in the case of approximation by algebraic polynomials, if orders of approximation of f are given using the $\mathbb{L}_p$ norms, then one can measure smoothness of f using either the Ivanov or the Ditzian-Totik moduli (we refer the reader to [4] for further discussion of recent developments in this area).

Corresponding problems for weighted polynomial approximation are much more complicated, especially if the weight has zeros and/or singularities inside $(-1,1)$. One cannot simply replace the usual $\mathbb{L}_p$ norm $\|\cdot\|_p$ by a corresponding weighted norm $\|\cdot\|_{w,p}$ everywhere, and there is a need to modify the definition of the moduli of smoothness near zeros/singularities of the weight (see, *e.g.*, [1,2,12] and the references therein for further details). Even the case for the classical Jacobi weights that have zeros/singularities "only" at the endpoints of $[-1,1]$ is rather involved, and requires a modification of the definition of the moduli.

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In this paper, we are interested in weighted approximation with Jacobi weights that is not only exact in the above-described sense but also preserves the shape of the functions being approximated. It is usually referred to as *shape preserving* or *constrained* approximation. Constrained approximation without weights has been relatively well studied in recent years (see, *e.g.,* our survey paper [10] for details). At the same time, we are not aware of any results on weighted polynomial approximation with constraints on a finite interval (with Jacobi or any other weights), and the purpose of this paper is to obtain the first result of this type. We prove Jackson type estimates in the $\mathbb{L}_p$, $1 \le p \le \infty$, norm weighted by the Jacobi weights in terms of the "correct" moduli of smoothness (*i.e.,* those moduli that yield matching direct and inverse estimates) in the case of monotone and convex polynomial approximation, and state several applications of our results. Some of the results may be had (with appropriate modifications) also for $0 < p < 1$, but we have limited ourselves to $1 \le p \le \infty$ in order to avoid complicated technicalities. Moreover, we have not considered the ℓ-monotone approximation with $\ell \ge 3$ in this paper, since this type of approximation is rather involved even in the unweighted case (see, *e.g.,* [10] for additional discussion). Furthermore, it is a natural question whether analogs of our results are valid in the case of approximation with more general doubling weights. However, this seems to be a rather complicated question, since the presence of internal zeros and singularities causes difficulties even in the case of approximation without any constraints (see [8, 9], for example). Still, we hope that the techniques developed in this paper and their modifications can be used to tackle more general problems on constrained weighted approximation in the future.

In order to continue our discussion, we need to introduce some notation and recall several definitions.

As usual, for a measurable $f\colon [-1,1] \mapsto \mathbb{R}$ and an interval $I \subseteq [-1,1]$, let

$$\|f\|_{\mathbb{L}_p(I)} := \left(\int_I |f(x)|^p dx \right)^{1/p}, \quad p < \infty, \quad \text{and} \quad \|f\|_{\mathbb{L}_\infty(I)} := \operatorname{ess\,sup}_{x\in I} |f(x)|.$$

For a weight function w, we also let

$$\mathbb{L}_{w,p}(I) := \left\{ f \mid \|wf\|_{\mathbb{L}_p(I)} < \infty \right\},$$

and, for $f \in \mathbb{L}_{w,p}(I)$, let

$$E_n(f, I)_{w,p} := \inf_{p_n \in \Pi_n} \|w(f - p_n)\|_{\mathbb{L}_p(I)}$$

be the degree of weighted approximation by polynomials in Π_n, the set of algebraic polynomials of degree $< n$. For $I = [-1,1]$, it is convenient to use the notation $\|f\|_p := \|f\|_{\mathbb{L}_p[-1,1]}$, $\mathbb{L}_{w,p} := \mathbb{L}_{w,p}[-1,1]$ and $E_n(f)_{w,p} := E_n(f, [-1,1])_{w,p}$.

Let

$$\Delta_h^k\big(f, x, [a,b]\big) := \begin{cases} \sum_{i=0}^k \binom{k}{i} (-1)^{k-i} f(x - kh/2 + ih) & \text{if } x \pm kh/2 \in [a,b], \\ 0 & \text{otherwise,} \end{cases}$$

be the k-th symmetric difference, $\Delta_h^k(f,x) := \Delta_h^k(f, x, [-1,1])$, and let

$$\overrightarrow{\Delta}_h^k(f,x) := \Delta_h^k(f, x + kh/2) \quad \text{and} \quad \overleftarrow{\Delta}_h^k(f,x) := \Delta_h^k(f, x - kh/2)$$

be the forward and backward k-th differences, respectively. Then the usual k-th modulus of smoothness of $f \in \mathbb{L}_p(I)$ on an interval I is defined as

$$\omega_k(f, \delta, I)_p := \sup_{0<h\le\delta} \left\| \Delta_h^k(f, \cdot, I) \right\|_{\mathbb{L}_p(I)} .$$

Now, let

$$w_{\alpha,\beta}(x) := (1+x)^\alpha (1-x)^\beta, \quad \alpha, \beta \in J_p := \begin{cases} (-1/p, \infty) & \text{if } p < \infty, \\ [0, \infty) & \text{if } p = \infty, \end{cases}$$

be the Jacobi weights, and denote $\mathbb{L}_p^{\alpha,\beta} := \mathbb{L}_{w_{\alpha,\beta},p}$.

The main part weighted modulus of smoothness (with Jacobi weights) is defined as

$$\Omega_\phi^k(f, A, \delta)_{w_{\alpha,\beta},p} := \sup_{0<h\le\delta} \left\| w_{\alpha,\beta}(\cdot) \Delta_{h\phi(\cdot)}^k \big(f, \cdot, [-1+Ah^2, 1-Ah^2]\big) \right\|_{\mathbb{L}_p[-1+Ah^2, 1-Ah^2]} ,$$

where A is a positive constant and $\phi(x) := \sqrt{1-x^2}$.

Following [1,12] (see also [5, Chapter 11]), we define

$$\omega_\phi^{\star k}(f, A, \delta)_{w_{\alpha,\beta},p} := \Omega_\phi^k(f, A, \delta)_{w_{\alpha,\beta},p} + E_k\big(f, [-1, -1+2A\delta^2]\big)_{w_{\alpha,\beta},p} + E_k\big(f, [1-2A\delta^2, 1]\big)_{w_{\alpha,\beta},p},$$

and note that $\omega_\phi^{\star k}(f, A, \delta)_{w_{\alpha,\beta},p}$ is bounded for all $f \in \mathbb{L}_p^{\alpha,\beta}$ with $\alpha, \beta \in J_p$.

If $1 \le p \le \infty$, it is possible to show (see [9, Corollary 11.1] for $1 \le p < \infty$, if $p = \infty$, the proof is analogous) that

$$\omega_\phi^{\star k}(f, A_1, \delta_1)_{w_{\alpha,\beta},p} \sim \omega_\phi^{\star k}(f, A_2, \delta_2)_{w_{\alpha,\beta},p} \text{ if } \quad A_1 \sim A_2 \quad \text{and} \quad \delta_1 \sim \delta_2, \tag{1.1}$$

where by $A \sim B$ we mean that there exist constants $0 < c_1 \le c_2$ such that $c_1 A \le B \le c_2 A$.

Here and in the sequel, the equivalence constants as well as the constants c, in general, depend on α, β, p and the order of the moduli. Constants c may be different on different occurrences even when they appear in the same line.

The weighted Ditzian–Totik (DT) modulus of smoothness (see [5, (8.1.2), (8.2.10) and Appendix B]) is defined as

$$\omega_\phi^k(f, A, \delta)_{w_{\alpha,\beta},p} := \Omega_\phi^k(f, A, \delta)_{w_{\alpha,\beta},p} + \overrightarrow{\Omega}_\phi^k(f, A, \delta)_{w,p} + \overleftarrow{\Omega}_\phi^k(f, A, \delta)_{w,p},$$

where

$$\overrightarrow{\Omega}_\phi^k(f, A, \delta)_{w_{\alpha,\beta},p} := \sup_{0<h\le 2A\delta^2} \left\| w_{\alpha,\beta} \overrightarrow{\Delta}_h^k(f) \right\|_{\mathbb{L}_p[-1, -1+2A\delta^2]}$$

and

$$\overleftarrow{\Omega}_\phi^k(f, A, \delta)_{w_{\alpha,\beta},p} := \sup_{0<h\le 2A\delta^2} \left\| w_{\alpha,\beta} \overleftarrow{\Delta}_h^k(f) \right\|_{\mathbb{L}_p[1-2A\delta^2, 1]} ,$$

where $\overrightarrow{\Delta}_h^k(f) := \overrightarrow{\Delta}_h^k(f,\cdot)$ and $\overleftarrow{\Delta}_h^k(f) := \overleftarrow{\Delta}_h^k(f,\cdot)$. If $w \equiv 1$, then $\omega_\phi^k(f,A,\delta)_{1,p}$ is equivalent to the usual DT modulus

$$\omega_\phi^k(f,\delta)_p := \sup_{0<h\le\delta} \|\Delta_{h\phi}^k(f)\|_p,$$

where $\Delta_{h\phi}^k(f) := \Delta_{h\phi(\cdot)}^k(f,\cdot)$.

We note that the moduli $\omega_\phi^k(f,\delta)_{w_{\alpha,\beta},p}$ are usually defined with the restriction $\alpha,\beta \ge 0$ for all $p \le \infty$ and not just for $p = \infty$. The reason for this is that if $\alpha < 0$ or $\beta < 0$, then there are functions f in $\mathbb{L}_p^{\alpha,\beta}$ for which $\omega_\phi^k(f,\delta)_{w_{\alpha,\beta},p} = \infty$ (see, *e.g.*, [7] for more discussion).

At the same time, if $\alpha,\beta \ge 0$, then for all $f \in \mathbb{L}_p^{\alpha,\beta}$, $1 \le p \le \infty$, $\omega_\phi^k(f,A,\delta)_{w_{\alpha,\beta},p}$ and $\omega_\phi^{\star k}(f,A,\delta)_{w_{\alpha,\beta},p}$ are equivalent. Namely, the following can be proved using the same method as in [2, Theorem 2.1 and Proposition 4.2].

Lemma 1.1 *If $k \in \mathbb{N}$, $A > 0$, $\alpha,\beta \ge 0$ and $f \in \mathbb{L}_p^{\alpha,\beta}$, $1 \le p \le \infty$, then there exists $\delta_0 > 0$ such that*

$$\omega_\phi^k(f,A,\delta)_{w_{\alpha,\beta},p} \sim \omega_\phi^{\star k}(f,A,\delta)_{w_{\alpha,\beta},p}$$

for all $0 < \delta \le \delta_0$.

The next theorem follows from [1, Theorem 3.1] (see also [12, Theorem 1.4] in the case where $p = \infty$ and [9, Theorems 5.2 and 9.1] if $1 \le p < \infty$).

Theorem 1.2 *Let $1 \le p \le \infty$, $\alpha,\beta \in J_p$, $k \in \mathbb{N}$, $A > 0$ and $f \in \mathbb{L}_p^{\alpha,\beta}$. Then*

$$E_n(f)_{w_{\alpha,\beta},p} \le c\omega_\phi^{\star k}(f,A,n^{-1})_{w_{\alpha,\beta},p}$$

and

$$\omega_\phi^{\star k}(f,A,n^{-1})_{w_{\alpha,\beta},p} \le cn^{-k}\sum_{i=1}^n i^{k-1}E_i(f)_{w_{\alpha,\beta},p},$$

where constants c depend only on k, p, α, β, and A.

Remark 1.3 With the moduli ω_ϕ^k instead of $\omega_\phi^{\star k}$ and $\alpha,\beta \ge 0$, Theorem 1.2 was proved in [11, Theorem 4].

Corollary 1.4 *Let $1 \le p \le \infty$, $\alpha,\beta \in J_p$, $k \in \mathbb{N}$, $A > 0$, and $f \in \mathbb{L}_p^{\alpha,\beta}$. Then for $0 < \gamma < k$,*

$$E_n(f)_{w_{\alpha,\beta},p} = O(n^{-\gamma}) \quad \Longleftrightarrow \quad \omega_\phi^{\star k}(f,A,\delta)_{w_{\alpha,\beta},p} = O(\delta^\gamma),\ \delta > 0.$$

Denote by Δ^ℓ the set of all ℓ-monotone functions on $(-1,1)$ (*i.e.*, $f \in \Delta^\ell$ if its ℓ-th order divided difference $[t_1,\dots,t_\ell;f] \ge 0$, for all collection of distinct points $(t_i)_{i=1}^\ell \subset (-1,1)$), where we view f as a representative of its class defined pointwise. Recall that $[t_1;f] := f(t_1)$ and for $\ell \ge 2$,

$$[t_1,\dots,t_\ell;f] := \big([t_1,\dots,t_{\ell-1};f] - [t_2,\dots,t_\ell;f]\big)/(t_1 - t_\ell).$$

In particular, note that Δ^1 and Δ^2 are the cones of all monotone and convex functions on $(-1,1)$, respectively. We also let

$$E_n^{(\ell)}(f,I)_{w,p} := \inf_{p_n\in\Pi_n\cap\Delta^\ell} \|w(f-p_n)\|_{\mathbb{L}_p(I)}, \quad E_n^{(\ell)}(f)_{w,p} := E_n^{(\ell)}\big(f,[-1,1]\big)_{w,p}.$$

The following theorem is the main result in this paper.

Theorem 1.5 *Let $\ell=1$ or $\ell=2$, $1\le p\le\infty$, $A>0$, $\alpha,\beta\in J_p$, and let $f\in\mathbb{L}_p^{\alpha,\beta}\cap\Delta^\ell$. Then*

$$E_n^{(\ell)}(f)_{w_{\alpha,\beta},p} \le c\omega_\phi^{\star(\ell+1)}(f,A,n^{-1})_{w_{\alpha,\beta},p}, \quad \text{for all } n\ge\ell+1. \tag{1.2}$$

Remark 1.6 Using exactly the same proof as in [13], the fact that all norms in finite dimensional spaces are equivalent (and so, for example, $\|P\|_\infty \sim \|w_{\alpha,\beta}P\|_\infty \sim \|w_{\alpha,\beta}P\|_p$ with equivalence constants depending on n, α, β and p), and the estimate

$$\omega_\phi^{\star k}(f,A,t)_{w_{\alpha,\beta},p} \le c\|w_{\alpha,\beta}f\|_p,$$

for $k\in\mathbb{N}$ and $A,t>0$, it is possible to show that the estimate (1.2) is exact in the sense that $\omega_\phi^{\star(\ell+1)}$ in (1.2) cannot be replaced by $\omega_\phi^{\star k}$ with $k\ge\ell+2$.

Remark 1.7 It suffices to prove Theorem 1.5 for sufficiently large n, since for small n, it immediately follows from the observation that for $f\in\mathbb{L}_p^{\alpha,\beta}\cap\Delta^\ell$, $\ell=1$ or $\ell=2$, $E_{\ell+1}^{(\ell)}(f)_{w_{\alpha,\beta},p} = E_{\ell+1}(f)_{w_{\alpha,\beta},p}$.

Corollary 1.8 *Let $\ell=1$ or $\ell=2$, $1\le p\le\infty$, $A>0$, $\alpha,\beta\ge 0$, and let $f\in\mathbb{L}_p^{\alpha,\beta}\cap\Delta^\ell$. Then*

$$E_n^{(\ell)}(f)_{w_{\alpha,\beta},p} \le c\omega_\phi^{\ell+1}(f,A,n^{-1})_{w_{\alpha,\beta},p}, \quad \text{for all } n\ge\ell+1. \tag{1.3}$$

Corollary 1.8 immediately follows from Theorem 1.5 and Lemma 1.1 for sufficiently large n (*i.e.*, $n\ge 1/\delta_0$). If $\ell+1\le n\le\lfloor 1/\delta_0\rfloor =: n_0$, then $n^{-1}\sim(n_0+1)^{-1}<\delta_0$, and using (1.1), Lemma 1.1, and monotonicity of $\omega_\phi^{\ell+1}(f,A,t)_{w_{\alpha,\beta},p}$ in t, we have

$$\begin{aligned} E_n^{(\ell)}(f)_{w_{\alpha,\beta},p} &\le c\omega_\phi^{\star(\ell+1)}(f,A,n^{-1})_{w_{\alpha,\beta},p} \le c\omega_\phi^{\star(\ell+1)}\big(f,A,(n_0+1)^{-1}\big)_{w_{\alpha,\beta},p} \\ &\le c\omega_\phi^{\ell+1}\big(f,A,(n_0+1)^{-1}\big)_{w_{\alpha,\beta},p} \le c\omega_\phi^{\ell+1}(f,A,n^{-1})_{w_{\alpha,\beta},p}, \end{aligned}$$

and so (1.3) is also verified for "small" n.

Theorems 1.2 and 1.5 imply the following result.

Corollary 1.9 *Let $\ell=1$ or $\ell=2$, $1\le p\le\infty$, $A>0$, $\alpha,\beta\in J_p$, and $f\in\mathbb{L}_p^{\alpha,\beta}\cap\Delta^\ell$. Then for $0<\gamma<\ell+1$, we have*

$$E_n^{(\ell)}(f)_{w_{\alpha,\beta},p} = O(n^{-\gamma}) \quad\Longleftrightarrow\quad \omega_\phi^{\star(\ell+1)}(f,A,\delta)_{w_{\alpha,\beta},p} = O(\delta^\gamma), \quad \delta>0.$$

Finally, we note that one additional application of our results is given in Section 7.

2 Auxiliary Results

Let $x_j := \cos(j\pi/n)$, $0 \le j \le n$, denote the Chebyshev nodes. We also put $x_j := 1$, $j < 0$ and $x_j := -1$, $j > n$, and denote $I_j := [x_j, x_{j-1}]$, $-\infty < j < \infty$ and $|I| := \operatorname{meas}(I)$. For $\nu \in \mathbb{N}_0$, it is convenient to define

$$I_j^{(\nu)} := [x_{j+\nu}, x_{j-1-\nu}] = \bigcup_{i=j-\nu}^{j+\nu} I_i.$$

Note that $I_j = I_j^{(0)}$. Finally, define

$$\psi_j(x) := \frac{|I_j|}{|x - x_j| + |I_j|}, \quad 1 \le j \le n.$$

The following lemma was essentially proved in [6] (see [6, Proposition 2, Lemmas 1 and 2, and estimates (59)]).

Lemma 2.1 *Let $n \in \mathbb{N}$, $1 \le j \le n-1$, and let $\mu \in \mathbb{N}$, $\mu \ge 10$. Then there exist constants c_0 and c, depending only on μ, and polynomials σ_j, δ_j, $\overline{\delta}_j$, R_j, and $\overline{R}_j$ of degree $\le c_0 n$ such that, for all $x \in [-1,1]$,*

$$\tag{2.1} \begin{gathered} \big|(x-x_j)_+ - \delta_j(x)\big| \le c|I_j|\psi_j^\mu(x), \qquad \big|(x-x_j)_+ - \overline{\delta}_j(x)\big| \le c|I_j|\psi_j^\mu(x), \\ \big|(x-x_j)_+ - \sigma_j(x)\big| \le c|I_j|\psi_j^\mu(x), \qquad \big|(x-x_j)_+^2 - R_j(x)\big| \le c|I_j|^2\psi_j^\mu(x), \\ \big|(x-x_j)_+^2 - \overline{R}_j(x)\big| \le c|I_j|^2\psi_j^\mu(x), \qquad \big|\chi_j(x) - \delta_j'(x)\big| \le c\psi_j^\mu(x), \\ \big|\chi_j(x) - \overline{\delta}_j'(x)\big| \le c\psi_j^\mu(x), \qquad \big|\chi_j(x)_+ - \sigma_j'(x)\big| \le c\psi_j^\mu(x), \\ \big|2(x-x_j)_+ - R_j'(x)\big| \le c|I_j|\psi_j^\mu(x), \qquad \big|2(x-x_j)_+ - \overline{R}_j'(x)\big| \le c|I_j|\psi_j^\mu(x), \end{gathered}$$

and

$$\tag{2.2} \begin{gathered} \sigma_j''(x) \ge 0, \quad -\delta_j'(x) \ge -\chi_{j+1}(x), \quad \overline{\delta}_j'(x) \ge \chi_{j-1}(x), \\ (x_{j-1} - x_j)\sigma_j''(x) - R_j''(x) \ge -2\chi_j(x), \\ (x_j - x_{j+1})\sigma_j''(x) + \overline{R}_j''(x) \ge 2\chi_j(x), \end{gathered}$$

where $\chi_j(x) := \chi_{[x_j,1]}(x)$ is the characteristic function of $[x_j,1]$ and $(x-x_j)_+^k := (x-x_j)^k\chi_{[x_j,1]}(x)$.

The following "restricted averaged main part modulus" (that we state here only for the Jacobi weights) was defined in [9]:

$$\widetilde{\Omega}_\phi^k(f,\delta)_{\mathbb{L}_p(S),w_{\alpha,\beta}} := \Big(\frac{1}{\delta}\int_0^\delta \int_S \big|w_{\alpha,\beta}(x)\Delta_{h\phi(x)}^k(f,x,S)\big|^p dx dh\Big)^{1/p},$$

where $S \subset [-1,1]$ is independent of h.

It is convenient to denote

$$\mathfrak{I}_{A,\delta} := [-1 + A\delta^2, 1 - A\delta^2].$$

The following lemma immediately follows from [9, Lemma 4.2] (with $\theta = 1$) taking into account that

$$\widetilde{\Omega}_\phi^k(f,\delta)_{\mathbb{L}_p(\mathfrak{I}_{A,\delta}),w_{\alpha,\beta}} \le \Omega_\phi^k(f,A,\delta)_{w_{\alpha,\beta},p}$$

and $\omega_k(f,|J|,J)_p \sim E_k(f,J)_{1,p} =: E_k(f,J)_p$, for $f \in \mathbb{L}_p(J)$.

Lemma 2.2 *Let $1 \le p < \infty$, $\alpha,\beta \in J_p$, $f \in \mathbb{L}_p^{\alpha,\beta}$, $n,k \in \mathbb{N}$, and let $A > 0$ be arbitrary. Denote*

$$I^* := \{1 \le i \le n \mid I_i \subset \mathfrak{I}_{A,1/n}\}, \tag{2.3}$$

and suppose that for each $i \in I^*$, the interval $\widehat{I}_i$ is such that $I_i \subset \widehat{I}_i \subset \mathfrak{I}_{A,1/n}$ and $|\widehat{I}_i| \le c_0|I_i|$. Then

$$\sum_{i\in I^*}\left[w_{\alpha,\beta}(x_i)E_k(f,\widehat{I}_i)_p\right]^p \le c\Omega_\phi^k(f,A,1/n)^p_{w_{\alpha,\beta},p},$$

where the constant c depends only on k, α, β, p, c_0, and A.

An analog of this lemma is also valid in the case $p = \infty$, and its proof is straightforward.

Lemma 2.3 *Let $\alpha,\beta \ge 0$, $f \in \mathbb{L}_\infty^{\alpha,\beta}$, $n,k \in \mathbb{N}$, and let $A > 0$ be arbitrary. Suppose that for each $i \in I^*$ (with I^* defined in (2.3)), the interval $\widehat{I}_i$ is such that $I_i \subset \widehat{I}_i \subset \mathfrak{I}_{A,1/n}$ and $|\widehat{I}_i| \le c_0|I_i|$. Then*

$$\sup_{i\in I^*} w_{\alpha,\beta}(x_i)E_k(f,\widehat{I}_i)_\infty \le c\Omega_\phi^k(f,A,1/n)_{w_{\alpha,\beta},\infty},$$

where the constant c depends only on k, α, β, c_0, and A.

Corollary 2.4 *Let $1 \le p \le \infty$, $A > 0$, $\alpha,\beta \in J_p$, $f \in \mathbb{L}_p^{\alpha,\beta}$, $k \in \mathbb{N}$, and $\nu \in \mathbb{N}_0$. Then for each $n \in \mathbb{N}$, we have*

$$\sup_{1\le i\le n} E_k(f,I_i^{(\nu)})_{w_{\alpha,\beta},\infty} \le c\omega_\phi^{\star k}(f,A,1/n)_{w_{\alpha,\beta},\infty}, \quad \text{if } p = \infty,$$
$$\sum_{i=1}^n E_k(f,I_i^{(\nu)})^p_{w_{\alpha,\beta},p} \le c\omega_\phi^{\star k}(f,A,1/n)^p_{w_{\alpha,\beta},p}, \quad \text{if } p < \infty,$$

where the constants c depend only on k, A, α, β, p and ν.

Proof In the case $p < \infty$, since $[x_{n-1},x_1] \subset \mathfrak{I}_{B,1/n}$ if $B \le B_0 := 2$, and $[x_{2\nu+1},1] \subset [1-2C/n^2,1]$ and $[-1,x_{n-2\nu-1}] \subset [-1,-1+2C/n^2]$ if $C \ge C_0 := 16(\nu+1)^2$, and taking into account that the sets $\{1,\dots,\nu+1\}$ and $\{n-\nu,\dots,n\}$ may have a non-empty intersection, we have by Lemma 2.2,

$$
\begin{aligned}
&\sum_{i=1}^{n} E_k(f, I_i^{(\nu)})_{w_{\alpha,\beta},p}^p \\
&\quad \le \Big(\sum_{i=1}^{\nu+1} + \sum_{i=\nu+2}^{n-\nu-1} + \sum_{i=n-\nu}^{n} \Big) E_k(f, I_i^{(\nu)})_{w_{\alpha,\beta},p}^p \\
&\quad \le cE_k(f, [x_{2\nu+1}, 1])_{w_{\alpha,\beta},p}^p + c \sum_{i=\nu+2}^{n-\nu-1} \Big[w_{\alpha,\beta}(x_i) E_k(f, I_i^{(\nu)})_p \Big]^p \\
&\qquad + cE_k(f, [-1, x_{n-2\nu-1}])_{w_{\alpha,\beta},p}^p \\
&\quad \le c\Omega_\phi^k(f, B_0, 1/n)_{w_{\alpha,\beta},p}^p + cE_k(f, [1 - 2C_0/n^2, 1])_{w_{\alpha,\beta},p}^p \\
&\qquad + cE_k(f, [-1, -1 + 2C_0/n^2])_{w_{\alpha,\beta},p}^p \\
&\quad \le c\omega_\phi^{\star k}(f, B_0, 1/n)_{w_{\alpha,\beta},p}^p + c\omega_\phi^{\star k}(f, C_0, 1/n)_{w_{\alpha,\beta},p}^p \\
&\quad \le c\omega_\phi^{\star k}(f, A, 1/n)_{w_{\alpha,\beta},p}^p.
\end{aligned}
$$

The case for $p = \infty$ is analogous. ■

3 Constrained Approximation by Splines

For $n, k \in \mathbb{N}$ and $r \in \mathbb{N}_0$, we write $s \in \mathbb{S}_{k,n}^r$ if $s\,|_{(x_i, x_{i-1})} \in \Pi_k$, $1 \le i \le n$, and $s \in \mathbb{C}^r[-1,1]$. It is also convenient to denote $\mathbb{S}_{k,n} := \mathbb{S}_{k,n}^{-1}$ the set of all piecewise polynomials with (possible) discontinuities at x_i, $1 \le i \le n-1$. We remark that the fact that a piecewise polynomial s is ℓ-monotone imposes smoothness on s and so, for example, $\mathbb{S}_{k,n} \cap \Delta^\ell \subset \mathbb{S}_{k,n}^{\ell-2}$, $\ell \ge 2$.

The following lemma follows immediately from (if $\ell = 2$) or can be proved similarly to (if $\ell = 1$) [3, Theorem 1.2].

Lemma 3.1 *Let $\ell = 1$ or $\ell = 2$, and $1 \le p \le \infty$. Then for every $g \in \Delta^\ell \cap \mathbb{L}_p$, $n \in \mathbb{N}$, there exists $\widetilde{S}(g) \in \mathbb{S}_{\ell+1,n}^{\ell-1} \cap \Delta^\ell$ and an absolute constant $\eta \in \mathbb{N}$ such that*

$$\big\| g - \widetilde{S}(g) \big\|_{\mathbb{L}_p(I_j)} \le cE_{\ell+1}(g, I_j^{(\eta)})_p, \quad 1 \le j \le n.$$

We will now prove the following theorem.

Theorem 3.2 *Let $\ell \in \mathbb{N}$, $\nu \in \mathbb{N}_0$, $r \in \mathbb{N}_0 \cup \{-1\}$ and $1 \le p \le \infty$. Suppose that for every $g \in \Delta^\ell \cap \mathbb{L}_p$ and $n \in \mathbb{N}$, there exists a spline $\widetilde{S}(g) \in \mathbb{S}_{\ell+1,n}^r \cap \Delta^\ell$ such that*

$$\big\| g - \widetilde{S}(g) \big\|_{\mathbb{L}_p(I_j)} \le c_0 E_{\ell+1}(g, I_j^{(\eta)})_p, \quad 1 \le j \le n.$$

Then for all $\alpha, \beta \in J_p$, $n \in \mathbb{N}$, and $f \in \mathbb{L}_p^{\alpha,\beta} \cap \Delta^\ell$, there exists $\nu \in \mathbb{N}$, depending only on η, and a spline $S \in \mathbb{S}_{\ell+1,n}^r \cap \Delta^\ell$ such that

$$\big\| w_{\alpha,\beta}(f - S) \big\|_{\mathbb{L}_p(I_j)} \le cE_{\ell+1}(f, I_j^{(\nu)})_{w_{\alpha,\beta},p}, \quad 1 \le j \le n, \tag{3.1}$$

where c depends only on α, β, η and c_0.

Theorem 3.2 and Lemma 3.1 immediately imply the following result.

Corollary 3.3 *Let $\ell = 1$ or $\ell = 2$, $1 \le p \le \infty$, $\alpha, \beta \in J_p$, $n \in \mathbb{N}$, and $f \in \mathbb{L}_p^{\alpha,\beta} \cap \Delta^\ell$. Then there exists $S \in \mathbb{S}_{\ell+1,n}^{\ell-1} \cap \Delta^\ell$ and an absolute constant $\nu \in \mathbb{N}$ such that*

$$\left\| w_{\alpha,\beta}(f - S) \right\|_{\mathbb{L}_p(I_j)} \le c E_{\ell+1}(f, I_j^{(\nu)})_{w_{\alpha,\beta},p}, \quad 1 \le j \le n,$$

where c depends only on α and β.

Proof of Theorem 3.2 First, we assume that $n \ge 2\eta + 3$.

Note that a function f from $\mathbb{L}_p^{\alpha,\beta} \cap \Delta^\ell$ does not have to belong to $\mathbb{L}_p$ (for example, a convex function $f(x) := (1+x)^{-1}$ is clearly not in $\mathbb{L}_1$, but it is in $\mathbb{L}_1^{\alpha,\beta}$ for any $\alpha > 0$ and $\beta > -1$). Hence, given $f \in \mathbb{L}_p^{\alpha,\beta} \cap \Delta^\ell$ we modify it near ± 1 so that it becomes an $\mathbb{L}_p$ function. Recalling that any function from Δ^ℓ has continuous derivatives of order i, $0 \le i \le \ell - 2$ on $(-1,1)$ and that $f^{(\ell-1)}(x\pm)$ exist for every $x \in (-1,1)$, we let

$$T_1(x) := \sum_{i=0}^{\ell-1} f^{(i)}(x_1)(x - x_1)^i / i! \quad \text{and} \quad T_{n-1}(x) := \sum_{i=0}^{\ell-1} f^{(i)}(x_{n-1})(x - x_{n-1})^i / i!,$$

where $f^{(\ell-1)}(x_1)$ and $f^{(\ell-1)}(x_{n-1})$ are understood as $f^{(\ell-1)}(x_1+)$ and $f^{(\ell-1)}(x_{n-1}-)$, respectively. In other words, T_1 and T_{n-1} are Taylor's polynomials from Π_ℓ for f at the points x_1 and x_{n-1}, respectively. We now define

$$\widetilde{f}(x) := \begin{cases} T_{n-1}(x) & \text{if } x \in I_n, \\ f(x) & \text{if } x \in [x_{n-1}, x_1], \\ T_1(x) & \text{if } x \in I_1, \end{cases}$$

and note that $f \in \Delta^\ell$ is necessarily bounded in $[x_{n-1}, x_1]$ so that $\widetilde{f} \in \mathbb{L}_p \cap \Delta^\ell$.

Suppose now that $\widetilde{S} := \widetilde{S}(\widetilde{f}) \in \mathbb{S}_{\ell+1,n}^r \cap \Delta^\ell$ is such that

$$\|\widetilde{f} - \widetilde{S}\|_{\mathbb{L}_p(I_j)} \le c_0 E_{\ell+1}(\widetilde{f}, I_j^{(\eta)})_p, \quad 1 \le j \le n. \tag{3.2}$$

For $j = \eta + 2$ and $j = n - 1 - \eta$, let $\widetilde{s}_j$ be the polynomials of degree $< \ell + 1$ defined on $[-1,1]$ such that $\widetilde{s}_j|_{I_j} = \widetilde{S}|_{I_j}$. Observing that $I_j^{(\eta)} \subset [x_{n-1}, x_1]$ if $\eta + 2 \le j \le n - 1 - \eta$, we define

$$S(x) := \begin{cases} \widetilde{s}_{n-1-\eta}(x) & \text{if } x \in [-1, x_{n-1-\eta}], \\ \widetilde{S}(x) & \text{if } x \in [x_{n-1-\eta}, x_{\eta+1}], \\ \widetilde{s}_{\eta+2}(x) & \text{if } x \in [x_{\eta+1}, 1]. \end{cases}$$

Evidently, $S \in \mathbb{S}_{\ell+1,n}^r \cap \Delta^\ell$.

We now note that

$$w_{\alpha,\beta}(x) \sim w_{\alpha,\beta}(x_j) \quad \forall x \in I_j^{(\eta)}, \quad \text{if} \quad I_j^{(\eta)} \subset [x_{n-1}, x_1]. \tag{3.3}$$

Note that this holds for $\eta + 2 \le j \le n - 1 - \eta$. Therefore, for all such η and j,

$$\|w_{\alpha,\beta} g\|_{\mathbb{L}_p(I_j^{(\eta)})} \sim w_{\alpha,\beta}(x_j) \, \|g\|_{\mathbb{L}_p(I_j^{(\eta)})}$$

and

$$E_{\ell+1}(f, I_j^{(\eta)})_{w_{\alpha,\beta},p} \sim w_{\alpha,\beta}(x_j) E_{\ell+1}(f, I_j^{(\eta)})_p.$$

Hence, (3.2) implies that for $\eta+2 \le j \le n-1-\eta$, we have

$$\begin{aligned} (3.4)\quad \|w_{\alpha,\beta}(f-S)\|_{\mathbb{L}_p(I_j)} &= \|w_{\alpha,\beta}(\widetilde{f}-\widetilde{S})\|_{\mathbb{L}_p(I_j)} \sim w_{\alpha,\beta}(x_j)\,\|\widetilde{f}-\widetilde{S}\|_{\mathbb{L}_p(I_j)} \\ &\le c_0 w_{\alpha,\beta}(x_j) E_{\ell+1}(\widetilde{f}, I_j^{(\eta)})_p = c_0 w_{\alpha,\beta}(x_j) E_{\ell+1}(f, I_j^{(\eta)})_p \\ &\sim E_{\ell+1}(f, I_j^{(\eta)})_{w_{\alpha,\beta},p}. \end{aligned}$$

In the case $j = \eta+2$, (3.4) becomes

$$\|w_{\alpha,\beta}(f-\widetilde{s}_{\eta+2})\|_{\mathbb{L}_p(I_{\eta+2})} \le c E_{\ell+1}(f, I_{\eta+2}^{(\eta)})_{w_{\alpha,\beta},p} = c E_{\ell+1}(f, [x_{2\eta+2}, x_1])_{w_{\alpha,\beta},p}.$$

Now, suppose that $1 \le j \le \eta+1$ (for $n-\eta \le j \le n$ the proof is similar). We will show that

$$(3.5)\qquad \|w_{\alpha,\beta}(f-S)\|_{\mathbb{L}_p(I_j)} \le c E_{\ell+1}\big(f, [x_{2\eta+2}, 1]\big)_{w_{\alpha,\beta},p}.$$

To this end, if $q \in \Pi_{\ell+1}$ is a polynomial of (near)best weighted approximation of f on $[x_{2\eta+2}, 1]$, *i.e.*,

$$\|w_{\alpha,\beta}(f-q)\|_{\mathbb{L}_p[x_{2\eta+2},1]} \le c E_{\ell+1}\big(f, [x_{2\eta+2}, 1]\big)_{w_{\alpha,\beta},p},$$

then we have

$$\begin{aligned} &\|w_{\alpha,\beta}(f-S)\|_{\mathbb{L}_p(I_j)} \\ &\quad = \|w_{\alpha,\beta}(f-\widetilde{s}_{\eta+2})\|_{\mathbb{L}_p(I_j)} \le c\,\|w_{\alpha,\beta}(f-q)\|_{\mathbb{L}_p(I_j)} + c\,\|w_{\alpha,\beta}(q-\widetilde{s}_{\eta+2})\|_{\mathbb{L}_p(I_j)} \\ &\quad \le c E_{\ell+1}\big(f, [x_{2\eta+2}, 1]\big)_{w_{\alpha,\beta},p} + c\,\|w_{\alpha,\beta}(q-\widetilde{s}_{\eta+2})\|_{\mathbb{L}_p[x_{\eta+2},1]} \\ &\quad \le c E_{\ell+1}\big(f, [x_{2\eta+2}, 1]\big)_{w_{\alpha,\beta},p} + c\,\|w_{\alpha,\beta}(q-\widetilde{s}_{\eta+2})\|_{\mathbb{L}_p(I_{\eta+2})} \\ &\quad \le c E_{\ell+1}\big(f, [x_{2\eta+2}, 1]\big)_{w_{\alpha,\beta},p} + c\,\|w_{\alpha,\beta}(f-q)\|_{\mathbb{L}_p(I_{\eta+2})} + c\,\|w_{\alpha,\beta}(f-\widetilde{s}_{\eta+2})\|_{\mathbb{L}_p(I_{\eta+2})} \\ &\quad \le c E_{\ell+1}\big(f, [x_{2\eta+2}, 1]\big)_{w_{\alpha,\beta},p} + c\,\|w_{\alpha,\beta}(f-\widetilde{s}_{\eta+2})\|_{\mathbb{L}_p(I_{\eta+2})} \\ &\quad \le c E_{\ell+1}\big(f, [x_{2\eta+2}, 1]\big)_{w_{\alpha,\beta},p}, \end{aligned}$$

which verifies (3.5). Here, for the third inequality we used the fact that

$$(3.6)\qquad \text{for } I \subset J \subset [-1,1],\ |J| \le c|I| \quad\text{and}\quad Q \in \Pi_{\ell+1},$$
$$\|w_{\alpha,\beta} Q\|_{\mathbb{L}_p(J)} \le c\,\|w_{\alpha,\beta} Q\|_{\mathbb{L}_p(I)},$$

which, in the case $p < \infty$, follows from the proof of [9, Lemma 12.1]. In the case $p = \infty$, we observe that since $\alpha, \beta \ge 0$, we have, for all $x \in [x_{\eta+2}, 1]$,

$$w_{\alpha,\beta}(x) \le 2^\alpha (1-x)^\beta \le 2^\alpha (1-x_{\eta+2})^\beta \le c \min_{u \in I_{\eta+2}} w_{\alpha,\beta}(u),$$
$$\|q-\widetilde{s}_{\eta+2}\|_{L_\infty[x_{\eta+2},1]} \le c\|q-\widetilde{s}_{\eta+2}\|_{L_\infty(I_{\eta+2})}.$$

Hence,

$$\|w_{\alpha,\beta}(q-\widetilde{s}_{\eta+2})\|_{L_\infty[x_{\eta+2},1]} \le c\|w_{\alpha,\beta}(q-\widetilde{s}_{\eta+2})\|_{L_\infty(I_{\eta+2})}.$$

Combining (3.4) and (3.5) we conclude that (3.1) is valid with $\nu = 2\eta+1$, if $n \ge 2\eta+3$.

For $1 \le n \le 2\eta + 2$, the statement of the theorem follows from the case for $n_0 := 2\eta + 3$. Indeed, suppose that $S \in \mathbb{S}^r_{\ell+1,n_0} \cap \Delta^\ell$ is such that (3.1) is satisfied with $n = n_0$. Let $s_1 := S|_{I_1}$ and define $Q(x) := s_1(x)$, $x \in [-1,1]$. Then, evidently, $Q \in \Pi_{\ell+1} \cap \Delta^\ell$ and

$$\left\| w_{\alpha,\beta}(f-Q) \right\|_{\mathbb{L}_p[\cos(\pi/n_0),1]} \le cE_{\ell+1}(f)_{w_{\alpha,\beta},p}.$$

Now, letting $P \in \Pi_{\ell+1}$ be such that

$$\left\| w_{\alpha,\beta}(f-P) \right\|_p \le cE_{\ell+1}(f)_{w_{\alpha,\beta},p},$$

and using (3.6), we have

$$\begin{aligned}
\left\| w_{\alpha,\beta}(f-Q) \right\|_p &\le c\left\| w_{\alpha,\beta}(f-P) \right\|_p + c\left\| w_{\alpha,\beta}(P-Q) \right\|_p \\
&\le cE_{\ell+1}(f)_{w_{\alpha,\beta},p} + c\left\| w_{\alpha,\beta}(P-Q) \right\|_{\mathbb{L}_p[\cos(\pi/n_0),1]} \\
&\le cE_{\ell+1}(f)_{w_{\alpha,\beta},p} + c\left\| w_{\alpha,\beta}(f-Q) \right\|_{\mathbb{L}_p[\cos(\pi/n_0),1]} \\
&\le cE_{\ell+1}(f)_{w_{\alpha,\beta},p},
\end{aligned}$$

and so (3.1) is verified with ν such that $I_j^{(\nu)} = [-1,1]$ for all $1 \le j \le n$. For example, $\nu = 2\eta + 1$ will do. ∎

4 Additional Auxiliary Statements

Lemma 4.1 $(1 \le p < \infty)$ *Let $n \in \mathbb{N}$, $1 \le p < \infty$, $\alpha, \beta \in J_p$, and $\gamma_j \ge 0$, $1 \le j \le n-1$. Then for*

$$\Sigma_p(x) := \Sigma_p\left(x, (\gamma_j)_{j=1}^{n-1}\right) := \sum_{j=1}^{n-1} \gamma_j |I_j|^{-1/p} \psi_j^\mu(x)$$

and sufficiently large μ, we have

$$\left\| w_{\alpha,\beta}(\cdot)\Sigma_p(\cdot) \right\|_p^p \le c \sum_{j=1}^{n-1} w_{\alpha,\beta}^p(x_j)\gamma_j^p.$$

Proof With the notation

$$w_n(x) := \rho_n^{-1}(x) \int_{x-\rho_n(x)}^{x+\rho_n(x)} w(u)du, \quad \rho_n(x) := \sqrt{1-x^2}\,n^{-1} + n^{-2},$$

we have (see, *e.g.*, [9, (5.1)])

$$w_{\alpha,\beta}^p(x) \sim \left(w_{\alpha,\beta}^p\right)_n(x) \sim \left(w_{\alpha,\beta}^p\right)_n(x_j), \quad \text{for each } x \in I_j,\ 2 \le j \le n-1. \tag{4.1}$$

Also, for any doubling weight w, $n \in \mathbb{N}$, $1 \le j \le n$, $x \in [-1,1]$, and $y \in I_j$,

$$w_n(x) \le c\psi_j^{-s}(x) w_n(y) \quad \text{and} \quad w_n(y) \le c\psi_j^{-s}(x) w_n(x),$$

where constants c and $s \ge 0$ depend only on the doubling constant of w (see [8, Lemma 2.5]), and note that $w_{\alpha,\beta}^p$, $\alpha, \beta \in J_p$, is a doubling weight.

Hence, for appropriate c and s, we have for all $x \in [-1,1]$,

$$\left(w_{\alpha,\beta}^p\right)_n(x) \le c\psi_j^{-s}(x)\left(w_{\alpha,\beta}^p\right)_n(y) \quad \text{and} \\ \left(w_{\alpha,\beta}^p\right)_n(y) \le c\psi_j^{-s}(x)\left(w_{\alpha,\beta}^p\right)_n(x), \quad y \in I_j. \tag{4.2}$$

For $1 \le j \le n-1$, we estimate

$$\int_{-1}^1 w_{\alpha,\beta}^p(x)\psi_j^{\mu p}(x)dx = \left(\int_{I_n} + \int_{x_{n-1}}^{x_1} + \int_{I_1}\right) w_{\alpha,\beta}^p(x)\psi_j^{\mu p}(x)dx =: \mathcal{I}_l + \mathcal{I}_c + \mathcal{I}_r.$$

By virtue of (4.1) and (4.2), if $\mu \ge (s+2)/p$, we have

$$\begin{aligned} \mathcal{I}_c &\le c\int_{x_{n-1}}^{x_1}\left(w_{\alpha,\beta}^p\right)_n(x)\psi_j^{\mu p}(x)dx \le c\int_{x_{n-1}}^{x_1}\left(w_{\alpha,\beta}^p\right)_n(x_j)\psi_j^{\mu p-s}(x)dx \\ &\le c\left(w_{\alpha,\beta}^p\right)_n(x_j)\int_{-1}^1 \psi_j^{\mu p-s}(x)dx \le c|I_j|\left(w_{\alpha,\beta}^p\right)_n(x_j) \sim |I_j|w_{\alpha,\beta}^p(x_j), \end{aligned}$$

since $\int_{-1}^1 \psi^\gamma(x)dx \le c|I_j|$ if $\gamma \ge 2$.

In order to estimate $\mathcal{I}_l$ (considerations for $\mathcal{I}_r$ are similar), we note that for each $x \in I_n$, $\psi_j(x) \sim \psi_j(-1)$, so that

$$\begin{aligned} \mathcal{I}_l &\le c\psi_j^{\mu p}(-1)\int_{I_n} w_{\alpha,\beta}^p(x)dx \le cn^{-2}\psi_j^{\mu p}(-1)\left(w_{\alpha,\beta}^p\right)_n(-1) \\ &\le cn^{-2}\psi_j^{\mu p-s}(-1)\left(w_{\alpha,\beta}^p\right)_n(x_j) \le c|I_j|w_{\alpha,\beta}^p(x_j), \end{aligned}$$

since

$$n^{-2}\psi_j^{\mu p-s}(-1) \le c|I_j|, \quad \text{if } \mu \ge s/p.$$

Combining the above estimates, we conclude that for $\mu \ge (s+2)/p$,

$$\int_{-1}^1 w_{\alpha,\beta}^p(x)\psi_j^{\mu p}(x)dx \le c|I_j|w_{\alpha,\beta}^p(x_j), \quad 1 \le j \le n-1.$$

Hence, taking into account that $\sum_{j=1}^{n-1}\psi_j^2(x) \le c$ and $\psi_j(x) \le 1$, $1 \le j \le n-1$, and denoting $p' := p/(p-1)$, we have, using Hölder's inequality,

$$\begin{aligned} \left\|w_{\alpha,\beta}(\cdot)\Sigma_p(\cdot)\right\|_p^p &\le c\int_{-1}^1 w_{\alpha,\beta}^p(x)\Big[\sum_{j=1}^{n-1}\gamma_j|I_j|^{-1/p}\psi_j^\mu(x)\Big]^p dx \\ &\le c\int_{-1}^1 w_{\alpha,\beta}^p(x)\Big(\sum_{j=1}^{n-1}\gamma_j^p|I_j|^{-1}\psi_j^{(\mu-2)p}(x)\Big)\Big(\sum_{j=1}^{n-1}\psi_j^{2p'}(x)\Big)^{p/p'}dx \\ &\le c\int_{-1}^1 w_{\alpha,\beta}^p(x)\sum_{j=1}^{n-1}\gamma_j^p|I_j|^{-1}\psi_j^{(\mu-2)p}(x)dx \le c\sum_{j=1}^{n-1}w_{\alpha,\beta}^p(x_j)\gamma_j^p, \end{aligned}$$

provided $\mu \ge 2 + (s+2)/p$. ■

Lemma 4.2 $(p = \infty)$ *Let $n \in \mathbb{N}$, $p = \infty$, $\alpha, \beta \ge 0$, and $\gamma_j \ge 0$, $1 \le j \le n-1$. Then for*

$$\Sigma_\infty(x) := \Sigma_\infty\left(x, (\gamma_j)_{j=1}^{n-1}\right) := \sum_{j=1}^{n-1}\gamma_j\psi_j^\mu(x)$$

and sufficiently large μ, we have

$$\left\| w_{\alpha,\beta}(\cdot)\Sigma_\infty(\cdot)\right\|_\infty \le c \sup_{1\le j\le n-1} w_{\alpha,\beta}(x_j)\gamma_j.$$

Proof It is convenient to denote $W := \sup_{1\le j\le n-1} w_{\alpha,\beta}(x_j)\gamma_j$. Then for every $x \in [-1,1]$, we have

$$\begin{aligned} w_{\alpha,\beta}(x)\Sigma_\infty(x) &= \sum_{j=1}^{n-1} \gamma_j w_{\alpha,\beta}(x)\psi_j^\mu(x) \le W \sum_{j=1}^{n-1} \frac{w_{\alpha,\beta}(x)}{w_{\alpha,\beta}(x_j)}\psi_j^\mu(x) \\ &\le W \sum_{j=1}^{n-1}\left(1+\frac{|x-x_j|}{1+x_j}\right)^\alpha \left(1+\frac{|x-x_j|}{1-x_j}\right)^\beta \psi_j^\mu(x) \\ &\le W \sum_{j=1}^{n-1}\left(1+\frac{|x-x_j|}{|I_{j+1}|}\right)^\alpha \left(1+\frac{|x-x_j|}{|I_j|}\right)^\beta \psi_j^\mu(x). \end{aligned}$$

Since $|I_{j+1}| \ge |I_j|/3$, this implies

$$w_{\alpha,\beta}(x)\Sigma_\infty(x) \le 3^\alpha W \sum_{j=1}^{n-1} \psi_j^{\mu-\alpha-\beta}(x) \le cW,$$

since $\sum_{j=1}^{n-1} \psi_j^{\mu-\alpha-\beta}(x) \le c$, provided $\mu \ge \alpha+\beta+2$. ∎

5 Convex Approximation of Quadratic Splines by Polynomials

In this section, suppose that $g \in \mathbb{S}_{3,n} \cap \Delta^2$. In other words, g is a continuous quadratic convex spline on the Chebyshev partition. We now construct a polynomial that approximates g.

Denote by $L_j(x,g)$ the quadratic polynomial interpolating g at x_j, x_{j-1} and x_{j-2}, *i.e.*,

$$L_j(x,g) := \sum_{j-2\le i\le j} g(x_i) \prod_{j-2\le l\le j, l\ne i} \frac{x-x_l}{x_i-x_l}.$$

For $n \ge 2$, let S be a continuous piecewise quadratic polynomial with knots at x_j, $1 \le j \le n-1$, such that

$$\begin{aligned} &S(x) := \max\left\{L_j(x,g), L_{j+1}(x,g)\right\}, \quad x \in I_j, \quad 2 \le j \le n-1, \\ &S(x) := L_2(x,g), \quad x \in I_1, \quad \text{and} \quad S(x) := L_n(x,g), \quad x \in I_n. \end{aligned}$$

Since g is convex, so is S.

We need the following lemma.

Lemma 5.1 *Let $1 \le p \le \infty$, $n \in \mathbb{N}$, $\kappa \in \mathbb{N}_0$, and $G \in \mathbb{S}_{l,n}$, $l \ge 1$. Then for all $1 \le j \le n-1$, we have*

$$w_{\alpha,\beta}(x_j)E_l(G, I_j^{(\kappa)})_p \le cE_l(G, I_j^{(\kappa)})_{w_{\alpha,\beta},p}. \tag{5.1}$$

Proof First, we note that (5.1) is obvious if $I_j^{(\kappa)} \subset [x_{n-1}, x_1]$. Thus, we assume that $1 \in I_j^{(\kappa)}$ (the case for $-1 \in I_j^{(\kappa)}$ is analogous). Then $I_j^{(\kappa)} = [x_{j+\kappa}, 1]$ with $j \le \kappa + 1$.

Suppose that $P \in \Pi_l$ is such that

$$\left\| w_{\alpha,\beta}(G-P) \right\|_{\mathbb{L}_p(I_j^{(\kappa)})} \le c E_l(G, I_j^{(\kappa)})_{w_{\alpha,\beta},p}.$$

Then for $1 \le p < \infty$, we have

$$\begin{aligned}
w_{\alpha,\beta}^p(x_j) E_l(G, I_j^{(\kappa)})_p^p &\le w_{\alpha,\beta}^p(x_j) \, \|G-P\|_{\mathbb{L}_p(I_j^{(\kappa)})}^p \\
&= w_{\alpha,\beta}^p(x_j) \Big(\sum_{i=2}^{j+\kappa} \|G-P\|_{\mathbb{L}_p(I_i)}^p + \|G-P\|_{\mathbb{L}_p(I_1)}^p \Big) \\
&\le w_{\alpha,\beta}^p(x_j) \Big(\sum_{i=2}^{j+\kappa} \|G-P\|_{\mathbb{L}_p(I_i)}^p + c\, \|G-P\|_{\mathbb{L}_p[x_1,(1+x_1)/2]}^p \Big) \\
&\le c \sum_{i=2}^{j+\kappa} \left\| w_{\alpha,\beta}(G-P) \right\|_{\mathbb{L}_p(I_i)}^p + c \left\| w_{\alpha,\beta}(G-P) \right\|_{\mathbb{L}_p[x_1,(1+x_1)/2]}^p \\
&\le c \left\| w_{\alpha,\beta}(G-P) \right\|_{\mathbb{L}_p(I_j^{(\kappa)})}^p \le c E_l(G, I_j^{(\kappa)})_{w_{\alpha,\beta},p}^p,
\end{aligned}$$

where for the second inequality we used the fact that $(G-P)\,|_{I_1}$ is a polynomial of degree l, and for the third inequality we have applied (3.3). The case for $p = \infty$ is analogous. This completes the proof. ∎

In particular, it follows from Lemma 5.1 that for any $1 \le j \le n-1$,

$$w_{\alpha,\beta}(x_j) E_3(g, I_j^{(1)})_p \le c E_3(g, I_j^{(1)})_{w_{\alpha,\beta},p}. \tag{5.2}$$

Now, in view of the fact that $g\,|_{I_i} \in \Pi_3$, $1 \le i \le n$, we have, with p_j denoting the best quadratic approximant to g in $\mathbb{L}_p(I_j^{(1)})$, $1 \le j \le n$,

$$\begin{aligned}
E_3(g, I_j^{(1)})_\infty &\le \|g-p_j\|_{\mathbb{L}_\infty(I_j^{(1)})} = \max_{j-1 \le i \le j+1} \|g-p_j\|_{\mathbb{L}_\infty(I_i)} \\
&\le c|I_j|^{-1/p} \max_{j-1 \le i \le j+1} \|g-p_j\|_{\mathbb{L}_p(I_i)} \le c|I_j|^{-1/p} E_3(g, I_j^{(1)})_p.
\end{aligned} \tag{5.3}$$

Similarly, for all $2 \le j \le n-1$, we have

$$\begin{aligned}
[x_{j+1}, x_j, x_{j-1}, x_{j-2}; g] &= [x_{j+1}, x_j, x_{j-1}, x_{j-2}; g-p_j] \\
&\le c|I_j|^{-3} \max_{j-1 \le i \le j+1} \left\| g-p_j \right\|_{\mathbb{L}_\infty(I_i)} \\
&\le c|I_j|^{-3-1/p} E_3(g, I_j^{(1)})_p.
\end{aligned} \tag{5.4}$$

Now, using (3.6), (5.2), (5.3), and Whitney's inequality, we obtain, for all $1 \le j \le n-1$,

$$\begin{aligned}
\left\| w_{\alpha,\beta}(g-S) \right\|_{\mathbb{L}_p(I_j)} &\le c w_{\alpha,\beta}(x_j) \, \|g-S\|_{\mathbb{L}_p(I_j)} \\
&\le c w_{\alpha,\beta}(x_j) |I_j|^{1/p} \, \|g-S\|_{\mathbb{L}_\infty(I_j)} \\
&\le c w_{\alpha,\beta}(x_j) |I_j|^{1/p} \omega_3\big(g, |I_j^{(1)}|, I_j^{(1)}\big)_\infty \\
&\le c w_{\alpha,\beta}(x_j) |I_j|^{1/p} E_3(g, I_j^{(1)})_\infty \\
&\le c w_{\alpha,\beta}(x_j) E_3(g, I_j^{(1)})_p \le c E_3(g, I_j^{(1)})_{w_{\alpha,\beta},p}.
\end{aligned} \tag{5.5}$$

Similarly, in the case $j = n$, we have

$$\begin{aligned}\|w_{\alpha,\beta}(g-S)\|_{\mathbb{L}_p(I_n)} &\le c w_{\alpha,\beta}(x_{n-1})\,\|g-S\|_{\mathbb{L}_p(I_n)} \\ &\le c w_{\alpha,\beta}(x_{n-1}) n^{-2/p}\,\|g-S\|_{\mathbb{L}_\infty(I_n)} \\ &\le c w_{\alpha,\beta}(x_{n-1}) n^{-2/p}\omega_3\big(g, |I_n^{(1)}|, I_n^{(1)}\big)_\infty \\ &\le c w_{\alpha,\beta}(x_{n-1}) n^{-2/p} E_3(g, I_n^{(1)})_\infty \\ &\le c w_{\alpha,\beta}(x_{n-1}) E_3(g, I_n^{(1)})_p \le c E_3(g, I_n^{(1)})_{w_{\alpha,\beta},p}.\end{aligned} \tag{5.6}$$

It was shown in [6, Section 4] that all knots x_j, $1 \le j \le n-1$, can be separated into classes **I**, **II**, **III**, and **IV** so that S has the following representation

$$\begin{aligned}S(x) = q_2(x) &+ \sum_{2\le j\le n-1, x_j\in \mathbf{I}\cup\mathbf{III}} A_j\big[(x_{j-1}-x_j)(x-x_j)_+ - (x-x_j)_+^2\big] \\ &+ \sum_{1\le j\le n-2, x_j\in \mathbf{II}\cup\mathbf{III}} (-A_{j+1})\big[(x_j - x_{j+1})(x-x_j)_+ + (x-x_j)_+^2\big],\end{aligned}$$

where

$$\begin{aligned}q_2(x) := g(-1) &+ \big([x_n, x_{n-1}; g] - [x_{n-1}, x_{n-2}; g] + [x_n, x_{n-2}; g]\big)(x+1) \\ &+ [x_n, x_{n-1}, x_{n-2}; g](x+1)^2,\end{aligned}$$

and

$$A_j := [x_{j+1}, x_j, x_{j-1}; g] - [x_j, x_{j-1}, x_{j-2}; g], \quad 2 \le j \le n-1,$$

is such that $A_j > 0$ if $x_j \in \mathbf{I}\cup\mathbf{III}$, and $A_{j+1} < 0$ if $x_j \in \mathbf{II}\cup\mathbf{III}$.

Then the polynomial

$$\begin{aligned}P_n(x) := P_n(x, g) := q_2(x) &+ \sum_{2\le j\le n-1, x_j\in \mathbf{I}\cup\mathbf{III}} A_j\big[(x_{j-1}-x_j)\sigma_j(x) - R_j(x)\big] \\ &+ \sum_{1\le j\le n-2, x_j\in \mathbf{II}\cup\mathbf{III}} (-A_{j+1})\big[(x_j - x_{j+1})\sigma_j(x) + \overline{R}_j(x)\big]\end{aligned}$$

of degree $\le cn$ is convex on $[-1, 1]$, since by (2.2),

$$P_n''(x) \ge S''(x) \ge 0, \quad x \in [-1,1], \quad x \ne x_j, \quad 1 \le j \le n-1.$$

Also, by (2.1) and (5.4),

$$\begin{aligned}|P_n(x) - S(x)| &\le c\sum_{j=2}^{n-1} |A_j||I_j|^2\psi_j^\mu(x) \\ &\le c\sum_{j=2}^{n-1} \big|[x_{j+1}, x_j, x_{j-1}, x_{j-2}; g]\big|\,|I_j|^3\psi_j^\mu(x) \\ &\le c\sum_{j=2}^{n-1} |I_j|^{-1/p}\psi_j^\mu(x) E_3(g, I_j^{(1)})_p.\end{aligned}$$

Hence, in the case $1 \le p < \infty$, using Lemma 4.1 and (5.2), we have

$$\|w_{\alpha,\beta}(P_n - S)\|_p^p \le c\sum_{j=1}^{n-1} w_{\alpha,\beta}^p(x_j) E_3(g, I_j^{(1)})_p^p \le c\sum_{j=1}^{n-1} E_3(g, I_j^{(1)})_{w_{\alpha,\beta},p}^p$$

and, combining with (5.5) and (5.6), we conclude that

$$\left\| w_{\alpha,\beta}(P_n - g) \right\|_p^p \le c \sum_{j=1}^{n} E_3(g, I_j^{(1)})_{w_{\alpha,\beta},p}^p. \tag{5.7}$$

In the case where $p = \infty$, using Lemma 4.2 and (5.2) we have

$$\left\| w_{\alpha,\beta}(P_n - S) \right\|_\infty \le c \sup_{1\le j\le n-1} w_{\alpha,\beta}(x_j) E_3(g, I_j^{(1)})_\infty \le c \sup_{1\le j\le n-1} E_3(g, I_j^{(1)})_{w_{\alpha,\beta},\infty}$$

and, combining with (5.5) and (5.6), we get

$$\left\| w_{\alpha,\beta}(P_n - g) \right\|_\infty \le c \sup_{1\le j\le n} E_3(g, I_j^{(1)})_{w_{\alpha,\beta},\infty}. \tag{5.8}$$

We will now show that the first derivative of the polynomial P_n approximates the first derivative of g (if it exists). This fact will be used to obtain the estimates in the monotone case as a corollary. Suppose that $g \in \mathbb{C}^1[-1,1]$.

For $1 \le j \le n-1$, we have

$$\begin{aligned}
\left\| w_{\alpha,\beta}(g' - S') \right\|_{\mathbb{L}_p(I_j)} &\le c w_{\alpha,\beta}(x_j) \left\| g' - S' \right\|_{\mathbb{L}_p(I_j)} \\
&\le c w_{\alpha,\beta}(x_j) |I_j|^{-1+1/p} \left\| g - S \right\|_{\mathbb{L}_\infty(I_j)} \\
&\le c w_{\alpha,\beta}(x_j) |I_j|^{-1+1/p} \omega_3\big(g, |I_j^{(1)}|, I_j^{(1)}\big)_\infty \\
&\le c w_{\alpha,\beta}(x_j) |I_j|^{1/p} \omega_2\big(g', |I_j^{(1)}|, I_j^{(1)}\big)_\infty \\
&\le c w_{\alpha,\beta}(x_j) |I_j|^{1/p} E_2(g', I_j^{(1)})_\infty \le c E_2(g', I_j^{(1)})_{w_{\alpha,\beta},p},
\end{aligned}$$

where for the second inequality we used the fact that $(g - S)\,|_{I_j} \in \Pi_3$. Similarly (as in (5.6)),

$$\left\| w_{\alpha,\beta}(g' - S') \right\|_{\mathbb{L}_p(I_n)} \le c E_2(g', I_n^{(1)})_{w_{\alpha,\beta},p}.$$

Also, for all $x \in [-1,1] \setminus \{x_j\}_{j=1}^{n-1}$,

$$\begin{aligned}
&|P_n'(x) - S'(x)| \\
&\quad \le \sum_{2\le j\le n-1, x_j \in \mathbf{I}\cup\mathbf{III}} |A_j| \Big[(x_{j-1} - x_j) \big| \sigma_j'(x) - \chi_j(x) \big| + \big| R_j'(x) - 2(x - x_j)_+ \big| \Big] \\
&\qquad + \sum_{1\le j\le n-2, x_j \in \mathbf{II}\cup\mathbf{III}} |A_{j+1}| \Big[(x_j - x_{j+1}) \big| \sigma_j'(x) - \chi_j(x) \big| + \big| \overline{R}_j'(x) - 2(x - x_j)_+ \big| \Big] \\
&\quad \le c \sum_{j=2}^{n-1} |A_j| |I_j| \psi_j^\mu(x) \le c \sum_{j=2}^{n-1} \big| [x_{j+1}, x_j, x_{j-1}, x_{j-2}; g] \big| |I_j|^2 \psi_j^\mu(x) \\
&\quad \le c \sum_{j=2}^{n-1} |I_j|^{-1-1/p} \psi_j^\mu(x) E_3(g, I_j^{(1)})_p \le c \sum_{j=2}^{n-1} |I_j|^{-1/p} \psi_j^\mu(x) E_2(g', I_j^{(1)})_p.
\end{aligned}$$

Therefore, Lemmas 4.1, 4.2 and 5.1 imply that

$$\left\| w_{\alpha,\beta}(P_n' - S') \right\|_p^p \le c \sum_{j=1}^{n-1} E_2(g', I_j^{(1)})_{w_{\alpha,\beta},p}^p, \qquad \text{if } 1 \le p < \infty,$$

and

$$\left\| w_{\alpha,\beta}(P_n' - S') \right\|_\infty \le c \sup_{1\le j\le n-1} E_2(g', I_j^{(1)})_{w_{\alpha,\beta},\infty}, \qquad \text{if } p = \infty.$$

Hence,

$$\left\| w_{\alpha,\beta}(P_n' - g') \right\|_p^p \le c \sum_{j=1}^n E_2(g', I_j^{(1)})_{w_{\alpha,\beta},p}^p, \qquad \text{if } 1 \le p < \infty, \tag{5.9}$$

and

$$\left\| w_{\alpha,\beta}(P_n' - g') \right\|_\infty \le c \sup_{1\le j\le n} E_2(g', I_j^{(1)})_{w_{\alpha,\beta},\infty}, \qquad \text{if } p = \infty. \tag{5.10}$$

6 Constrained Approximation by Polynomials

Suppose that $\ell = 1$ or $\ell = 2$, $1 \le p \le \infty$, $\alpha, \beta \in J_p$, $f \in \mathbb{L}_p^{\alpha,\beta} \cap \Delta^\ell$, and let $n \in \mathbb{N}$ be sufficiently large. Corollary 3.3 implies that there exists $g_\ell \in \mathbb{S}_{\ell+1,n} \cap \mathbb{C}[-1,1] \cap \Delta^\ell$ and $\nu \in \mathbb{N}$ such that

$$\left\| w_{\alpha,\beta}(f - g_\ell) \right\|_{\mathbb{L}_p(I_j)} \le c E_{\ell+1}(f, I_j^{(\nu)})_{w_{\alpha,\beta},p}, \quad 1 \le j \le n. \tag{6.1}$$

Therefore,

$$\left\| w_{\alpha,\beta}(f - g_\ell) \right\|_p^p = \sum_{j=1}^n \left\| w_{\alpha,\beta}(f - g_\ell) \right\|_{\mathbb{L}_p(I_j)}^p \le c \sum_{j=1}^n E_{\ell+1}(f, I_j^{(\nu)})_{w_{\alpha,\beta},p}^p, \tag{6.2}$$

if $1 \le p < \infty$, and

$$\left\| w_{\alpha,\beta}(f - g_\ell) \right\|_\infty \le c \sup_{1\le j\le n} E_{\ell+1}(f, I_j^{(\nu)})_{w_{\alpha,\beta},\infty}, \tag{6.3}$$

if $p = \infty$.

We note that (6.1) implies, for any $\kappa \in \mathbb{N}_0$ and $1 \le j \le n$,

$$\begin{aligned} E_{\ell+1}(g_\ell, I_j^{(\kappa)})_{w_{\alpha,\beta},p} &\le c E_{\ell+1}(f, I_j^{(\kappa)})_{w_{\alpha,\beta},p} + c \left\| w_{\alpha,\beta}(g_\ell - f) \right\|_{\mathbb{L}_p(I_j^{(\kappa)})} \\ &\le c E_{\ell+1}(f, I_j^{(\nu+\kappa)})_{w_{\alpha,\beta},p}. \end{aligned} \tag{6.4}$$

6.1 Convex Approximation

Suppose that $\ell = 2$, thus g_2 is a continuous piecewise quadratic convex spline on $[-1,1]$. Let $P_n(\cdot, g_2)$ denote the polynomial associated with g_2 satisfying (5.7) or (5.8). Then it follows from (6.2), (6.3), and (6.4) that

$$\begin{aligned} \left\| w_{\alpha,\beta}(f - P_n(g_2)) \right\|_p^p &\le c \sum_{j=1}^n E_3(f, I_j^{(\nu)})_{w_{\alpha,\beta},p}^p + c \sum_{j=1}^n E_3(g_2, I_j^{(1)})_{w_{\alpha,\beta},p}^p \\ &\le c \sum_{j=1}^n E_3(f, I_j^{(\nu+1)})_{w_{\alpha,\beta},p}^p, \end{aligned}$$

if $1 \le p < \infty$, and

$$\left\| w_{\alpha,\beta}(f - P_n(g_2)) \right\|_\infty \le c \sup_{1\le j\le n} E_3(f, I_j^{(\nu)})_{w_{\alpha,\beta},\infty} + c \sup_{1\le j\le n} E_3(g_2, I_j^{(1)})_{w_{\alpha,\beta},\infty}$$
$$\le c \sup_{1\le j\le n} E_3(f, I_j^{(\nu+1)})_{w_{\alpha,\beta},\infty},$$

if $p = \infty$.

The statement of Theorem 1.5 in the case $\ell = 2$ now follows from Corollary 2.4.

6.2 Monotone Approximation

If $\ell = 1$, then g_1 is a continuous piecewise linear monotone spline on $[-1,1]$.

We now define $G(x) := \int_{-1}^x g_1(u)du$. Then $G \in \mathbb{S}_{3,n} \cap \mathbb{C}^1[-1,1] \cap \Delta^2$. Let $P_n(\cdot, G)$ be the polynomial associated with G satisfying (5.7) and denote $Q_n(x) := P_n'(x, G)$.

It follows that $Q_n \in \Delta^1$, and estimates (5.9) and (5.10) imply

$$\left\| w_{\alpha,\beta}(Q_n - g_1) \right\|_p^p = \left\| w_{\alpha,\beta}\big(P_n'(G) - G'\big) \right\|_p^p \le c \sum_{j=1}^n E_2(g_1, I_j^{(1)})_{w_{\alpha,\beta},p}^p,$$

if $1 \le p < \infty$, and

$$\left\| w_{\alpha,\beta}(Q_n - g_1) \right\|_\infty = \left\| w_{\alpha,\beta}\big(P_n'(G) - G'\big) \right\|_\infty \le c \sup_{1\le j\le n} E_2(g_1, I_j^{(1)})_{w_{\alpha,\beta},\infty},$$

if $p = \infty$.

Hence,

$$\left\| w_{\alpha,\beta}(f - Q_n) \right\|_p^p \le c \sum_{j=1}^n E_2(f, I_j^{(\nu)})_{w_{\alpha,\beta},p}^p + c \sum_{j=1}^n E_2(g_1, I_j^{(1)})_{w_{\alpha,\beta},p}^p$$
$$\le c \sum_{j=1}^n E_2(f, I_j^{(\nu+1)})_{w_{\alpha,\beta},p}^p,$$

if $1 \le p < \infty$, and

$$\left\| w_{\alpha,\beta}(f - Q_n) \right\|_\infty \le c \sup_{1\le j\le n} E_2(f, I_j^{(\nu)})_{w_{\alpha,\beta},\infty} + c \sup_{1\le j\le n} E_2(g_1, I_j^{(1)})_{w_{\alpha,\beta},\infty}$$
$$\le c \sup_{1\le j\le n} E_2(f, I_j^{(\nu+1)})_{w_{\alpha,\beta},\infty},$$

if $p = \infty$.

The statement of Theorem 1.5 in the case where $\ell = 1$ now follows from Corollary 2.4.

7 Weighted $\mathbb{L}_q$ Approximation of Constrained Unit Spheres in $\mathbb{L}_p^{\alpha,\beta}$, $1 \le q < p \le \infty$

The following result was proved in [7].

Theorem 7.1 ([7, Theorem 1.1]) *Let* $\ell \in \mathbb{N}$, $1 \le q < p \le \infty$, $\alpha, \beta \in J_p$, $f \in \mathbb{L}_p^{\alpha,\beta} \cap \Delta^\ell$ *and* $0 < \delta < 1/4$. *Then,*

$$\omega_\phi^\ell(f, 1, \delta)_{w_{\alpha,\beta},q} \le c\Upsilon_\delta^{\alpha,\beta}(\ell, q, p) \left\| w_{\alpha,\beta} f \right\|_p, \tag{7.1}$$

where

$$\Upsilon_\delta^{\alpha,\beta}(\ell, q, p) := \begin{cases} \delta^{2/q-2/p} & \text{if } \ell \ge 2, \text{ and } (\ell, q, p) \ne (2, 1, \infty), \\ \delta^2 |\ln \delta| & \text{if } \ell = 2,\ q = 1,\ p = \infty, \text{ and } (\alpha, \beta) \ne (0, 0), \\ \delta^2 & \text{if } \ell = 2,\ q = 1,\ p = \infty, \text{ and } (\alpha, \beta) = (0, 0), \\ \delta^{2/q-2/p} & \text{if } \ell = 1 \text{ and } p < 2q, \\ \delta^{1/q} |\ln \delta|^{1/(2q)} & \text{if } \ell = 1 \text{ and } p = 2q, \\ \delta^{1/q} & \text{if } \ell = 1 \text{ and } p > 2q. \end{cases}$$

is best possible in the sense that (7.1) *is no longer valid if one increases (resp. decreases) any of the powers of* δ *(resp.* $|\ln \delta|$*) in its definition.*

Now, let $\mathbb{S}_p^{\alpha,\beta}$ be the unit sphere in $\mathbb{L}_p^{\alpha,\beta}$, *i.e.*, $f \in \mathbb{S}_p^{\alpha,\beta}$ if and only if $\left\| w_{\alpha,\beta} f \right\|_p = 1$, and denote

$$\mathcal{E}(X, \Pi_n)_{w,q} := \sup_{f \in X} E_n(f)_{w,q} \quad \text{and} \quad \mathcal{E}^{(\ell)}(X, \Pi_n)_{w,q} := \sup_{f \in X} E_n^{(\ell)}(f)_{w,q}.$$

Theorem 7.1 and Corollary 1.8 imply the following result.

Corollary 7.2 *Let* $\ell = 1$ *or* $\ell = 2$, $1 \le q < p \le \infty$ *and* $\alpha, \beta \ge 0$. *Then*

$$\mathcal{E}^{(\ell)}(\Delta^\ell \cap \mathbb{S}_p^{\alpha,\beta}, \Pi_n)_{w_{\alpha,\beta},q} \le c\Upsilon_{1/n}^{\alpha,\beta}(\ell, q, p), \quad \text{for all } n \in \mathbb{N}.$$

Finally, applying [7, Theorem 1.5] and using the fact that

$$\mathcal{E}(X, \Pi_n)_{w,q} \le \mathcal{E}^{(\ell)}(X, \Pi_n)_{w,q},$$

we obtain the following corollary.

Corollary 7.3 *Let* $\ell = 1$ *or* $\ell = 2$, $1 \le q < p \le \infty$, *and* $\alpha, \beta \ge 0$. *Then for any* $n \in \mathbb{N}$,

$$\mathcal{E}^{(\ell)}(\Delta^\ell \cap \mathbb{S}_p^{\alpha,\beta}, \Pi_n)_{w_{\alpha,\beta},q} \sim \mathcal{E}(\Delta^\ell \cap \mathbb{S}_p^{\alpha,\beta}, \Pi_n)_{w_{\alpha,\beta},q}$$
$$\sim \begin{cases} n^{-2/q+2/p} & \text{if } \ell = 2 \text{ and } (q, p) \ne (1, \infty), \\ n^{-2} & \text{if } \ell = 2,\ q = 1,\ p = \infty, \text{ and } \alpha = \beta = 0, \\ n^{-\min\{2/q-2/p, 1/q\}} & \text{if } \ell = 1 \text{ and } p \ne 2q. \end{cases}$$

If $\ell = 2$, $q = 1$, $p = \infty$, *and* $(\alpha, \beta) \ne (0, 0)$, *then*

$$cn^{-2} \le \mathcal{E}(\Delta^2 \cap \mathbb{S}_\infty^{\alpha,\beta}, \Pi_n)_{w_{\alpha,\beta},1} \le \mathcal{E}^{(2)}(\Delta^2 \cap \mathbb{S}_\infty^{\alpha,\beta}, \Pi_n)_{w_{\alpha,\beta},1} \le cn^{-2} \ln(n + 1).$$

If $\ell = 1$ *and* $p = 2q$, *then*

$$cn^{-1/q} \le \mathcal{E}(\Delta^1 \cap \mathbb{S}_{2q}^{\alpha,\beta}, \Pi_n)_{w_{\alpha,\beta},q} \le \mathcal{E}^{(1)}(\Delta^1 \cap \mathbb{S}_{2q}^{\alpha,\beta}, \Pi_n)_{w_{\alpha,\beta},q} \le cn^{-1/q} [\ln(n + 1)]^{1/(2q)}.$$

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