Constrained Approximation with Jacobi Weights

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Abstract. In this paper, we prove that for ℓ = 1 or 2 the rate of best ℓ-monotone polynomial approximation in the $L_p$ norm (1 ≤ p ≤ ∞) weighted by the Jacobi weight $w_{\alpha, \beta}(x) := (1 + x)^\alpha (1 - x)^\beta$ with $\alpha, \beta > -1/p$ if $p < \infty$, or $\alpha, \beta \geq 0$ if $p = \infty$, is bounded by an appropriate $(\ell + 1)$-st modulus of smoothness with the same weight, and that this rate cannot be bounded by the $(\ell + 2)$-nd modulus. Related results on constrained weighted spline approximation and applications of our estimates are also given.

1 Introduction and Main Results

One of the central topics in Approximation Theory is the investigation of the connection between the rate with which a function can be approximated and the smoothness of this function. The goal is to prove direct and matching inverse estimates in terms of the right measure of smoothness. In other words, one strives to obtain results of the following type: “a function can be approximated with a given order if and only if it belongs to a certain smoothness class”. In order to describe these smoothness classes, one usually needs to introduce appropriate moduli of smoothness that correspond to the way approximation orders are measured. For example, in the case of approximation by algebraic polynomials, if orders of approximation of $f$ are given using the $L_p$ norms, then one can measure smoothness of $f$ using either the Ivanov or the Ditzian-Totik moduli (we refer the reader to [4] for further discussion of recent developments in this area).

Corresponding problems for weighted polynomial approximation are much more complicated, especially if the weight has zeros and/or singularities inside $(-1, 1)$. One cannot simply replace the usual $L_p$ norm $\| \cdot \|_p$ by a corresponding weighted norm $\| \cdot \|_{w_p}$ everywhere, and there is a need to modify the definition of the moduli of smoothness near zeros/singularities of the weight (see, e.g., [1,2,12] and the references therein for further details). Even the case for the classical Jacobi weights that have zeros/singularities “only” at the endpoints of $[-1, 1]$ is rather involved, and requires a modification of the definition of the moduli.

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In this paper, we are interested in weighted approximation with Jacobi weights that is not only exact in the above-described sense but also preserves the shape of the functions being approximated. It is usually referred to as shape preserving or constrained approximation. Constrained approximation without weights has been relatively well studied in recent years (see, e.g., our survey paper [10] for details). At the same time, we are not aware of any results on weighted polynomial approximation with constraints on a finite interval (with Jacobi or any other weights), and the purpose of this paper is to obtain the first result of this type. We prove Jackson type estimates in the $L_p$, $1 \leq p \leq \infty$, norm weighted by the Jacobi weights in terms of the “correct” moduli of smoothness (i.e., those moduli that yield matching direct and inverse estimates) in the case of monotone and convex polynomial approximation, and state several applications of our results. Some of the results may be had (with appropriate modifications) also for $0 < p < 1$, but we have limited ourselves to $1 \leq p \leq \infty$ in order to avoid complicated technicalities. Moreover, we have not considered the $\ell$-monotone approximation with $\ell \geq 3$ in this paper, since this type of approximation is rather involved even in the unweighted case (see, e.g., [10] for additional discussion). Furthermore, it is a natural question whether analogs of our results are valid in the case of approximation with more general doubling weights. However, this seems to be a rather complicated question, since the presence of internal zeros and singularities causes difficulties even in the case of approximation without any constraints (see [8, 9], for example). Still, we hope that the techniques developed in this paper and their modifications can be used to tackle more general problems on constrained weighted approximation in the future.

In order to continue our discussion, we need to introduce some notation and recall several definitions.

As usual, for a measurable $f: [-1, 1] \to \mathbb{R}$ and an interval $I \subseteq [-1, 1]$, let
\[
\|f\|_{L_p(I)} := \left( \int_I |f(x)|^p \, dx \right)^{1/p}, \quad p < \infty, \quad \text{and} \quad \|f\|_{L_\infty(I)} := \text{ess sup}_{x \in I} |f(x)|.
\]
For a weight function $w$, we also let
\[
L_{w, p}(I) := \{ f \| \| w f \|_{L_p(I)} < \infty \},
\]
and, for $f \in L_{w, p}(I)$, let
\[
E_n(f, I)_{w, p} := \inf_{p_n \in \Pi_n} \| w(f - p_n) \|_{L_p(I)}
\]
be the degree of weighted approximation by polynomials in $\Pi_n$, the set of algebraic polynomials of degree $< n$. For $I = [-1, 1]$, it is convenient to use the notation $\|f\|_p := \|f\|_{L_p([-1, 1]), L_{w, p} := L_{w, p}([-1, 1])}$ and $E_n(f)_{w, p} := E_n(f, [-1, 1])_{w, p}$.

Let
\[
\Delta_h^k(f, x, [a, b]) := \begin{cases}
\sum_{i=0}^k \binom{k}{i} (-1)^{k-i} f(x - kh/2 + ih) & \text{if } x \pm kh/2 \in [a, b], \\
0 & \text{otherwise},
\end{cases}
\]
be the $k$-th symmetric difference, $\Delta_h^k(f, x) := \Delta_h^k(f, x, [-1, 1])$, and let
\[
\overline{\Delta_h^k}(f, x) := \Delta_h^k(f, x + kh/2) \quad \text{and} \quad \underline{\Delta_h^k}(f, x) := \Delta_h^k(f, x - kh/2)
\]
be the forward and backward k-th differences, respectively. Then the usual k-th modulus of smoothness of \( f \in \mathbb{L}_p(I) \) on an interval \( I \) is defined as
\[
\omega_k(f, \delta, I)_p := \sup_{0<h \leq \delta} \| \Delta_h^k(f, \cdot, I) \|_{\mathbb{L}_p(I)} .
\]

Now, let
\[
w_{\alpha, \beta}(x) := (1 + x)^{\alpha}(1 - x)^{\beta}, \quad \alpha, \beta \in \mathbb{R}_+ := \begin{cases} (-1/p, \infty) & \text{if } p < \infty, \\ (0, \infty) & \text{if } p = \infty,
\end{cases}
\]
be the Jacobi weights, and denote \( \omega_{\alpha, \beta} := L_{\alpha, \beta, p} \).

The main part weighted modulus of smoothness (with Jacobi weights) is defined as
\[
\Omega^k_{\phi}(f, A, \delta)_{\alpha, \beta, p} := \\
\sup_{0<h \leq \delta} \| w_{\alpha, \beta}(\cdot) \Delta_h^k(\cdot)(f, \cdot, [-1 + Ah^2, 1 - Ah^2]) \|_{\mathbb{L}_p([-1 + Ah^2, 1 - Ah^2])},
\]
where \( A \) is a positive constant and \( \phi(x) := \sqrt{1 - x^2} \).

Following [1,2] (see also [5, Chapter 11]), we define
\[
\omega^k_{\phi}(f, A, \delta)_{\alpha, \beta, p} := \Omega^k_{\phi}(f, A, \delta)_{\alpha, \beta, p} + E_k(f, [-1, -1 + 2A\delta^2])_{\alpha, \beta, p} + E_k(f, [-1 - 2A\delta^2, 1])_{\alpha, \beta, p},
\]
and note that \( \omega^k_{\phi}(f, A, \delta)_{\alpha, \beta, p} \) is bounded for all \( f \in \mathbb{L}^\alpha_{\beta, p} \) with \( \alpha, \beta \in \mathbb{R}_+ \).

If \( 1 \leq p \leq \infty \), it is possible to show (see [9, Corollary II.1]) for \( 1 \leq p < \infty \), if \( p = \infty \), the proof is analogous) that
\[
\omega_{\phi}^k(f, A, \delta_1)_{\alpha, \beta, p} \sim \omega_{\phi}^k(f, A, \delta_2)_{\alpha, \beta, p} \quad \text{if } A_1 \sim A_2 \quad \text{and } \delta_1 \sim \delta_2,
\]
where by \( A \sim B \) we mean that there exist constants \( 0 < c_1 \leq c_2 \) such that \( c_1A \leq B \leq c_2A \).

Here and in the sequel, the equivalence constants as well as the constants \( c_i \) in general, depend on \( \alpha, \beta, p \) and the order of the moduli. Constants \( c \) may be different on different occurrences even when they appear in the same line.

The weighted Ditzian–Totik (DT) modulus of smoothness (see [5, (8.1.2), (8.2.10) and Appendix B]) is defined as
\[
\omega_{\phi}^k(f, A, \delta)_{\alpha, \beta, p} := \Omega_{\phi}^k(f, A, \delta)_{\alpha, \beta, p} + \Omega_{\phi}^k(f, A, \delta)_{\alpha, \beta, p} + \Omega_{\phi}^k(f, A, \delta)_{\alpha, \beta, p},
\]
where
\[
\Omega_{\phi}^k(f, A, \delta)_{\alpha, \beta, p} := \sup_{0<h \leq 2A\delta^2} \| w_{\alpha, \beta}(\cdot) \Delta_h^k(\cdot)(f) \|_{\mathbb{L}_p([-1, -1 + 2A\delta^2])}
\]
and
\[
\Omega_{\phi}^k(f, A, \delta)_{\alpha, \beta, p} := \sup_{0<h \leq 2A\delta^2} \| w_{\alpha, \beta}(\cdot) \Delta_h^k(\cdot)(f) \|_{\mathbb{L}_p([-1 - 2A\delta^2, 1])}.
\]
where $\Delta_h^k(f) := \Delta_h^k(f, \cdot)$ and $\Delta_h^k(f) := \Delta_h^k(f, \cdot)$. If $w \equiv 1$, then $\omega^k_\varphi (f, A, \delta)_{1, p}$ is equivalent to the usual DT modulus

$$\omega^k_\varphi (f, \delta) := \sup_{0 < h \leq \delta} \| \Delta_h^k (f) \|_p,$$

where $\Delta_h^k (f) := \Delta_h^k (\cdot, \cdot)$. We note that the moduli $\omega^k_\varphi (f, \delta)_{w, p, \beta}$ are usually defined with the restriction $\alpha, \beta \geq 0$ for all $p \leq \infty$ and not just for $p = \infty$. The reason for this is that if $\alpha < 0$ or $\beta < 0$, then there are functions $f$ in $\mathbb{F}_p^{a, \beta}$ for which $\omega^k_\varphi (f, \delta)_{w, p, \beta} = \infty$ (see, e.g., [7] for more discussion).

At the same time, if $\alpha, \beta \geq 0$, then for all $f \in \mathbb{F}_p^{a, \beta}$, $1 \leq p \leq \infty$, $\omega^k_\varphi (f, A, \delta)_{w, p}$ and $\omega^k_\varphi (f, A, \delta)_{w, p}$ are equivalent. Namely, the following can be proved using the same method as in [2, Theorem 2.1 and Proposition 4.2].

**Lemma 1.1** If $k \in \mathbb{N}$, $A > 0$, $\alpha, \beta \geq 0$ and $f \in \mathbb{F}_p^{a, \beta}$, $1 \leq p \leq \infty$, then there exists $\delta_0 > 0$ such that

$$\omega^k_\varphi (f, A, \delta)_{w, p} \sim \omega^k_\varphi (f, A, \delta)_{w, p}$$

for all $0 < \delta \leq \delta_0$.

The next theorem follows from [1, Theorem 3.1] (see also [12, Theorem 1.4]) in the case where $p = \infty$ and [9, Theorems 5.2 and 9.1] if $1 \leq p < \infty$.

**Theorem 1.2** Let $1 \leq p \leq \infty$, $\alpha, \beta \in J_p$, $k \in \mathbb{N}$, $A > 0$ and $f \in \mathbb{F}_p^{a, \beta}$. Then

$$E_n (f)_{w, p} \leq c \omega^k_\varphi (f, A, n^{-1})_{w, p}$$

and

$$\omega^k_\varphi (f, A, n^{-1})_{w, p} \leq cn^{-k} \sum_{i=1}^n i^{k-1} E_i (f)_{w, p}$$

where constants $c$ depend only on $k$, $p$, $\alpha$, $\beta$, and $A$.

**Remark 1.3** With the moduli $\omega^k_\varphi$ instead of $\omega^k_\varphi$ and $\alpha, \beta \geq 0$, Theorem 1.2 was proved in [11, Theorem 4].

**Corollary 1.4** Let $1 \leq p \leq \infty$, $\alpha, \beta \in J_p$, $k \in \mathbb{N}$, $A > 0$, and $f \in \mathbb{F}_p^{a, \beta}$. Then for $0 < y < k$,

$$E_n (f)_{w, p} = O(n^{-y}) \iff \omega^k_\varphi (f, A, \delta)_{w, p} = O(\delta^{-y})$$

where $\Delta$ denotes the set of all $\ell$-monotone functions on $(-1, 1)$ (i.e., $f \in \Delta$ if its $\ell$-th order divided difference $[t_1, \ldots, t_\ell; f] \geq 0$, for all collection of distinct points $(t_i)_{i=1}^\ell \subset (-1, 1)$, where we view $f$ as a representative of its class defined pointwise. Recall that $[t_1; f] := f (t_1)$ and for $\ell \geq 2$,

$$[t_1, \ldots, t_\ell; f] := \left( \left[ t_1, \ldots, t_{\ell-1}; f \right] - \left[ t_2, \ldots, t_\ell; f \right] \right) / (t_1 - t_\ell).$$
In particular, note that $\Delta^1$ and $\Delta^2$ are the cones of all monotone and convex functions on $(-1,1)$, respectively. We also let

$$E_n^\ell(f, I)_{w_p} := \inf_{p \in \Pi_n \cap \Delta^\ell} \| w(f - p_n) \|_{L_p(I)}, \quad E_n^\ell(f)_{w_p} := E_n^\ell(f, [-1,1])_{w_p}.$$ 

The following theorem is the main result in this paper.

**Theorem 1.5** Let $\ell = 1$ or $\ell = 2, 1 \leq p \leq \infty$, $A > 0$, $\alpha, \beta \in I_p$, and let $f \in L_p^{a,\beta} \cap \Delta^\ell$. Then

$$E_n^\ell(f)_{w_{a,\beta}, p} \leq c \omega^\ell(f, A, n^{-1})_{w_{a,\beta}, p}, \quad \text{for all } n \geq \ell + 1. \quad (1.2)$$

**Remark 1.6** Using exactly the same proof as in [13], the fact that all norms in finite dimensional spaces are equivalent (and so, for example, $\| P \|_\infty \sim \| u P \|_\infty \sim \| w_{a,\beta} P \|_p$ with equivalence constants depending on $n, \alpha, \beta$ and $p$), and the estimate

$$\omega_k^\ell(f, A, t)_{w_{a,\beta}, p} \leq c \| w_{a,\beta} f \|_p,$$

for $k \in \mathbb{N}$ and $A, t > 0$, it is possible to show that the estimate (1.2) is exact in the sense that $\omega^\ell(f, A, t)$ in (1.2) cannot be replaced by $\omega_k^\ell$ with $k \geq \ell + 2$.

**Remark 1.7** It suffices to prove Theorem 1.5 for sufficiently large $n$, since for small $n$, it immediately follows from the observation that for $f \in L_p^{a,\beta} \cap \Delta^\ell, \ell = 1$ or $\ell = 2$,

$$E_n^\ell(f)_{w_{a,\beta}, p} = E_{n+1}^\ell(f)_{w_{a,\beta}, p}.$$ 

**Corollary 1.8** Let $\ell = 1$ or $\ell = 2, 1 \leq p \leq \infty$, $A > 0$, $\alpha, \beta \geq 0$, and let $f \in L_p^{a,\beta} \cap \Delta^\ell$. Then

$$E_n^\ell(f)_{w_{a,\beta}, p} \leq c \omega^\ell(f, A, n^{-1})_{w_{a,\beta}, p}, \quad \text{for all } n \geq \ell + 1. \quad (1.3)$$

**Corollary 1.9** immediately follows from Theorem 1.5 and Lemma 1.1 for sufficiently large $n$ (i.e., $n \geq [1/\delta_0]$). If $\ell + 1 \leq n \leq [1/\delta_0] + n_0$, then $n^{-1} \sim (n_0 + 1)^{-1} \sim \delta_0$, and using (1.1), Lemma 1.1, and monotonicity of $\omega^\ell(f, A, t)_{w_{a,\beta}, p}$ in $t$, we have

$$E_n^\ell(f)_{w_{a,\beta}, p} \leq c \omega^\ell(f, A, n^{-1})_{w_{a,\beta}, p} \leq c \omega^\ell(f, A, (n_0 + 1)^{-1})_{w_{a,\beta}, p} \leq c \omega^\ell(f, A, n^{-1})_{w_{a,\beta}, p},$$

and so (1.3) is also verified for "small" $n$.

Theorems 1.2 and 1.5 imply the following result.

**Corollary 1.9** Let $\ell = 1$ or $\ell = 2, 1 \leq p \leq \infty$, $A > 0$, $\alpha, \beta \in I_p$, and $f \in L_p^{a,\beta} \cap \Delta^\ell$. Then for $0 < \gamma < 1 + \ell$, we have

$$E_n^\ell(f)_{w_{a,\beta}, p} = O(n^{-\gamma}) \iff \omega^\ell(f, A, \delta)_{w_{a,\beta}, p} = O(\delta^\gamma), \quad \delta > 0.$$ 

Finally, we note that one additional application of our results is given in Section 7.
2 Auxiliary Results

Let \( x_j := \cos(j\pi/n) \), \( 0 \leq j \leq n \), denote the Chebyshev nodes. We also put \( x_j := 1 \), \( j < 0 \) and \( x_j := -1 \), \( j > n \), and denote \( I_j := [x_j, x_{j-1}] \), \( -\infty < j < \infty \) and \( |I| := \text{meas}(I) \). For \( \nu \in \mathbb{N}_0 \), it is convenient to define

\[
I_j^{(\nu)} := [x_{j+\nu}, x_{j-\nu}] = \bigcup_{i=j-\nu}^{j+\nu} I_i.
\]

Note that \( I_j = I_j^{(0)} \). Finally, define

\[
\psi_j(x) := \frac{|I_j|}{|x - x_j| + |I_j|}, \quad 1 \leq j \leq n.
\]

The following lemma was essentially proved in [6] (see [6, Proposition 2, Lemmas 1 and 2, and estimates (59)])).

**Lemma 2.1** Let \( n \in \mathbb{N}, 1 \leq j \leq n-1 \), and let \( \mu \in \mathbb{N}, \mu \geq 10 \). Then there exist constants \( c_0 \) and \( c \), depending only on \( \mu \), and polynomials \( \sigma_j, \delta_j, \tilde{\sigma}_j, R_j \), and \( \overline{R}_j \) of degree \( \leq c_0n \) such that, for all \( x \in [-1,1] \),

\[
\begin{align*}
|\psi_j'(x)| &\leq c|I_j|\psi_j''(x), & |\psi_j''(x)| &\leq c|I_j|\psi_j'(x), \\
|\psi_j'(x)| &\leq c|I_j|\psi_j''(x), & \left|\chi_j(x) - \delta_j(x)\right| &\leq c\psi_j''(x), \\
\left|\chi_j(x) - \tilde{\sigma}_j(x)\right| &\leq c\psi_j''(x), & \left|\chi_j(x) - \delta_j(x)\right| &\leq c\psi_j''(x), \\
\left|\chi_j(x) - \sigma_j(x)\right| &\leq c\psi_j''(x), & \left|\chi_j(x) - \sigma'_j(x)\right| &\leq c\psi_j''(x), \\
\left|2\chi_j(x) - R_j(x)\right| &\leq c|I_j|\psi_j''(x), & \left|2\chi_j(x) - \overline{R}_j(x)\right| &\leq c|I_j|\psi_j''(x),
\end{align*}
\]

and

\[
\begin{align*}
\sigma_j''(x) &\geq 0, & -\delta_j'(x) &\geq -\chi_{j+1}(x), & \tilde{\sigma}_j'(x) &\geq \chi_{j-1}(x), \\
(x_j - x_{j-1})\sigma_j''(x) - R_j''(x) &\geq -2\chi_j(x), & (x_j - x_{j+1})\sigma_j''(x) + \overline{R}_j''(x) &\geq 2\chi_j(x),
\end{align*}
\]

where \( \chi_j(x) := \chi_{[x_j,1]}(x) \) is the characteristic function of \([x_j,1]\) and \( (x-x_j)_+^k := (x-x_j)^k \chi_{[x_j,1]}(x) \).

The following “restricted averaged main part modulus” (that we state here only for the Jacobi weights) was defined in [9]:

\[
\widetilde{\Omega}_h^k(f, \delta)_{\mathcal{L}_p(x), w_{a,\delta}} := \left( \frac{1}{\delta} \int_0^\delta \int_S |w_{a,\delta}(x) \Delta_h^k(x) (f, x, S)|^p dx dh \right)^{1/p},
\]

where \( S \subset [-1,1] \) is independent of \( h \).

It is convenient to denote

\[
\mathcal{I}_{A,\delta} := [-1 + A\delta^2, 1 - A\delta^2].
\]
The following lemma immediately follows from [9, Lemma 4.2] (with $\theta = 1$) taking into account that
\[
\Omega^k_{\phi}(f, \delta)_{L_p(J_{\alpha,B}, w_{\alpha,B})} \leq \Omega^k_{\phi}(f, A, \delta)_{w_{\alpha,B}, p}
\]
and $\omega_k(f, J, I)_p \sim E_k(f, J)_{1,p} =: E_k(f, J)_p$, for $f \in L_p(f)$.

**Lemma 2.2** Let $1 \leq p < \infty$, $\alpha, \beta \in \mathbb{L}_p^{a,B}$, $n, k \in \mathbb{N}$, and let $A > 0$ be arbitrary. Denote
\[
I^* := \{1 \leq i \leq n \mid I_i \subset J_{A,1/n}\},
\]
and suppose that for each $i \in I^*$, the interval $I_i$ is such that $I_i \subset \tilde{I}_i \subset J_{A,1/n}$ and $|\tilde{I}_i| \leq c_0 |I_i|$. Then
\[
\sum_{i \in I^*} \left[ w_{\alpha,B}(x_i) E_k(f, \tilde{I}_i)_p \right]^p \leq c \Omega^k_{\phi}(f, A, 1/n)_w_{\alpha,B, p},
\]
where the constant $c$ depends only on $k$, $\alpha$, $\beta$, $p$, $c_0$, and $A$.

An analog of this lemma is also valid in the case $p = \infty$, and its proof is straightforward.

**Lemma 2.3** Let $\alpha, \beta \geq 0$, $f \in \mathbb{L}_p^a$, $n, k \in \mathbb{N}$, and let $A > 0$ be arbitrary. Suppose that for each $i \in I^*$ (with $I^*$ defined in (2.3)), the interval $I_i$ is such that $I_i \subset \tilde{I}_i \subset J_{A,1/n}$ and $|\tilde{I}_i| \leq c_0 |I_i|$. Then
\[
\sup_{i \in I^*} w_{\alpha,B}(x_i) E_k(f, \tilde{I}_i)_\infty \leq c \Omega^k_{\phi}(f, A, 1/n)_{w_{\alpha,B}, \infty},
\]
where the constant $c$ depends only on $k$, $\alpha$, $\beta$, $c_0$, and $A$.

**Corollary 2.4** Let $1 \leq p \leq \infty$, $A > 0$, $\alpha, \beta \in \mathbb{L}_p^{a,B}$, $k \in \mathbb{N}$, and $v \in \mathbb{N}_0$. Then for each $n \in \mathbb{N}$, we have
\[
\sup_{1 \leq \ell \leq n} E_k(f, I_1^{(v)})_{w_{\alpha,B}, \infty} \leq c \omega^k_{\phi}(f, A, 1/n)_{w_{\alpha,B}, \infty}, \quad \text{if } p = \infty,
\]
\[
\sum_{i=1}^n E_k(f, I_1^{(v)})_{w_{\alpha,B}, p}^p \leq c \omega^k_{\phi}(f, A, 1/n)_{w_{\alpha,B}, p}^p, \quad \text{if } p < \infty,
\]
where the constants $c$ depend only on $k$, $A$, $\alpha$, $\beta$, $p$, and $v$.

**Proof** In the case $p < \infty$, since $[x_{n-1}, x_1] \subset J_{B,1/n}$ if $B \leq B_0 := 2$, and $[x_{2v+1}, 1] \subset [1 - 2C/n^2, 1]$ and $[-1, x_{n-2v-1}] \subset [-1 - 1 + 2C/n^2]$ if $C \geq C_0 := 16(v+1)^2$, and taking into account that the sets $\{1, \ldots, v+1\}$ and $\{n - v, \ldots, n\}$ may have a non-empty
intersection, we have by Lemma 2.2,
\[
\sum_{i=1}^{n} E_k(f, I_j^{(r)})^p_{w_{a,p}}
\leq \left( \sum_{i=1}^{n} \sum_{i=1}^{n-1} \sum_{i=n-v}^{n} \right) E_k(f, I_j^{(r)})^p_{w_{a,p}}
\leq c E_k(f, [x_{2v+1}, 1])^p_{w_{a,p}} + c \sum_{i=1}^{n-v} \left[ w_{a,p}(x_i) E_k(f, I_j^{(r)})^p_{w_{a,p}} \right]
+ c E_k(f, [-1, x_{n-2v-1}])^p_{w_{a,p}}
\leq c \Omega_k^p(f, B_0, 1/n)^p_{w_{a,p}} + c E_k(f, [-1, 2C_0/n^2])^p_{w_{a,p}}
+ c E_k(f, [1, 1] - 2C_0/n^2]^p_{w_{a,p}}
\leq c \omega_k^p(f, B_0, 1/n)^p_{w_{a,p}} + c \omega_k^p(f, C_0, 1/n)^p_{w_{a,p}}
\leq c \omega_k^p(f, A, 1/n)^p_{w_{a,p}}.
\]
The case for \( p = \infty \) is analogous. \hfill \blacksquare

3 Constrained Approximation by Splines

For \( n, k \in \mathbb{N} \) and \( r \in \mathbb{N}_0 \), we write \( s \in \mathbb{S}_{k,n}^r \) if \( s|_{(x_i, x_{i+1})} \in \Pi_k, 1 \leq i \leq n \), and \( s \in \mathbb{C}^r[-1, 1] \). It is also convenient to denote \( \mathbb{S}_{k,n}^{r-2} \) the set of all piecewise polynomials with (possible) discontinuities at \( x_i, 1 \leq i \leq n - 1 \). We remark that the fact that a piecewise polynomial \( s \) is \( \ell \)-monotone imposes smoothness on \( s \) and so, for example, \( \mathbb{S}_{k,n} \cap \Delta^\ell \subset \mathbb{S}_{k,n}^{r-2}, \ell \geq 2 \).

The following lemma follows immediately from (if \( \ell = 2 \)) or can be proved similarly to (if \( \ell = 1 \)) [3, Theorem 1.2].

**Lemma 3.1** Let \( \ell = 1 \) or \( \ell = 2 \), and \( 1 \leq p \leq \infty \). Then for every \( g \in \Delta^\ell \cap \mathbb{L}_p, n \in \mathbb{N} \), there exists \( \tilde{S}(g) \in \mathbb{S}_{k,n}^{r-1} \cap \Delta^\ell \) and an absolute constant \( \eta \in \mathbb{N} \) such that
\[
\| g - \tilde{S}(g) \|_{L_p(I_j)} \leq c E_{\ell+1}(g, I_j^{(r)})^p, \quad 1 \leq j \leq n.
\]

We will now prove the following theorem.

**Theorem 3.2** Let \( \ell \in \mathbb{N}, \nu \in \mathbb{N}_0, r \in \mathbb{N}_0 \cup \{-1\} \) and \( 1 \leq p \leq \infty \). Suppose that for every \( g \in \Delta^\ell \cap \mathbb{L}_p \) and \( n \in \mathbb{N} \), there exists a spline \( \tilde{S}(g) \in \mathbb{S}_{k,n}^{r-1} \cap \Delta^\ell \) such that
\[
\| g - \tilde{S}(g) \|_{L_p(I_j)} \leq c_0 E_{\ell+1}(g, I_j^{(r)})^p, \quad 1 \leq j \leq n.
\]

Then for all \( \alpha, \beta \in I_p, \nu \in \mathbb{N}, r \in \mathbb{L}_p^{\alpha, \beta} \cap \Delta^\ell \), there exists \( \nu \in \mathbb{N} \), depending only on \( \eta \), and a spline \( S \in \mathbb{S}_{k,n}^{r-1} \cap \Delta^\ell \) such that
\[
\| w_{\alpha, \beta}(f - S) \|_{L_p(I_j)} \leq c E_{\ell+1}(f, I_j^{(r)})^p_{w_{a,p}}, \quad 1 \leq j \leq n,
\]
where \( c \) depends only on \( \alpha, \beta, \eta \) and \( c_0 \).
Corollary 3.3 Let $\ell = 1$ or $\ell = 2$, $1 \leq p < \infty$, $x, y \in I_p$, $n \in \mathbb{N}$, and $f \in L_p^{\alpha, \beta} \cap \Delta^\ell$. Then there exists $S \in \mathbb{G}_{\ell+1,n} \cap \Delta^\ell$ and an absolute constant $c \in \mathbb{R}$ such that
\[
\left\| w_{a_\beta}(f - S) \right\|_{L_p(I_j)} \leq c E_{\ell+1}(f, I_j^{(n)}) \right\|_{w_{a_\beta, p}}, \quad 1 \leq j \leq n,
\]
where $c$ depends only on $\alpha$ and $\beta$.

Proof of Theorem 3.2 First, we assume that $n \geq 2\eta + 3$.

Note that a function $f$ from $L_p^{\alpha, \beta} \cap \Delta^\ell$ does not have to belong to $L_p$ (for example, a convex function $f(x) = (1 + x)^{-1}$ is clearly not in $L_1$, but it is in $L_1^{\alpha, \beta}$ for any $\alpha > 0$ and $\beta > 1$). Hence, given $f \in L_p^{\alpha, \beta} \cap \Delta^\ell$ we modify it near $\pm 1$ so that it becomes an $L_p$ function. Recalling that any function from $\Delta^\ell$ has continuous derivatives of order $i$, $0 \leq i \leq \ell - 2$ on $(-1, 1)$ and that $f^{(\ell-1)}(\pm 1)$ exist for every $x \in (-1, 1)$, we let
\[
T_i(x) := \sum_{i=0}^{\ell-1} f^{(i)}(x_i)(x - x_i)^i / i! \quad \text{and} \quad T_{n-1}(x) := \sum_{i=0}^{\ell-1} f^{(i)}(x_{n-1})(x - x_{n-1})^i / i!,
\]
where $f^{(\ell-1)}(x_i)$ and $f^{(\ell-1)}(x_{n-1})$ are understood as $f^{(\ell-1)}(x_i^+)$ and $f^{(\ell-1)}(x_{n-1}^-)$, respectively. In other words, $T_i$ and $T_{n-1}$ are Taylor's polynomials from $\Pi_\ell$ for $f$ at the points $x_i$ and $x_{n-1}$, respectively. We now define
\[
\tilde{f}(x) := \begin{cases} 
T_{n-1}(x) & \text{if } x \in I_n, \\
f(x) & \text{if } x \in [x_{n-1}, x_1], \\
T_i(x) & \text{if } x \in I_i,
\end{cases}
\]
and note that $f \in \Delta^\ell$ is necessarily bounded in $[x_{n-1}, x_1]$ so that $\tilde{f} \in L_p \cap \Delta^\ell$.

Suppose now that $\tilde{S} := \tilde{S}(\tilde{f}) \in \mathbb{G}_{\ell+1,n} \cap \Delta^\ell$ is such that
\[
\left\| \tilde{f} - \tilde{S} \right\|_{L_p(I_j)} \leq c_9 E_{\ell+1}(\tilde{f}, I_j^{(\eta)})_p, \quad 1 \leq j \leq n.
\]

For $j = \eta + 2$ and $j = n - 1 - \eta$, let $\tilde{S}_j$ be the polynomials of degree $\ell + 1$ defined on $[-1, 1]$ such that $\tilde{S}_j | I_j = \tilde{S} | I_j$. Observing that $I_j^{(\eta)} \subset [x_{n-1}, x_1]$ if $2 \leq j < n - 1 - \eta$, we define
\[
S(x) := \begin{cases} 
\tilde{S}_{n-1 - \eta}(x) & \text{if } x \in [-1, x_{n-1 - \eta}], \\
\tilde{S}_{n-1}(x) & \text{if } x \in [x_{n-1 - \eta}, x_{n-1 + 1}], \\
\tilde{S}_{n-2}(x) & \text{if } x \in [x_{n-1 + 1}, 1].
\end{cases}
\]
Evidently, $S \in \mathbb{G}_{\ell+1,n} \cap \Delta^\ell$.

We now note that
\[
w_{a_\beta}(x) \sim w_{a_\beta}(x_j) \quad \forall x \in I_j^{(\eta)}, \quad \text{if } I_j^{(\eta)} \subset [x_{n-1}, x_1].
\]
Note that this holds for $\eta + 2 \leq j \leq n - 1 - \eta$. Therefore, for all such $\eta$ and $j$,
\[
\left\| w_{a_\beta} g \right\|_{L_p(I_j^{(\eta)})} \sim \left\| w_{a_\beta}(x_j) \right\|_{L_p(I_j^{(\eta)})}.
\]
and $$E_{\varepsilon+1}(f, I_j^{(n)})_{w_{a,\beta}p} \sim w_{a,\beta}(x_j)E_{\varepsilon+1}(f, I_j^{(n)})_p.$$ Hence, (3.2) implies that for $$\eta + 2 \leq j \leq n - 1 - \eta,$$ we have

$$\begin{align*}
\|w_{a,\beta}(f - S)\|_{L_p(I_j)} &= \|w_{a,\beta}(\tilde{f} - \tilde{S})\|_{L_p(I_j)} \\
&\leq c_0w_{a,\beta}(x_j)E_{\varepsilon+1}(\tilde{f}, I_j^{(n)})_p = c_0w_{a,\beta}(x_j)E_{\varepsilon+1}(f, I_j^{(n)})_p \\
&\sim E_{\varepsilon+1}(f, I_j^{(n)})_{w_{a,\beta}p}.
\end{align*}$$

In the case $$j = \eta + 2,$$ (3.4) becomes

$$\|w_{a,\beta}(f - \tilde{S}_{\eta+2})\|_{L_p(I_{\eta+2})} \leq cE_{\varepsilon+1}(f, I_{\eta+2}^{(n)})_{w_{a,\beta}p} = cE_{\varepsilon+1}(f, [x_{2\eta+2}, x_1])_{w_{a,\beta}p}.$$ Now, suppose that $$1 \leq j \leq \eta + 1$$ (for $$n - \eta \leq j \leq n$$ the proof is similar). We will show that

$$\|w_{a,\beta}(f - q)\|_{L_p[x_{\eta+1}, 1]} \leq cE_{\varepsilon+1}(f, [x_{2\eta+2}, 1])_{w_{a,\beta}p},$$

To this end, if $$q \in \Pi_{\varepsilon+1}$$ is a polynomial of (near)best weighted approximation of $$f$$ on $$[x_{2\eta+2}, 1],$$ i.e.,

$$\|w_{a,\beta}(f - q)\|_{L_p[x_{\eta+1}, 1]} \leq cE_{\varepsilon+1}(f, [x_{2\eta+2}, 1])_{w_{a,\beta}p},$$

then we have

$$\begin{align*}
\|w_{a,\beta}(f - S)\|_{L_p(I_j)} &= \|w_{a,\beta}(f - \tilde{S}_{\eta+2})\|_{L_p(I_j)} \\
&\leq cE_{\varepsilon+1}(f, [x_{2\eta+2}, 1])_{w_{a,\beta}p} + c\|w_{a,\beta}(q - \tilde{S}_{\eta+2})\|_{L_p[I_{\eta+2}]} \\
&\leq cE_{\varepsilon+1}(f, [x_{2\eta+2}, 1])_{w_{a,\beta}p} + c\|w_{a,\beta}(q - \tilde{S}_{\eta+2})\|_{L_p[I_{\eta+2}]} \\
&\leq cE_{\varepsilon+1}(f, [x_{2\eta+2}, 1])_{w_{a,\beta}p} + c\|w_{a,\beta}(f - \tilde{S}_{\eta+2})\|_{L_p[I_{\eta+2}]} \\
&\leq cE_{\varepsilon+1}(f, [x_{2\eta+2}, 1])_{w_{a,\beta}p} + c\|w_{a,\beta}(f - \tilde{S}_{\eta+2})\|_{L_p[I_{\eta+2}]} \\
&\leq cE_{\varepsilon+1}(f, [x_{2\eta+2}, 1])_{w_{a,\beta}p},
\end{align*}$$

which verifies (3.5). Here, for the third inequality we used the fact that

(3.6) $$\|w_{a,\beta}Q\|_{L_p(I_j)} \leq c\|w_{a,\beta}Q\|_{L_p(I_j)},$$

which, in the case $$p < \infty,$$ follows from the proof of [9, Lemma 12.1]. In the case $$p = \infty,$$ we observe that since $$a, \beta \geq 0,$$ we have, for all $$x \in [x_{\eta+2}, 1],$$

$$w_{a,\beta}(x) \leq 2^a(1 - x) \beta \leq 2^a(1 - x_{\eta+2}) \beta \leq c \min_{u \in [x_{\eta+2}, 1]} w_{a,\beta}(u),$$

$$\|q - \tilde{S}_{\eta+2}\|_{L_\infty[I_{\eta+2}]} \leq c\|q - \tilde{S}_{\eta+2}\|_{L_\infty[I_{\eta+2}]}.$$ Hence,

$$\|w_{a,\beta}(q - \tilde{S}_{\eta+2})\|_{L_\infty[I_{\eta+2}]} \leq c\|w_{a,\beta}(q - \tilde{S}_{\eta+2})\|_{L_\infty[I_{\eta+2}]}.$$ Combining (3.4) and (3.5) we conclude that (3.1) is valid with $$\nu = 2\eta + 1,$$ if $$n \geq 2\eta + 3.$$
For $1 \leq n \leq 2\eta + 2$, the statement of the theorem follows from the case for $n_0 := 2\eta + 3$. Indeed, suppose that $S \in S_{t+1,n_0} \cap \Delta^t$ is such that (3.1) is satisfied with $n = n_0$. Let $s_1 := S \mid_{t}$, and define $Q(x) := s_1(x), x \in [-1,1]$. Then, evidently, $Q \in \Pi_{t+1} \cap \Delta^t$ and
\[ \|w_{\alpha,\beta}(f - Q)\|_{L_p[\cos(\pi/n_0),1]} \leq cE_{t+1}(f)_{w_{\alpha,\beta},p}. \]

Now, letting $P \in \Pi_{t+1}$ be such that
\[ \|w_{\alpha,\beta}(f - P)\|_p \leq cE_{t+1}(f)_{w_{\alpha,\beta},p}, \]
and using (3.6), we have
\[ \|w_{\alpha,\beta}(f - Q)\|_p \leq c \|w_{\alpha,\beta}(f - P)\|_p + c \|w_{\alpha,\beta}(P - Q)\|_p \leq cE_{t+1}(f)_{w_{\alpha,\beta},p} + c \|w_{\alpha,\beta}(P - Q)\|_{L_p[\cos(\pi/n_0),1]} \leq cE_{t+1}(f)_{w_{\alpha,\beta},p}, \]
and so (3.1) is verified with $\nu$ such that $\nu^{(v)} = [-1,1]$ for all $1 \leq j \leq n$. For example, $\nu = 2\eta + 1$ will do.

4 Additional Auxiliary Statements

**Lemma 4.1** ($1 \leq p < \infty$) Let $n \in \mathbb{N}, 1 \leq p < \infty, \alpha, \beta \in I_p$, and $y_j \geq 0, 1 \leq j \leq n - 1$. Then for
\[ \Sigma_p(x) := \Sigma_p(x, (y_j)_{j=1}^{n-1}) := \sum_{j=1}^{n-1} y_j |I_j|^{-1/p} \psi_j^p(x) \]
and sufficiently large $\mu$, we have
\[ \|w_{\alpha,\beta}(\cdot)\Sigma_p(\cdot)\|_p^p \leq c \sum_{j=1}^{n-1} w_{\alpha,\beta}^p(x_j) y_j^p. \]

**Proof** With the notation
\[ w_n(x) := \rho_n^{-1}(x) \int_{x_{j-1}}^{x_{j+1}} w(u) du, \quad \rho_n(x) := \sqrt{1-x^2} - n^{-1} + n^{-2}, \]
we have (see, e.g., [9, (5.1)])
\[ (4.1) \quad w_{\alpha,\beta}^p(x) \sim \left(w_{\alpha,\beta}^p\right)_n(x) \sim \left(w_{\alpha,\beta}^p\right)_n(x_j), \quad \text{for each } x \in I_j, \quad 2 \leq j \leq n - 1. \]
Also, for any doubling weight $w, n \in \mathbb{N}, 1 \leq j \leq n, x \in [-1,1]$, and $y \in I_j$,
\[ w_n(x) \leq c\psi_j^p(x) w_n(y) \quad \text{and} \quad w_n(y) \leq c\psi_j^p(x) w_n(x), \]
where constants $c$ and $s \geq 0$ depend only on the doubling constant of $w$ (see [8, Lemma 2.5]), and note that $w_{\alpha,\beta}^p, \alpha, \beta \in I_p$, is a doubling weight.
Hence, for appropriate $c$ and $s$, we have for all $x \in [-1, 1]$,

\begin{align*}
(4.2) \quad (w_{a, \beta}^p)_n(x) & \leq c \psi_j^s(x)(w_{a, \beta}^p)_n(y) \quad \text{and} \\
(w_{a, \beta}^p)_n(y) & \leq c \psi_j^s(x)(w_{a, \beta}^p)_n(x), \quad y \in I_j.
\end{align*}

For $1 \leq j \leq n - 1$, we estimate

\[
\int_{-1}^1 w_{a, \beta}^p(x) \psi_j^p(x) \, dx = \left( \int_a^b + \int_{x_{a-1}}^{x_a} + \int_h \right) w_{a, \beta}^p(x) \psi_j^p(x) \, dx =: J_l + J_c + J_r.
\]

By virtue of (4.1) and (4.2), if $\mu \geq (s + 2)/p$, we have

\[
J_c \leq c \int_{x_{a-1}}^{x_a} (w_{a, \beta}^p)_n(x) \psi_j^p(x) \, dx \leq c \int_{x_{a-1}}^{x_a} (w_{a, \beta}^p)_n(x_j) \psi_j^{\mu - s}(x) \, dx \\
\leq c (w_{a, \beta}^p)_n(x_j) \int_{-1}^1 \psi_j^{\mu - s}(x) \, dx \leq c |I_j| (w_{a, \beta}^p)_n(x_j) \sim |I_j| w_{a, \beta}^p(x_j),
\]

since $\int_{-1}^1 \psi_j^p(x) \, dx \leq c |I_j|$ if $j \geq 2$.

In order to estimate $J_l$ (considerations for $J_r$ are similar), we note that for each $x \in I_n, \psi_j(x) \sim \psi_j(-1)$, so that

\[
J_l \leq c \psi_j^p(-1) \int_{-1}^1 w_{a, \beta}^p(x) \, dx \leq c n^{-2} \psi_j^p(-1) (w_{a, \beta}^p)_n(-1) \\
\leq c n^{-2} \psi_j^{\mu - s}(-1) (w_{a, \beta}^p)_n(x_j) \leq c |I_j| w_{a, \beta}^p(x_j),
\]

since

\[
n^{-2} \psi_j^{\mu - s}(-1) \leq c |I_j|, \quad \text{if } \mu \geq s/p.
\]

Combining the above estimates, we conclude that for $\mu \geq (s + 2)/p$,

\[
\int_{-1}^1 w_{a, \beta}^p(x) \psi_j^p(x) \, dx \leq c |I_j| w_{a, \beta}^p(x_j), \quad 1 \leq j \leq n - 1.
\]

Hence, taking into account that $\sum_{j=1}^{n-1} \psi_j^p(x) \leq c$ and $\psi_j(x) \leq 1, 1 \leq j \leq n - 1$, and denoting $p' := p/(p - 1)$, we have, using Hölder’s inequality,

\[
\left\| w_{a, \beta}(\cdot) \Sigma_P(\cdot) \right\|_p^p \leq c \int_{-1}^1 w_{a, \beta}^p(x) \left( \sum_{j=1}^{n-1} y_j |I_j|^{-1/p} \psi_j^p(x) \right)^p \, dx \\
\leq c \int_{-1}^1 w_{a, \beta}^p(x) \left( \sum_{j=1}^{n-1} y_j |I_j|^{-1/p} \psi_j^{(\mu - 2)p}(x) \right) \left( \sum_{j=1}^{n-1} \psi_j^{p'}(x) \right)^{p/p'} \, dx \\
\leq c \int_{-1}^1 w_{a, \beta}^p(x) \sum_{j=1}^{n-1} y_j |I_j|^{-1/p} \psi_j^{(\mu - 2)p}(x) \, dx \leq c \sum_{j=1}^{n-1} w_{a, \beta}^p(x_j) y_j^{p'},
\]

provided $\mu \geq 2 + (s + 2)/p$.

**Lemma 4.2** ($p = \infty$) Let $n \in \mathbb{N}$, $p = \infty$, $\alpha, \beta \geq 0$, and $\gamma_j \geq 0$, $1 \leq j \leq n - 1$. Then for

\[
\Sigma_{\infty}(x) := \Sigma_{\infty}(x, (\gamma_j)_{j=1}^{n-1}) := \sum_{j=1}^{n-1} \gamma_j \psi_j^p(x)
\]
and sufficiently large $\mu$, we have

$$ \left\| w_{a,\beta}(\cdot)\Sigma_{\infty}(\cdot) \right\|_{\infty} \leq c \sup_{1 \leq j \leq n-1} w_{a,\beta}(x_j) y_j. $$

**Proof.** It is convenient to denote $W := \sup_{1 \leq j \leq n-1} w_{a,\beta}(x_j) y_j$. Then for every $x \in [-1,1]$, we have

$$ w_{a,\beta}(x)\Sigma_{\infty}(x) = \sum_{j=1}^{n-1} y_j w_{a,\beta}(x) \psi_j^\beta(x) \leq W \sum_{j=1}^{n-1} w_{a,\beta}(x_j) \psi_j^\beta(x) $$

$$ \leq W \sum_{j=1}^{n-1} \left(1 + \frac{|x-x_j|}{1+x_j}\right)^\alpha \left(1 + \frac{|x-x_j|}{1-x_j}\right)^\beta \psi_j^\beta(x) $$

$$ \leq W \sum_{j=1}^{n-1} \left(1 + \frac{|x-x_j|}{|I_{j+1}|}\right)^\alpha \left(1 + \frac{|x-x_j|}{|I_j|}\right)^\beta \psi_j^\beta(x). $$

Since $|I_{j+1}| \geq |I_j|/3$, this implies

$$ w_{a,\beta}(x)\Sigma_{\infty}(x) \leq 3^\alpha W \sum_{j=1}^{n-1} \psi_j^{\mu-a-\beta}(x) \leq cW, $$

since $\sum_{j=1}^{n-1} \psi_j^{\mu-a-\beta}(x) \leq c$, provided $\mu \geq \alpha + \beta + 2$. \hfill \blacksquare

## 5 Convex Approximation of Quadratic Splines by Polynomials

In this section, suppose that $g \in S_{3,\Delta^2}$. In other words, $g$ is a continuous quadratic convex spline on the Chebyshev partition. We now construct a polynomial that approximates $g$.

Denote by $L_j(x,g)$ the quadratic polynomial interpolating $g$ at $x_j, x_{j-1}$ and $x_{j-2}$, i.e.,

$$ L_j(x,g) := \sum_{j-2 \leq i \leq j} g(x_i) \prod_{j-2 \leq i \leq j, i \neq j} \frac{x-x_i}{x_i-x_l}. $$

For $n \geq 2$, let $S$ be a continuous piecewise quadratic polynomial with knots at $x_j, 1 \leq j \leq n-1$, such that

$$ S(x) := \max \{ L_j(x,g), L_{j+1}(x,g) \}, \quad x \in I_j, \quad 2 \leq j \leq n-1, $$

$$ S(x) := L_2(x,g), \quad x \in I_1, \quad \text{and} \quad S(x) := L_n(x,g), \quad x \in I_n. $$

Since $g$ is convex, so is $S$.

We need the following lemma.

**Lemma 5.1.** Let $1 \leq p \leq \infty$, $n \in \mathbb{N}$, $\kappa \in \mathbb{N}_0$, and $G \in S_{l,n}$, $l \geq 1$. Then for all $1 \leq j \leq n-1$, we have

$$ w_{a,\beta}(x_j)E_l(G, I_j^{(x)})_p \leq cE_l(G, I_j^{(x)})_{w_{a,\beta},p}. $$

**Proof.** First, we note that (5.1) is obvious if $I_j^{(x)} \subset [x_{n-1}, x_1]$. Thus, we assume that $1 \in I_j^{(x)}$ (the case for $-1 \in I_j^{(x)}$ is analogous). Then $I_j^{(x)} = [x_{j+\kappa}, 1]$ with $j \leq \kappa + 1$. 
Suppose that \( P \in \Pi_j \) is such that
\[
\left\| w_{a,\beta}(G - P) \right\|_{L_p(I_j)} \leq cE_l(G, I_j)w_{a,\beta, p}.
\]
Then for \( 1 \leq p < \infty \), we have
\[
w_{a,\beta}(x_j)E_l(G, I_j) \leq w_{a,\beta}(x_j) \left\| G - P \right\|_{L_p(I_j)}^{p}
\]
\[
= w_{a,\beta}(x_j)\left( \sum_{i=1}^{j} \left\| G - P \right\|_{L_p(I_i)}^{p} + \left\| G - P \right\|_{L_p(I_j)}^{p} \right)
\]
\[
\leq w_{a,\beta}(x_j)\left( \sum_{i=1}^{j} \left\| G - P \right\|_{L_p(I_i)}^{p} + c \left\| G - P \right\|_{L_p(I_{j-1})}^{p} \right)
\]
\[
\leq c \sum_{i=1}^{j+1} w_{a,\beta}(G - P) \left\| L_p(I_i) \right\| + c \left\| w_{a,\beta}(G - P) \right\|_{L_p(I_{j-1})}^{p}
\]
where for the second inequality we used the fact that \( (G - P) |_{I_i} \) is a polynomial of degree \( l \), and for the third inequality we have applied (3.3). The case for \( p = \infty \) is analogous. This completes the proof.

In particular, it follows from Lemma 5.1 that for any \( 1 \leq j \leq n - 1 \),
\[
w_{a,\beta}(x_j)E_3(g, I_j) \leq cE_3(g, I_j)w_{a,\beta, p}.
\]
Now, in view of the fact that \( g |_{I_j} \in \Pi_j \), \( 1 \leq i \leq n \), we have, with \( p_j \) denoting the best quadratic approximant to \( g \) in \( L_p(I_j) \), \( 1 \leq j \leq n \),
\[
E_3(g, I_j) \leq \max_{1 \leq i \leq j+1} \| g - p_j \|_{L_{\infty}(I_i)}
\]
\[
\leq c|I_j|^{-1/p} \max_{1 \leq i \leq j+1} \| g - p_j \|_{L_{\infty}(I_i)} \leq c|I_j|^{-1/p}E_3(g, I_j).
\]
Similarly, for all \( 2 \leq j \leq n - 1 \), we have
\[
[x_{j+1}, x_j, x_{j-1}, x_{j-2}; g] = [x_{j+1}, x_j, x_{j-1}, x_{j-2}; g - p_j]
\]
\[
\leq c|I_j|^{-3} \max_{1 \leq i \leq j+1} \| g - p_j \|_{L_{\infty}(I_i)}
\]
\[
\leq c|I_j|^{-3-1/p}E_3(g, I_j).
\]
Now, using (3.6), (5.2), (5.3), and Whitney's inequality, we obtain, for all \( 1 \leq j \leq n - 1 \),
\[
w_{a,\beta}(g - S) \leq \| w_{a,\beta}(x_j) \|_{L_p(I_j)} \leq \| g - S \|_{L_p(I_j)}
\]
\[
\leq c w_{a,\beta}(x_j) \left\| I_j \right\|_{L_{\infty}(I_j)}^{1/p} \left\| g - S \right\|_{L_{\infty}(I_j)}
\]
\[
\leq c w_{a,\beta}(x_j) |I_j|^{1/p} w_3 \left( g, I_j \right) \leq c w_{a,\beta}(x_j) |I_j|^{1/p} E_3(g, I_j)
\]
\[
\leq c w_{a,\beta}(x_j) E_3(g, I_j) \leq cE_3(g, I_j)w_{a,\beta, p}.
\]
Similarly, in the case $j = n$, we have
\begin{equation}
\|w_{a,\beta}(g - S)\|_{L_p(I_n)} \leq c w_{a,\beta}(x_{n-1}) \|g - S\|_{L_p(I_n)} \\
\leq c w_{a,\beta}(x_{n-1}) n^{-2/p} \|g - S\|_{L_\infty(I_n)} \\
\leq c w_{a,\beta}(x_{n-1}) n^{-2/p} a_3\left( g, \|f^{(1)}_n\|_{L_1(I_n)} \right) \\
\leq c w_{a,\beta}(x_{n-1}) n^{-2/p} E_3(g, I^{(1)}_n) \\
\leq c w_{a,\beta}(x_{n-1}) E_3(g, I^{(1)}_n) \leq c E_3(g, I^{(1)}_n) w_{a,\beta,p}.
\end{equation}

It was shown in [6, Section 4] that all knots $x_j, 1 \leq j \leq n - 1$, can be separated into classes I, II, III, and IV so that $S$ has the following representation
\begin{align*}
S(x) &= q_2(x) + \sum_{2 \leq j \leq n-1, j \neq 1, n, x_j \in \text{II} \cup \text{III}} A_j \left( (x_{j-1} - x_j)(x - x_j) + (x - x_j)^2 \right) \\
&\quad + \sum_{1 \leq j \leq n-2, x_j \in \text{III} \cap \text{II}} (-A_{j+1}) \left( (x_{j+1} - x_j)(x - x_j) + (x - x_j)^2 \right),
\end{align*}
where
\begin{align*}
q_2(x) &= g(-1) + (\left[x_n, x_{n-1}; g\right] - \left[x_{n-1}, x_{n-2}; g\right] + \left[x_n, x_{n-2}; g\right])(x + 1) \\
&\quad + \left[x_n, x_{n-1}, x_{n-2}; g\right](x + 1)^2,
\end{align*}
and
\begin{align*}
A_j &= \left[x_{j+1}, x_j, x_{j-1}; g\right] - \left[x_j, x_{j-1}, x_{j-2}; g\right], \quad 2 \leq j \leq n - 1,
\end{align*}
is such that $A_j > 0$ if $x_j \in \text{I} \cup \text{III}$, and $A_{j+1} < 0$ if $x_j \in \text{II} \cup \text{III}$.

Then the polynomial
\begin{align*}
P_n(x) := P_n(x, g) := q_2(x) + \sum_{2 \leq j \leq n-1, j \neq 1, n, x_j \in \text{II} \cup \text{III}} A_j \left( (x_{j-1} - x_j) \sigma_j(x) - R_j(x) \right) \\
&\quad + \sum_{1 \leq j \leq n-2, x_j \in \text{III} \cap \text{II}} (-A_{j+1}) \left( (x_{j+1} - x_j) \sigma_j(x) + \overline{R}_j(x) \right)
\end{align*}
of degree $\leq cn$ is convex on $[-1, 1]$, since by (2.2),
\begin{align*}
P''_n(x) &\geq S''(x) \geq 0, \quad x \in [-1, 1], \quad x \neq x_j, \quad 1 \leq j \leq n - 1.
\end{align*}

Also, by (2.1) and (5.4),
\begin{align*}
\left| P_n(x) - S(x) \right| &\leq c \sum_{j=2}^{n-1} |A_j| |I_j|^2 \psi_j^\mu(x) \\
&\leq c \sum_{j=2}^{n-1} \left| \left[x_{j+1}, x_j, x_{j-1}, x_{j-2}; g\right] \right| |I_j|^3 \psi_j^\mu(x) \\
&\leq c \sum_{j=2}^{n-1} |I_j|^{-1/p} \psi_j^\mu(x) E_3(g, I^{(1)}_j) \leq c E_3(g, I^{(1)}_j) w_{a,\beta,p}.
\end{align*}

Hence, in the case $1 \leq p < \infty$, using Lemma 4.1 and (5.2), we have
\begin{align*}
\left\| w_{a,\beta}(P_n - S) \right\|_p^p &\leq c \sum_{j=1}^{n-1} w_{a,\beta}^p(x_j) E_3(g, I^{(1)}_j)^p \leq c \sum_{j=1}^{n-1} E_3(g, I^{(1)}_j)^p w_{a,\beta,p}.
\end{align*}
and, combining with (5.5) and (5.6), we conclude that

\[ \left\| w_{a,b}(P_n - g) \right\|_p^p \leq c \sum_{j=1}^{n} E_3(g, I_j^{(1)})^p w_{a,b,p}. \]

In the case where \( p = \infty \), using Lemma 4.2 and (5.2) we have

\[ \left\| w_{a,b}(P_n - S) \right\|_{\infty} \leq c \sup_{1 \leq j \leq n} w_{a,b}(x_j) E_3(g, I_j^{(1)}) \leq c \sup_{1 \leq j \leq n} E_3(g, I_j^{(1)}) w_{a,b,\infty} \]

and, combining with (5.5) and (5.6), we get

\[ \left\| w_{a,b}(P_n - g) \right\|_{\infty} \leq c \sup_{1 \leq j \leq n} E_3(g, I_j^{(1)}) w_{a,b,\infty}. \]

We will now show that the first derivative of the polynomial \( P_n \) approximates the first derivative of \( g \) (if it exists). This fact will be used to obtain the estimates in the monotone case as a corollary. Suppose that \( g \in \mathbb{C}^1[-1,1] \).

For \( 1 \leq j \leq n - 1 \), we have

\[ \left\| w_{a,b}(g' - S') \right\|_{L_p(I_j)} \leq c w_{a,b}(x_j) \left\| g' - S' \right\|_{L_p(I_j)} \]

\[ \leq c w_{a,b}(x_j) \left| I_j \right|^{-1/p} \left\| g - S \right\|_{L_p(I_j)} \]

\[ \leq c w_{a,b}(x_j) \left| I_j \right|^{-1/p} \omega_3(g, I_j^{(1)}, I_j^{(1)}) \]

\[ \leq c w_{a,b}(x_j) \left| I_j \right|^{1/p} \omega_2(g', I_j^{(1)}) \]

\[ \leq c w_{a,b}(x_j) \left| I_j \right|^{1/p} E_2(g', I_j^{(1)}) \leq c E_2(g', I_j^{(1)}) w_{a,b,p}, \]

where for the second inequality we used the fact that \( (g - S) \mid_{I_j} \in \Pi_3 \). Similarly (as in (5.6)),

\[ \left\| w_{a,b}(g' - S') \right\|_{L_p(I_n)} \leq c E_2(g', I_n^{(1)}) w_{a,b,p}. \]

Also, for all \( x \in [-1,1] \setminus \{ x_j \}_{j=1}^{n-1} \),

\[ \left| P_n'(x) - S'(x) \right| \]

\[ \leq \sum_{2 \leq j \leq n-1, j \in \Pi_3} |A_j| \left[ (x_{j-1} - x_j) \left| \sigma'_j(x) - \chi_j(x) \right| + \left| R'_j(x) - 2(x - x_j) \right| \right] \]

\[ + \sum_{1 \leq j \leq n-2, j \in \Pi_3} |A_{j+1}| \left[ (x_j - x_{j+1}) \left| \sigma'_j(x) - \chi_j(x) \right| + \left| \overline{R'_j}(x) - 2(x - x_j) \right| \right] \]

\[ \leq c \sum_{j=2}^{n-1} |A_j| \left| I_j \right| \psi_j(x) \leq c \sum_{j=2}^{n-1} \left| x_{j+1} - x_j \right| \left| \sigma_j(x) - \chi_j(x) \right| \left| I_j \right| \psi_j(x) \]

\[ \leq c \sum_{j=2}^{n-1} \left| I_j \right|^{-1/\rho} \psi_j(x) E_3(g, I_j^{(1)}) \leq c \sum_{j=2}^{n-1} \left| I_j \right|^{-1/\rho} \psi_j(x) E_2(g', I_j^{(1)}) \]

Therefore, Lemmas 4.1, 4.2 and 5.1 imply that

\[ \left\| w_{a,b}(P_n - S') \right\|_{p}^p \leq c \sum_{j=1}^{n-1} E_2(g', I_j^{(1)})^p w_{a,b,p}, \quad \text{if} \ 1 \leq p < \infty, \]
and
\[
\|w_{\alpha, \beta}(P'_n - g')\|_{\infty} \leq c \sup_{1 \leq j \leq n} E_2(g', I^{(1)}_j)_{w_{\alpha, \beta}, \infty}, \quad \text{if } p = \infty.
\]

Hence,
\[
(5.9) \quad \|w_{\alpha, \beta}(P'_n - g')\|_p^p \leq c \sum_{j=1}^n E_2(g', I^{(1)}_j)_{w_{\alpha, \beta}, p}^p, \quad \text{if } 1 \leq p < \infty,
\]
and
\[
(5.10) \quad \|w_{\alpha, \beta}(P'_n - g')\|_{\infty} \leq c \sup_{1 \leq j \leq n} E_2(g', I^{(1)}_j)_{w_{\alpha, \beta}, \infty}, \quad \text{if } p = \infty.
\]

6 Constrained Approximation by Polynomials

Suppose that \( \ell = 1 \) or \( \ell = 2, 1 \leq p \leq \infty, \alpha, \beta \in J_p, f \in L^p_{\alpha, \beta} \cap \Delta^\ell \), and let \( n \in \mathbb{N} \) be sufficiently large. Corollary 3.3 implies that there exists \( g \in S_{\ell+1, n} \cap \mathbb{C}[-1, 1] \cap \Delta^\ell \) and \( v \in \mathbb{N} \) such that
\[
(6.1) \quad \|w_{\alpha, \beta}(f - g)\|_{L^p(t_j)} \leq c E_{\ell+1}(f, I^{(v)}_j)_{w_{\alpha, \beta}, p}, \quad 1 \leq j \leq n.
\]

Therefore,
\[
(6.2) \quad \|w_{\alpha, \beta}(f - g)\|_p^p = \sum_{j=1}^n \|w_{\alpha, \beta}(f - g)\|_{L^p(t_j)}^p \leq c \sum_{j=1}^n E_{\ell+1}(f, I^{(v)}_j)_{w_{\alpha, \beta}, p}^p,
\]
if \( 1 \leq p < \infty \), and
\[
(6.3) \quad \|w_{\alpha, \beta}(f - g)\|_{\infty} \leq c \sup_{1 \leq j \leq n} E_{\ell+1}(f, I^{(v)}_j)_{w_{\alpha, \beta}, \infty},
\]
if \( p = \infty \).

We note that (6.1) implies, for any \( \kappa \in \mathbb{N}_0 \) and \( 1 \leq j \leq n \),
\[
(6.4) \quad E_{\ell+1}(g\ell, I^{(\kappa)}_j)_{w_{\alpha, \beta}, p} \leq c E_{\ell+1}(f, I^{(v)}_j)_{w_{\alpha, \beta}, p} + c \|w_{\alpha, \beta}(g\ell - f)\|_{L^p(t_j^{(v)})}
\]
\[
\leq c E_{\ell+1}(f, I^{(v+\kappa)}_j)_{w_{\alpha, \beta}, p}.
\]

6.1 Convex Approximation

Suppose that \( \ell = 2 \), thus \( g \) is a continuous piecewise quadratic convex spline on \([-1, 1]\). Let \( P_n(\cdot, g) \) denote the polynomial associated with \( g \) satisfying (5.7) or (5.8). Then it follows from (6.2), (6.3), and (6.4) that
\[
\|w_{\alpha, \beta}(f - P_n(g))\|_p^p \leq c \sum_{j=1}^n E_3(f, I^{(v)}_j)_{w_{\alpha, \beta}, p}^p + c \sum_{j=1}^n E_3(g, I^{(1)}_j)_{w_{\alpha, \beta}, p}^p
\]
\[
\leq c \sum_{j=1}^n E_3(f, I^{(v+1)}_j)_{w_{\alpha, \beta}, p}^p,
\]
if $1 \leq p < \infty$, and
\[
\left\| w_{a,\beta}(f - P_n(g_2)) \right\|_\infty \leq c \sup_{1 \leq j \leq n} E_3(f, I_j^{(v)})_{w_{a,\beta},\infty} + c \sup_{1 \leq j \leq n} E_3(g_2, I_j^{(1)})_{w_{a,\beta},\infty}
\]
\[
\leq c \sup_{1 \leq j \leq n} E_3(f, I_j^{(v+1)})_{w_{a,\beta},\infty},
\]
if $p = \infty$.

The statement of Theorem 1.5 in the case $\ell = 2$ now follows from Corollary 2.4.

### 6.2 Monotone Approximation

If $\ell = 1$, then $g_1$ is a continuous piecewise linear monotone spline on $[-1,1]$. We now define $G(x) := \int_{x}^{\frac{1}{x}} g_1(u) du$. Then $G \in \mathbb{S}_{3\alpha} \cap \mathbb{C}’[-1,1] \cap \Delta^2$. Let $P_n(\cdot, G)$ be the polynomial associated with $G$ satisfying (5.7) and denote $Q_n(x) := P_n(x, G)$.

It follows that $Q_n \in \Delta^1$, and estimates (5.9) and (5.10) imply
\[
\left\| w_{a,\beta}(Q_n - g_1) \right\|_p = \left\| w_{a,\beta}(P_n(G) - G') \right\|_p \leq c \sum_{j=1}^{n} E_2(g_1, I_j^{(1)})_{w_{a,\beta},p},
\]
if $1 \leq p < \infty$, and
\[
\left\| w_{a,\beta}(Q_n - g_1) \right\|_\infty = \left\| w_{a,\beta}(P_n(G) - G') \right\|_\infty \leq c \sup_{1 \leq j \leq n} E_2(g_1, I_j^{(1)})_{w_{a,\beta},\infty},
\]
if $p = \infty$.

Hence,
\[
\left\| w_{a,\beta}(f - Q_n) \right\|_p \leq c \sum_{j=1}^{n} E_2(f, I_j^{(v)})_{w_{a,\beta},p} + c \sum_{j=1}^{n} E_2(g_1, I_j^{(1)})_{w_{a,\beta},p}
\]
\[
\leq c \sum_{j=1}^{n} E_2(f, I_j^{(v+1)})_{w_{a,\beta},p},
\]
if $1 \leq p < \infty$, and
\[
\left\| w_{a,\beta}(f - Q_n) \right\|_\infty \leq c \sup_{1 \leq j \leq n} E_2(f, I_j^{(v)})_{w_{a,\beta},\infty} + c \sup_{1 \leq j \leq n} E_2(g_1, I_j^{(1)})_{w_{a,\beta},\infty}
\]
\[
\leq c \sup_{1 \leq j \leq n} E_2(f, I_j^{(v+1)})_{w_{a,\beta},\infty},
\]
if $p = \infty$.

The statement of Theorem 1.5 in the case where $\ell = 1$ now follows from Corollary 2.4.

### 7 Weighted $L_q$ Approximation of Constrained Unit Spheres in $\mathbb{L}_p^{a,\beta}$

$1 \leq q < p \leq \infty$

The following result was proved in [7].
Theorem 7.1 ([7, Theorem 1.1]) Let \( \ell \in \mathbb{N}, 1 \leq q < p \leq \infty, \alpha, \beta \in I_p, f \in \mathbb{L}_p^{\alpha, \beta} \cap \Delta^\ell \) and \( 0 < \delta < 1/4 \). Then,

\[
\omega^\ell_p(f, 1, \delta)_{w_{\alpha, \beta}, q} \leq c Y^{\alpha, \beta}_p(\ell, q, p) \left\| w_{\alpha, \beta}f \right\|_p,
\]

where

\[
Y^{\alpha, \beta}_p(\ell, q, p) := \begin{cases} 
\delta^{2/4 - 2/p} \quad &\text{if } \ell \geq 2, \text{ and } (\ell, q, p) \neq (2, 1, \infty), \\
\delta^2 |\ln \delta| \quad &\text{if } \ell = 2, q = 1, p = \infty, \text{ and } (\alpha, \beta) \neq (0, 0), \\
\delta^2 \quad &\text{if } \ell = 2, q = 1, p = \infty, \text{ and } (\alpha, \beta) = (0, 0), \\
\delta^{2/4 - 2/p} \quad &\text{if } \ell = 1 \text{ and } p < 2q, \\
\delta^{1/q} |\ln \delta|^{1/(2q)} \quad &\text{if } \ell = 1 \text{ and } p = 2q, \\
\delta^{1/q} \quad &\text{if } \ell = 1 \text{ and } p > 2q.
\end{cases}
\]

is best possible in the sense that (71) is no longer valid if one increases (resp. decreases) any of the powers of \( \delta \) (resp. \( |\ln \delta| \)) in its definition.

Now, let \( \mathbb{S}_p^{\alpha, \beta} \) be the unit sphere in \( \mathbb{L}_p^{\alpha, \beta} \), i.e., \( f \in \mathbb{S}_p^{\alpha, \beta} \) if and only if \( \left\| w_{\alpha, \beta}f \right\|_p = 1 \), and denote

\[
\mathcal{E}(X, \Pi_n)_{w, q} := \sup_{f \in X} E_n(f)_{w, q} \quad \text{and} \quad \mathcal{E}^{(\ell)}(X, \Pi_n)_{w, q} := \sup_{f \in X} E_n^{(\ell)}(f)_{w, q}.
\]

Theorem 7.1 and Corollary 1.8 imply the following result.

Corollary 7.2 Let \( \ell = 1 \) or \( \ell = 2, 1 \leq q < p \leq \infty \) and \( \alpha, \beta \geq 0 \). Then

\[
\mathcal{E}^{(\ell)}(\Delta^\ell \cap \mathbb{S}_p^{\alpha, \beta}, \Pi_n)_{w_{\alpha, \beta}, q} \leq c Y^{\alpha, \beta}_n(\ell, q, p), \quad \text{for all } n \in \mathbb{N}.
\]

Finally, applying [7, Theorem 1.5] and using the fact that

\[
\mathcal{E}(X, \Pi_n)_{w, q} \leq \mathcal{E}^{(\ell)}(X, \Pi_n)_{w, q},
\]

we obtain the following corollary.

Corollary 7.3 Let \( \ell = 1 \) or \( \ell = 2, 1 \leq q < p \leq \infty, \) and \( \alpha, \beta \geq 0 \). Then for any \( n \in \mathbb{N} \),

\[
\mathcal{E}^{(\ell)}(\Delta^\ell \cap \mathbb{S}_p^{\alpha, \beta}, \Pi_n)_{w_{\alpha, \beta}, q} \sim \mathcal{E}(\Delta^\ell \cap \mathbb{S}_p^{\alpha, \beta}, \Pi_n)_{w_{\alpha, \beta}, q} \sim \begin{cases} 
\mu^{-2/4 - 2/p} \quad &\text{if } \ell = 2 \text{ and } (q, p) \neq (1, \infty), \\
\mu^2 \quad &\text{if } \ell = 2, q = 1, p = \infty, \text{ and } \alpha = \beta = 0, \\
\mu^{-\min\{2/4 - 2/p, 1/q\}} \quad &\text{if } \ell = 1 \text{ and } p \neq 2q.
\end{cases}
\]

If \( \ell = 2, q = 1, p = \infty, \) and \( (\alpha, \beta) \neq (0, 0) \), then

\[
c n^{-2} \leq \mathcal{E}(\Delta^\ell \cap \mathbb{S}_\infty^{\alpha, \beta}, \Pi_n)_{w_{\alpha, \beta}, 1} \leq \mathcal{E}^{(2)}(\Delta^\ell \cap \mathbb{S}_\infty^{\alpha, \beta}, \Pi_n)_{w_{\alpha, \beta}, 1} \leq c n^{-2} \ln(n + 1).
\]

If \( \ell = 1 \) and \( p = 2q \), then

\[
c n^{-1/q} \leq \mathcal{E}(\Delta^\ell \cap \mathbb{S}_{2q}^{\alpha, \beta}, \Pi_n)_{w_{\alpha, \beta}, q} \leq \mathcal{E}^{(1)}(\Delta^\ell \cap \mathbb{S}_{2q}^{\alpha, \beta}, \Pi_n)_{w_{\alpha, \beta}, q} \leq c n^{-1/q}[\ln(n + 1)]^{1/(2q)}.
\]
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