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Full Length Article

On weighted approximation with Jacobi weights

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Abstract

We obtain matching direct and inverse theorems for the degree of weighted L_p-approximation by polynomials with the Jacobi weights $(1-x)^\alpha(1+x)^\beta$. Combined, the estimates yield a constructive characterization of various smoothness classes of functions via the degree of their approximation by algebraic polynomials. In addition, we prove Whitney type inequalities which are of independent interest.

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1. Introduction and main results

In this paper, we are interested in weighted polynomial approximation with the Jacobi weights

$$w_{\alpha,\beta}(x) := (1-x)^\alpha(1+x)^\beta, \quad \alpha,\beta \in J_p := \begin{cases} (-1/p,\infty), & \text{if } 0<p<\infty, \\ [0,\infty), & \text{if } p=\infty. \end{cases}$$

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Let $L_p^{\alpha,\beta}(I) := \left\{ f \mid \left\| w_{\alpha,\beta} f \right\|_{L_p(I)} < \infty \right\}$, where $\|\cdot\|_{L_p(I)}$ is the usual L_p (quasi)norm on the interval $I \subseteq [-1, 1]$, and, for $f \in L_p^{\alpha,\beta}(I)$, denote by

$$E_n(f, I)_{\alpha,\beta,p} := \inf_{p_n \in \mathbb{P}_n} \left\| w_{\alpha,\beta}(f - p_n) \right\|_{L_p(I)}$$

the error of best weighted approximation of f by polynomials in $\mathbb{P}_n$, the set of algebraic polynomials of degree not more than $n - 1$. For $I = [-1, 1]$, we denote $\|\cdot\|_p := \|\cdot\|_{L_p[-1,1]}$, $L_p^{\alpha,\beta} := L_p^{\alpha,\beta}([-1, 1])$, $E_n(f)_{\alpha,\beta,p} := E_n(f, [-1, 1])_{\alpha,\beta,p}$, etc.

Definition 1.1 (*[10]*)**.** For $r \in \mathbb{N}_0$ and $0 < p \le \infty$, denote $\mathbb{B}_p^0(w_{\alpha,\beta}) := L_p^{\alpha,\beta}$ and

$$\mathbb{B}_p^r(w_{\alpha,\beta}) := \left\{ f \mid f^{(r-1)} \in AC_{loc}(-1, 1) \quad \text{and} \quad \varphi^r f^{(r)} \in L_p^{\alpha,\beta} \right\}, \quad r \ge 1,$$

where $\varphi(x) := \sqrt{1 - x^2}$ and $AC_{loc}(-1, 1)$ denotes the set of functions which are locally absolutely continuous in $(-1, 1)$.

We remark that, in the case $p < 1$, our definition of derivatives is understood in the classical sense, *i.e.,* the assumption $f^{(r-1)} \in AC_{loc}(-1, 1)$ in the case $r \ge 2$ is understood in the sense that f is the $(r - 1)$st integral of a locally absolutely continuous $f^{(r-1)}$ plus a polynomial of degree $r - 2$.

As is common when dealing with L_p spaces, we will not distinguish between a function in $\mathbb{B}_p^r(w_{\alpha,\beta})$ and all functions which are equivalent to it in $L_p^{\alpha,\beta}$.

Definition 1.2 (*[10]*)**.** For $k, r \in \mathbb{N}$ and $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$, $0 < p \le \infty$, define

$$\omega_{k,r}^{\varphi}(f^{(r)}, t)_{\alpha,\beta,p} := \sup_{0 \le h \le t} \left\| \mathcal{W}_{kh}^{r/2+\alpha, r/2+\beta}(\cdot) \Delta_{h\varphi(\cdot)}^k (f^{(r)}, \cdot) \right\|_p, \tag{1.1}$$

where

$$\mathcal{W}_\delta^{\xi,\zeta}(x) := (1 - x - \delta\varphi(x)/2)^\xi (1 + x - \delta\varphi(x)/2)^\zeta,$$

and

$$\Delta_h^k(f, x) := \begin{cases} \displaystyle\sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f(x - \frac{kh}{2} + ih), & \text{if } [x - \frac{kh}{2}, x + \frac{kh}{2}] \subseteq [-1, 1], \\ 0, & \text{otherwise}, \end{cases}$$

is the kth symmetric difference.

For $\delta > 0$, denote (see [9])

$$\mathfrak{D}_\delta := \left\{ x \;\middle|\; 1 - \delta\varphi(x)/2 \ge |x| \right\} \setminus \{\pm 1\} = [-1 + \mu(\delta), 1 - \mu(\delta)],$$

where

$$\mu(\delta) := 2\delta^2/(4 + \delta^2).$$

We note that $\mathfrak{D}_{\delta_1} \subset \mathfrak{D}_{\delta_2}$ if $\delta_2 < \delta_1 \le 2$, and that $\mathfrak{D}_\delta = \emptyset$ if $\delta > 2$. Also, since $\Delta_{h\varphi(x)}^k(f, x) = 0$ if $x \notin \mathfrak{D}_{kh}$,

$$\omega_{k,r}^{\varphi}(f^{(r)}, t)_{\alpha,\beta,p} = \sup_{0 < h \le t} \left\| \mathcal{W}_{kh}^{r/2+\alpha, r/2+\beta}(\cdot) \Delta_{h\varphi(\cdot)}^k (f^{(r)}, \cdot) \right\|_{L_p(\mathfrak{D}_{kh})}. \tag{1.2}$$

In particular, $\omega_{k,r}^{\varphi}(f^{(r)}, t)_{\alpha,\beta,p} = \omega_{k,r}^{\varphi}(f^{(r)}, 2/k)_{\alpha,\beta,p}$, for all $t \ge 2/k$.

Following [10] we also define the weighted averaged moduli.

Definition 1.3 (*[10]*). For $k \in \mathbb{N}$, $r \in \mathbb{N}_0$ and $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$, $0 < p < \infty$, the kth weighted averaged modulus of smoothness of f is defined as

$$\omega_{k,r}^{*\varphi}(f^{(r)}, t)_{\alpha,\beta,p} := \left(\frac{1}{t} \int_0^t \int_{\mathfrak{D}_{k\tau}} \left| \mathcal{W}_{k\tau}^{r/2+\alpha, r/2+\beta}(x) \Delta_{\tau\varphi(x)}^k (f^{(r)}, x) \right|^p dx\, d\tau \right)^{1/p}.$$

If $p = \infty$ and $f \in \mathbb{B}_\infty^r(w_{\alpha,\beta})$, we write

$$\omega_{k,r}^{*\varphi}(f^{(r)}, t)_{\alpha,\beta,\infty} := \omega_{k,r}^{\varphi}(f^{(r)}, t)_{\alpha,\beta,\infty}. \tag{1.3}$$

Clearly,

$$\omega_{k,r}^{*\varphi}(f^{(r)}, t)_{\alpha,\beta,p} \le \omega_{k,r}^{\varphi}(f^{(r)}, t)_{\alpha,\beta,p}, \quad t > 0. \tag{1.4}$$

Moreover, it was proved in [10] that if $r/2 + \alpha, r/2 + \beta \ge 0$, then the weighted moduli and the weighted averaged moduli are equivalent.

Throughout this paper, all constants c may depend only on k, r, p, α and β, unless a specific dependence on an additional parameter is mentioned.

We have the following direct (Jackson-type) theorem.

Theorem 1.4. *Let $k \in \mathbb{N}$, $0 < p \le \infty$, $\alpha \ge 0$ and $\beta \ge 0$. If $f \in L_p^{\alpha,\beta}$, then for every $n \ge k$ and $0 < \vartheta \le 1$,*

$$E_n(f)_{\alpha,\beta,p} \le c\omega_{k,0}^{\varphi}(f, \vartheta/n)_{\alpha,\beta,p}, \tag{1.5}$$

where the constant c depends only on k, α, β, p and ϑ.

It follows from [10, Lemma 1.11] that, if $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $r/2 + \alpha \ge 0$, $r/2 + \beta \ge 0$, $1 \le p \le \infty$ and $f \in \mathbb{B}_p^{r+1}(w_{\alpha,\beta})$, then

$$\omega_{k+1,r}^{\varphi}(f^{(r)}, t)_{\alpha,\beta,p} \le ct\omega_{k,r+1}^{\varphi}(f^{(r+1)}, t)_{\alpha,\beta,p}, \quad t > 0.$$

Hence, (1.5) implies that, for $f \in B_p^r(w_{\alpha,\beta})$, $1 \le p \le \infty$,

$$E_n(f)_{\alpha,\beta,p} \le c\omega_{k+r,0}^{\varphi}(f, 1/n)_{\alpha,\beta,p} \le cn^{-r}\omega_{k,r}^{\varphi}(f^{(r)}, 1/n)_{\alpha,\beta,p}, \quad n \ge k + r,$$

provided $\alpha, \beta \ge 0$. We strengthen this result by showing that the last estimate is, in fact, valid for all $\alpha, \beta \ge -r/2$. Namely,

Theorem 1.5. *Let $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $1 \le p \le \infty$, and $\alpha, \beta \in J_p$ be such that $r/2 + \alpha \ge 0$ and $r/2 + \beta \ge 0$. If $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$, then*

$$E_n(f)_{\alpha,\beta,p} \le \frac{c}{n^r}\omega_{k,r}^{\varphi}(f^{(r)}, 1/n)_{\alpha,\beta,p}, \quad n \ge k + r. \tag{1.6}$$

We remark that Theorem 1.5 is not valid if $r \ge 1$ and $0 < p < 1$ (one can show this using the same construction that was used in the proof of [5, Theorem 3 and Corollary 4]).

Jackson type estimates of the form (1.5) and (1.6) frequently appear with the inequalities being valid for n sufficiently large. In order to have these inequalities for small n, we need certain Whitney type results. We devote Section 3 to Whitney type estimates, and we feel that the results in this section are of independent interest by themselves.

Next, we have the following inverse result in the case $1 \le p \le \infty$.

Theorem 1.6. *Suppose that $r \in \mathbb{N}_0$, $1 \le p \le \infty$, $\alpha, \beta \in J_p$ are such that $r/2 + \alpha \ge 0$ and $r/2 + \beta \ge 0$, and $f \in L_p^{\alpha,\beta}$. If*

$$\sum_{n=1}^{\infty} rn^{r-1} E_n(f)_{\alpha,\beta,p} < +\infty \tag{1.7}$$

(i.e., *if $r = 0$ then this condition is vacuous), then $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$, and for $k \in \mathbb{N}$ and $N \in \mathbb{N}$,*

$$\begin{aligned} \omega_{k,r}^{\varphi}(f^{(r)}, t)_{\alpha,\beta,p} \le c &\sum_{n > \max\{N,1/t\}} rn^{r-1} E_n(f)_{\alpha,\beta,p} \\ &+ ct^k \sum_{N \le n \le \max\{N,1/t\}} n^{k+r-1} E_n(f)_{\alpha,\beta,p} \\ &+ c(N) t^k E_{k+r}(f)_{\alpha,\beta,p}, \quad t > 0. \end{aligned} \tag{1.8}$$

In particular, if $N \le k + r$, then

$$\begin{aligned} \omega_{k,r}^{\varphi}(f^{(r)}, t)_{\alpha,\beta,p} \le c &\sum_{n > \max\{N,1/t\}} rn^{r-1} E_n(f)_{\alpha,\beta,p} \\ &+ ct^k \sum_{N \le n \le \max\{N,1/t\}} n^{k+r-1} E_n(f)_{\alpha,\beta,p}, \ t > 0. \end{aligned}$$

Remark 1.7. (i) Note that the first term in (1.8) disappears if $r = 0$.
(ii) If $\alpha = \beta = 0$, Theorem 1.6 was proved in [9].
(iii) The case $\alpha, \beta \ge 0$, $N = 1$ and $r = 0$ follows from [3, Theorem 8.2.4] by virtue of (4.2).

Denote by Φ the set of nondecreasing functions $\phi : [0, \infty) \to [0, \infty)$, satisfying $\lim_{t \to 0+} \phi(t) = 0$. The following is an immediate corollary of Theorem 1.6 (in fact, it is a restatement of Theorem 1.6 in terms of ϕ).

Corollary 1.8. *Suppose that $r \in \mathbb{N}_0$, $N \in \mathbb{N}$, $1 \le p \le \infty$, $\alpha, \beta \in J_p$ are such that $r/2 + \alpha \ge 0$ and $r/2 + \beta \ge 0$, and $\phi \in \Phi$ is such that*

$$\int_0^1 \frac{r\phi(u)}{u^{r+1}} du < +\infty$$

(i.e., *if $r = 0$ then this condition is vacuous). Then, if $f \in L_p^{\alpha,\beta}$ is such that*

$$E_n(f)_{\alpha,\beta,p} \le \phi\left(\frac{1}{n+1}\right), \quad \textit{for all} \quad n \ge N,$$

then $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$, and for $k \in \mathbb{N}$ and $0 < t \le 1/2$,

$$\omega_{k,r}^{\varphi}(f^{(r)}, t)_{\alpha,\beta,p} \le c \int_0^t \frac{r\phi(u)}{u^{r+1}} du + ct^k \int_t^1 \frac{\phi(u)}{u^{k+r+1}} du + c(N) t^k E_{k+r}(f)_{\alpha,\beta,p}.$$

In particular, if $N \le k + r$, then

$$\omega_{k,r}^{\varphi}(f^{(r)}, t)_{\alpha,\beta,p} \le c \int_0^t \frac{r\phi(u)}{u^{r+1}} du + ct^k \int_t^1 \frac{\phi(u)}{u^{k+r+1}} du.$$

Remark 1.9. We take this opportunity to correct an inadvertent misprint in three of our earlier papers where the inverse theorems of this type were proved in the case $\alpha = \beta = 0$. Namely, the inequality $E_n(f)_p \le \phi(1/n)$ in [7, Theorem 3.2] (the case $p = \infty$), and in [9, Theorem 9.1] and [8, Theorem I_r] (the case $1 \le p \le \infty$), should be replaced by $E_n(f)_p \le \phi(1/(n+1))$. Otherwise, the last estimates in these results are not justified/valid if $N = 1$, $k = 1$ and $r = 0$ since $E_{k+r}(f)_p = E_1(f)_p \le \phi(1)$ cannot be estimated above by $\int_t^1 \phi(u)u^{-2}du$ without any extra assumptions on the function ϕ.

It immediately follows from Theorem 1.4 that if $\alpha, \beta \in J_p$, $r/2 + \alpha \ge 0$, $r/2 + \beta \ge 0$ and $\omega^{\varphi}_{k,r}(f^{(r)}, t)_{\alpha,\beta,p} \le t^{\gamma}$, then $E_n(f)_{\alpha,\beta,p} \le cn^{-r-\gamma}$. Conversely, an immediate consequence of Theorem 1.6 (Corollary 1.8) is the following result which, for $\alpha, \beta \ge 0$, was proved by a different method in [8, Theorem 5.3].

Corollary 1.10. *Suppose that $r \in \mathbb{N}_0$, $N \in \mathbb{N}$, $1 \le p \le \infty$, and $\alpha, \beta \in J_p$ are such that $r/2 + \alpha \ge 0$ and $r/2 + \beta \ge 0$. If $f \in L_p^{\alpha,\beta}$ is such that, for some $N \in \mathbb{N}$ and $r < \gamma < k + r$,*

$$E_n(f)_{\alpha,\beta,p} \le n^{-\gamma}, \quad n \ge N, \tag{1.9}$$

then $f \in \mathbb{B}^r_p(w_{\alpha,\beta})$, and

$$\omega^{\varphi}_{k,r}(f^{(r)}, t)_{\alpha,\beta,p} \le ct^{\gamma-r} + c(N)t^k E_{k+r}(f)_{\alpha,\beta,p}, \quad t > 0.$$

In particular, if $N \le k + r$, then

$$\omega^{\varphi}_{k,r}(f^{(r)}, t)_{\alpha,\beta,p} \le ct^{\gamma-r}, \quad t > 0.$$

Finally, we have the following inverse theorem for $0 < p < 1$ which is an immediate corollary of [6, Theorem 10.1] and [7, Lemma 4.5].

Theorem 1.11. *Let $k \in \mathbb{N}$, $\alpha \ge 0$, $\beta \ge 0$ and $f \in L_p^{\alpha,\beta}$, $0 < p < 1$. Then there exists a positive constant $\vartheta \le 1$ depending only on k, p, α and β such that, for any $n \in \mathbb{N}$,*

$$\omega^{\varphi}_{k,0}(f, \vartheta n^{-1})^p_{\alpha,\beta,p} \le cn^{-kp} \sum_{m=1}^{n} m^{kp-1} E_m(f)^p_{\alpha,\beta,p}.$$

2. Auxiliary lemma

Lemma 2.1. *Let $0 < \delta \le 2$, and let $y := y(x)$, $y : [-1, 1] \to \mathbb{R}$ be such that*

$$y(x) + \delta\varphi(y(x))/2 = x, \quad x \in [-1, 1].$$

Then,

(i) *y is strictly increasing on $[-1, 1]$, and $y'(x) \le 2$, $x \in [-1, 1]$,*
(ii) *$y([-1 + 2\mu(\delta), 1]) = \mathfrak{D}_\delta$,*
(iii) *$y'(x) \ge 2/3$, $x \in [-1 + 2\mu(\delta), 1]$,*
(iv) *if $y_\lambda(x) := y(x) + \lambda\varphi(y(x))$, then $1/3 \le y'_\lambda(x) \le 3$, for all $|\lambda| \le \delta/2$ and $x \in [-1 + 2\mu(\delta), 1]$,*
(v) *for all $x \in [-1 + 2\mu(\delta), 1]$,*

$$\mu(\delta) + 2(1 - x)/3 \le 1 - y(x) \le \mu(\delta) + 2(1 - x) \tag{2.1}$$

and

$$(1 + x)/2 \le 1 + y(x) \le 1 + x. \tag{2.2}$$

Note that it is not difficult to see that the function $y = y(x)$ in the statement of Lemma 2.1 is well defined for all $x \in [-1, 1]$ and, in fact,

$$y(x) = \frac{4x - \delta\sqrt{4 - 4x^2 + \delta^2}}{4 + \delta^2}, \quad -1 \le x \le 1.$$

However, we will not be using this explicit formula in this paper.

Proof. Since $x \le 1$, we have $y + \delta\varphi(y)/2 \le 1$ which can be rewritten as $\delta/(2\varphi(y)) \le 1/(1 + y)$, and so, if $y \ge 0$, then

$$1 - \frac{\delta y}{2\varphi(y)} \ge \frac{1}{1 + y} \ge \frac{1}{2},$$

and, clearly, $1 - \delta y/(2\varphi(y)) \ge 1/2$ if $y < 0$ as well.

Therefore, since

$$\frac{dy}{dx} = \left(1 - \frac{\delta y}{2\varphi(y)}\right)^{-1},$$

we immediately conclude that (i) holds.

Now, since y is nondecreasing, $y([-1 + 2\mu(\delta), 1]) = [y(-1 + 2\mu(\delta)), y(1)]$, and (ii) follows because $y(1) = 1 - \mu(\delta)$ and $y(-1 + 2\mu(\delta)) = -1 + \mu(\delta)$.

It follows from (ii) that, for $x \in [-1 + 2\mu(\delta), 1]$, we have $y - \delta\varphi(y)/2 \ge -1$, which can be rewritten as $\delta/(2\varphi(y)) \le 1/(1 - y)$, and so, if $y \le 0$, then

$$1 - \frac{\delta y}{2\varphi(y)} \le \frac{1 - 2y}{1 - y} \le \frac{3}{2},$$

and, clearly, $1 - \delta y/(2\varphi(y)) \le 3/2$ if $y > 0$ as well. This implies (iii).

Now, it follows from the above estimates that $\delta/(2\varphi(y)) \le 1/(1+|y|)$, for $x \in [-1+2\mu(\delta), 1]$, which implies

$$y_\lambda'(x) = \left(1 - \frac{\lambda y}{\varphi(y)}\right) y'(x) \le 2\left(1 + \frac{\delta|y|}{2\varphi(y)}\right) \le \frac{2 + 4|y|}{1 + |y|} \le 3,$$

and

$$y_\lambda'(x) \ge \frac{2}{3}\left(1 - \frac{\delta|y|}{2\varphi(y)}\right) \ge \frac{2}{3(1 + |y|)} \ge \frac{1}{3},$$

and so (iv) is verified.

Now, by

$$\frac{dy}{dx}(\xi) = \frac{y(1) - y(x)}{1 - x} = \frac{1 - \mu(\delta) - y(x)}{1 - x}, \quad \xi \in (x, 1),$$

(i) and (iii) imply (2.1), for $x \in [-1+2\mu(\delta), 1]$. Finally, the second inequality in (2.2) is obvious, and the first one immediately follows from (ii) which implies

$$1 + x = 1 + y + \delta\varphi(y)/2 \le 2(1 + y).$$

Thus, (v) is verified. □

3. Whitney-type estimates

In this section, we prove Whitney-type estimates, which we feel are of independent interest, and which we need in order to prove the direct (Jackson-type) theorem (Theorem 1.4) for small n.

Recall that the celebrated Whitney inequalities for the ordinary moduli of smoothness were first proved by Whitney [16] for functions in $C[a,b]$. Later Brudnyi [1] extended the inequalities to $L_p[a,b]$, $1 \le p < \infty$ and, finally, Storozhenko [15] proved the inequalities for $L_p[a,b]$, $0 < p < 1$.

Theorem 3.1. *Let $k \in \mathbb{N}$, $\alpha \ge 0$, $\beta \ge 0$, $0 < p \le \infty$, $f \in L_p^{\alpha,\beta}$, $0 < h \le 2$ and $x_0 \in \mathfrak{D}_h$. Then, for any $\theta \in (0,1]$, we have*

$$E_k(f,[x_0 - h\varphi(x_0)/2, x_0 + h\varphi(x_0)/2])_{\alpha,\beta,p} \le c\omega_{k,0}^{*\varphi}(f,\theta h)_{\alpha,\beta,p} \le c\omega_{k,0}^{\varphi}(f,\theta h)_{\alpha,\beta,p},$$

where c depends only on k, α, β, p and θ.

Choosing $x_0 = 0$ and $h = 2$ in Theorem 3.1 we immediately get the following corollary.

Corollary 3.2. *Let $k \in \mathbb{N}$, $\alpha \ge 0$, $\beta \ge 0$, $0 < p \le \infty$ and $f \in L_p^{\alpha,\beta}$. Then, for any $\theta \in (0,1]$, we have*

$$E_k(f)_{\alpha,\beta,p} \le c\omega_{k,0}^{*\varphi}(f,\theta)_{\alpha,\beta,p} \le c\omega_{k,0}^{\varphi}(f,\theta)_{\alpha,\beta,p}, \tag{3.1}$$

where c depends only on k, α, β, p and θ.

Also, if $x_0 \pm h\varphi(x_0)/2 = \pm 1$, Theorem 3.1 immediately gives the following result (by letting $h := t\sqrt{4A/(4-At^2)}$, $x_0 := \pm(1-\mu(h))$, $\theta := \min\{1, 1/\sqrt{2A}\}$, and using monotonicity of the moduli with respect to t).

Corollary 3.3. *Let $k \in \mathbb{N}$, $\alpha \ge 0$, $\beta \ge 0$, $A > 0$, $0 < p \le \infty$ and $f \in L_p^{\alpha,\beta}$. Then, for any $0 < t \le \sqrt{2/A}$, we have*

$$E_k(f,[1-At^2,1])_{\alpha,\beta,p} \le c\omega_{k,0}^{*\varphi}(f,t)_{\alpha,\beta,p} \le c\omega_{k,0}^{\varphi}(f,t)_{\alpha,\beta,p},$$

and

$$E_k(f,[-1,-1+At^2])_{\alpha,\beta,p} \le c\omega_{k,0}^{*\varphi}(f,t)_{\alpha,\beta,p} \le c\omega_{k,0}^{\varphi}(f,t)_{\alpha,\beta,p},$$

where c depends on k, p, α, β and A.

Proof of Theorem 3.1. Theorem 3.1 follows from the classical (non-weighted) Whitney's inequality (see [2, Theorem 6.4.2 and Theorem 12.5.5]), which readily implies (see *e.g.* [13, Sections 3.1 and 7.1]), for each interval $J \subset [-1,1]$, the existence of a polynomial $p_k \in \mathbb{P}_k$, such that

$$\begin{aligned}\|f - p_k\|_{L_p(J)} &\le c\omega_k(f,|J|;J)_p \\ &\le c\frac{|J|^{k-1+1/\min\{1,p\}}}{\delta^{k-1+1/\min\{1,p\}}}\omega_k(f,\delta;J)_p, \quad 0 < \delta \le |J|,\end{aligned} \tag{3.2}$$

where $|J|$ is the length of the interval J.

In order to prove Theorem 3.1, we assume, without loss of generality, that $x_0 \ge 0$, and denote

$$[a,b] := [x_0 - h\varphi(x_0)/2, x_0 + h\varphi(x_0)/2], \qquad W_p := \omega_{k,0}^{*\varphi}(f,\theta h)_{\alpha,\beta,p},$$

Note that

$$1 - x \le 2(1-x_0) \quad \text{and} \quad 1 + x \le 2(1+x_0), \quad x \in [a,b], \tag{3.3}$$

since x_0 is the middle of $[a,b]$, and so

$$\varphi(b) \le \varphi(x) \le 2\varphi(x_0), \quad \text{for all} \quad x \in [a,b], \tag{3.4}$$

where the first inequality is valid since $|x| \le b$ (because x_0 is assumed to be nonnegative).

We will consider two cases: (i) $\varphi(x_0) \le 2\varphi(b)$ and (ii) $\varphi(x_0) > 2\varphi(b)$.

In the case (i), x_0 and $[a,b]$ are "far away" from the endpoints of $[-1,1]$ and so $w_{\alpha,\beta}(x) \sim w_{\alpha,\beta}(x_0)$, for all $x \in [a,b]$. This is a simpler case, and we can reduce the proof to the classical non-weighted Whitney's inequality. The case (ii) is a bit more involved since the right endpoint b is now "close to 1" (in fact, we will show that it is sufficient to assume that it is equal to 1), and so the weight $w_{\alpha,\beta}(x)$ is no longer equivalent to a constant over $[a,b]$. The idea of the proof in this case is to consider the interval $J := [a,b] \cap [-\tilde{b}, \tilde{b}]$, where $\tilde{b} := 1-(c\theta h)^2$ with some small constant c depending only on k. Then, $w_{\alpha,\beta}(x) \sim w_{\alpha,\beta}(x_0)$, for all $x \in J$, and the main task remaining in this case is to show that a polynomial $p_k \in \mathbb{P}_k$ that approximates f well on J also approximates it with the right order on $[a,b] \setminus J$.

Case (i): $\varphi(x_0) \le 2\varphi(b)$.

Then, for all $x \in [a,b]$,

$$1 - x_0 \le \varphi^2(x_0) \le 4\varphi^2(b) \le 4\varphi^2(x) \le 8(1-x) \tag{3.5}$$

and

$$1 + x_0 = \frac{\varphi^2(x_0)}{1-x_0} \le \frac{4\varphi^2(b)}{1-x_0} \le \frac{8\varphi^2(x)}{1-x} = 8(1+x). \tag{3.6}$$

Note that (3.3), (3.5) and (3.6) imply that

$$w_{\alpha,\beta}(x_0) \sim w_{\alpha,\beta}(x) \le c\mathcal{W}_{k\tau}^{\alpha,\beta}(x), \quad \text{if} \quad x \pm k\tau\varphi(x)/2 \in [a,b]. \tag{3.7}$$

Now, let $J := [a,b]$ and $\delta := \theta h \varphi(b)$, and note that

$$\frac{\theta}{2}|J| = \frac{\theta}{2} h\varphi(x_0) \le \delta \le \theta h \varphi(x_0) \le |J|. \tag{3.8}$$

So, for $p = \infty$, we have

$$\begin{aligned}
\omega_k(f,\delta;J)_\infty &= \sup_{0<s\le\delta} \sup_{x\in J} \left|\Delta_s^k(f,x;J)\right| = \sup_{0<\tau\le\delta/\varphi(b)} \sup_{x\in J} \left|\Delta_{\tau\varphi(b)}^k(f,x;J)\right| \\
&= \sup_{0<\tau\le\theta h} \sup_{x\in J} \left|\Delta_{\tau\varphi(b)}^k(f,x;J)\right| \le \sup_{0<\tau\le\theta h} \sup_{x\in J} \left|\Delta_{\tau\varphi(x)}^k(f,x;J)\right| \\
&\le c w_{\alpha,\beta}^{-1}(x_0) \sup_{0<\tau\le\theta h} \sup_{x\in J} \left|\mathcal{W}_{k\tau}^{\alpha,\beta}(x)\Delta_{\tau\varphi(x)}^k(f,x;J)\right| \\
&= c w_{\alpha,\beta}^{-1}(x_0) W_\infty,
\end{aligned}$$

where in the last inequality we used (3.7).

If $p < \infty$, then it is well known (see *e.g.* [13, Lemma 7.2]) that

$$\omega_k(f,t;J)_p^p \le c\frac{1}{t}\int_0^t \int_J |\Delta_s^k(f,x;J)|^p \, dx\, ds, \quad 0 < t \le |J|/k.$$

Hence, using (3.4) and (3.7) we have

$$\begin{aligned}
c\delta\omega_k(f,\delta;J)_p^p &\le \int_J\int_0^\delta \left|\Delta_s^k(f,x,J)\right|^p ds\,dx \\
&= \int_J\int_0^{\delta/\varphi(x)} \varphi(x)\left|\Delta_{\tau\varphi(x)}^k(f,x,J)\right|^p d\tau\,dx \\
&\le \int_J\int_0^{\theta h} \varphi(x)\left|\Delta_{\tau\varphi(x)}^k(f,x,J)\right|^p d\tau\,dx \\
&\le c w_{\alpha,\beta}^{-p}(x_0)\varphi(b)\int_J\int_0^{\theta h} \left|\mathcal{W}_{k\tau}^{\alpha,\beta}(x)\Delta_{\tau\varphi(x)}^k(f,x,J)\right|^p d\tau\,dx \\
&\le c w_{\alpha,\beta}^{-p}(x_0)\varphi(b)\int_0^{\theta h}\int_{\mathfrak{D}_{k\tau}} \left|\mathcal{W}_{k\tau}^{\alpha,\beta}(x)\Delta_{\tau\varphi(x)}^k(f,x)\right|^p dx\,d\tau \\
&= c w_{\alpha,\beta}^{-p}(x_0)\theta h\varphi(b)W_p^p.
\end{aligned}$$

Thus, for all $0 < p \le \infty$, we have

$$\omega_k(f,\delta;J)_p \le c w_{\alpha,\beta}^{-1}(x_0)W_p,$$

which, by virtue of (3.3), yields

$$\left\| w_{\alpha,\beta}(f-p_k)\right\|_{L_p(J)} \le c w_{\alpha,\beta}(x_0)\left\|f-p_k\right\|_{L_p(J)} \le c w_{\alpha,\beta}(x_0)\omega_k(f,\delta;J)_p \le cW_p,$$

and so the proof is complete in Case (i).

Case (ii): $\varphi(x_0) > 2\varphi(b)$.

We first note that, in this case, it suffices to assume that $b = 1$. Indeed, suppose that the theorem is proved for all $\hat{x}_0$ and $\hat{h}$ such that $\hat{x}_0 + \hat{h}\varphi(\hat{x}_0)/2 = 1$, and let x_0 and h be such that $\varphi(x_0) > 2\varphi(b)$ (recall that $b = x_0 + h\varphi(x_0)/2$). We let $\hat{x}_0 := x_0$ and $\hat{h} := 2(1-x_0)/\varphi(x_0)$ so that $x_0 + \hat{h}\varphi(x_0)/2 = 1$. Now, since

$$1 - x_0 = \frac{\varphi^2(x_0)}{1+x_0} > \frac{4\varphi^2(b)}{1+x_0} = 4(1-b)\frac{1+b}{1+x_0} \ge 4(1-b),$$

we have

$$h\varphi(x_0) = 2(b-x_0) = 2(1-x_0) - 2(1-b) > 3(1-x_0)/2.$$

Therefore, $h \le \hat{h} \le 4h/3$, and so

$$\begin{aligned}
&E_k(f,[x_0-h\varphi(x_0)/2,x_0+h\varphi(x_0)/2])_{\alpha,\beta,p} \\
&\quad\le E_k(f,[x_0-\hat{h}\varphi(x_0)/2,x_0+\hat{h}\varphi(x_0)/2])_{\alpha,\beta,p} \\
&\quad\le c\omega_{k,0}^{*\varphi}(f,\theta_1\hat{h})_{\alpha,\beta,p} \le cW_p,
\end{aligned}$$

where $\theta_1 := 3\theta/4$.

Hence, for the rest of this proof, we assume that $b = 1$. Note that

$$b - a = h\varphi(x_0) = 2(1-x_0) = 2\mu(h) \in [h^2/2, h^2]. \tag{3.9}$$

Define

$$\tilde{h} := \frac{\theta h}{10k}, \quad \tilde{b} := 1 - \tilde{h}^2 \quad \text{and} \quad J := [a,b]\cap[-\tilde{b},\tilde{b}].$$

Then $x_0 \in J$, and, for all $x \in J$, we have

$$\frac{1-x_0}{1-x} \le \frac{\mu(h)}{\tilde{h}^2} < c, \qquad \frac{1+x_0}{1+x} \le \frac{2}{\max\{\tilde{h}^2, 1+a\}} \le \frac{c}{\max\{h^2, 4-h^2\}} < c, \tag{3.10}$$

and, recalling (3.4),

$$\varphi(\tilde{b}) \le \varphi(x) \le 2\varphi(x_0) \le c\varphi(\tilde{b}).$$

We now let $\delta := \theta h \varphi(\tilde{b})$, note that

$$c|J| \le c(b-a) \le \delta \le b-a \le c|J|,$$

and conclude using the same argument that was used in Case (i) and using (3.10) instead of (3.5) and (3.6), that there is a polynomial $p_k \in \mathbb{P}_k$, such that

$$\left\| w_{\alpha,\beta}(f-p_k) \right\|_{L_p(J)} \le cW_p. \tag{3.11}$$

So, to finish the proof in Case (ii) we have to show that, for the function $g := f - p_k$, the inequalities

$$\left\| w_{\alpha,\beta}\, g \right\|_{L_p[\tilde{b},1]} \le cW_p. \tag{3.12}$$

and, if $a < -\tilde{b}$,

$$\left\| w_{\alpha,\beta}\, g \right\|_{L_p[a,-\tilde{b}]} \le cW_p. \tag{3.13}$$

hold. We prove (3.12), the proof of (3.13) being similar and simpler, since $a < -\tilde{b}$ only holds for “large” h (*i.e.,* those h that are close to 2). More precisely,

$$a < -\tilde{b} \quad \text{if and only if} \quad \frac{\theta^2 h^2}{100k^2} + \frac{4h^2}{4+h^2} > 2.$$

We let $t \in [2\tilde{h}/\sqrt{k}, 4\tilde{h}/\sqrt{k}]$ be fixed for now, and denote by $y = y(x)$ and $y_i = y_i(x)$, $1 \le i \le k$, the functions such that

$$y(x) + kt\varphi(y(x))/2 = x \quad \text{and} \quad y_i(x) := x - it\varphi(y(x)) = y(x) + (k/2 - i)t\varphi(y(x)).$$

Note that functions y and y_i are well defined (see remark after the statement of Lemma 2.1).

We now note that $[\tilde{b}, 1] \subset [-1 + 2\mu(kt), 1]$, since

$$-1 + 2\mu(kt) \le -1 + k^2t^2 \le -1 + 16k\tilde{h}^2 \le 1 - \tilde{h}^2 = \tilde{b}$$

and so Lemma 2.1 with $\delta = kt$ implies that, for all $x \in [\tilde{b}, 1]$, $2/3 \le y'(x) \le 2$, $1/3 \le y_i'(x) \le 3$, and

$$\varphi^2(y(x)) \le (1+x)\,(\mu(kt) + 2(1-x)) \le 2(\mu(kt) + 2\tilde{h}^2) \le k^2t^2 + 4\tilde{h}^2 \le 25k\tilde{h}^2. \tag{3.14}$$

Additionally, note that

$$y_i(x) \in J, \quad x \in [\tilde{b}, 1] \quad \text{and} \quad 1 \le i \le k. \tag{3.15}$$

Indeed, since $y(1) = 1 - \mu(kt)$, we have, for $x \in [\tilde{b}, 1]$,

$$y_i(x) \le y_1(x) \le y_1(1) = 1 - t\varphi(y(1)) = 1 - 2\mu(kt)/k \le 1 - kt^2/2 \le 1 - 2\tilde{h}^2 < \tilde{b},$$

and, using (3.14) and (3.9),

$$\begin{aligned} y_i(x) \ge y_k(x) \ge y_k(\tilde{b}) &= \tilde{b} - kt\varphi(y(\tilde{b})) \ge 1 - \tilde{h}^2 - 5k^{3/2}t\tilde{h} \ge 1 - 21k\tilde{h}^2 \\ &\ge \max\{-1 + \tilde{h}^2, a\}, \end{aligned}$$

which yields (3.15). Note also that (3.14) and inequalities $t \le 4\tilde{h}/\sqrt{k}$ and $\tilde{h} \le (5k)^{-1}$ imply that $1 + y(x) \ge 3kt\varphi(y(x))/2$, for $x \in [\tilde{b}, 1]$.

Hence,

$$w_{\alpha,\beta}(x) = w_{\alpha,\beta}\left(y(x) + kt\varphi(y(x))/2\right) \le 2^{\beta} w_{\alpha,\beta}(y_i(x)), \quad x \in [\tilde{b}, 1],$$

and using (3.15) and (3.11) we get, for $0 < p < \infty$,

$$\begin{aligned}\left\|w_{\alpha,\beta} g(y_i)\right\|_{L_p[\tilde{b},1]} &\le 2^{\beta} \left\|w_{\alpha,\beta}(y_i) g(y_i)\right\|_{L_p[\tilde{b},1]} \le c \left\|w_{\alpha,\beta}(y_i) g(y_i)(y_i')^{1/p}\right\|_{L_p[\tilde{b},1]} \\ &\le c \left\|w_{\alpha,\beta} g\right\|_{L_p(J)} \le c W_p.\end{aligned}$$

If $p = \infty$, then similar (and, in fact, simpler) arguments yield

$$\|w_{\alpha,\beta} g(y_i)\|_{L_\infty[\tilde{b},1]} \le c W_\infty, \quad 1 \le i \le k.$$

Now, for $x \in [\tilde{b}, 1]$,

$$\begin{aligned} g(x) &= \Delta^k_{t\varphi(y(x))}(g, y(x)) - \sum_{i=0}^{k-1} (-1)^{k-i} \binom{k}{i} g\left(y(x) + (i - \frac{k}{2}) t\varphi(y(x))\right) \\ &= \Delta^k_{t\varphi(y(x))}(g, y(x)) - \sum_{i=1}^{k} (-1)^{i} \binom{k}{i} g\left(y_i(x)\right),\end{aligned}$$

and so

$$\begin{aligned}\|w_{\alpha,\beta} g\|_{L_p[\tilde{b},1]} &\le c \left\|w_{\alpha,\beta} \Delta^k_{t\varphi(y)}(g, y)\right\|_{L_p[\tilde{b},1]} + c \sum_{i=1}^{k} \binom{k}{i} \left\|w_{\alpha,\beta} g(y_i)\right\|_{L_p[\tilde{b},1]} \\ &\le c \left\|w_{\alpha,\beta} \Delta^k_{t\varphi(y)}(g, y)\right\|_{L_p[\tilde{b},1]} + c W_p \\ &\le c \left\|\mathcal{W}^{\alpha,\beta}_{tk}(y) \Delta^k_{t\varphi(y)}(g, y)\right\|_{L_p[\tilde{b},1]} + c W_p \\ &\le c \left\|\mathcal{W}^{\alpha,\beta}_{tk} \Delta^k_{t\varphi}(g, \cdot)\right\|_{L_p(\mathfrak{D}_{kt})} + c W_p.\end{aligned}$$

This completes the proof in the case $p = \infty$. If $p < \infty$, then integrating with respect to t over $[2\tilde{h}/\sqrt{k}, 4\tilde{h}/\sqrt{k}]$ we get

$$\left\|w_{\alpha,\beta} g\right\|^p_{L_p[\tilde{b},1]} \le \frac{c}{\tilde{h}} \int_{2\tilde{h}/\sqrt{k}}^{4\tilde{h}/\sqrt{k}} \left\|\mathcal{W}^{\alpha,\beta}_{tk} \Delta^k_{t\varphi}(g, \cdot)\right\|^p_{L_p(\mathfrak{D}_{kt})} dt + c W_p^p \le c W_p^p.$$

The proof is now complete. □

We now prove a Whitney-type result for functions from $f \in \mathbb{B}^r_p(w_{\alpha,\beta})$, $r \in \mathbb{N}$.

Theorem 3.4. *Let $k \in \mathbb{N}$, $r \in \mathbb{N}$, $1 \le p \le \infty$, and let $\alpha, \beta \in J_p$ be such that $r/2 + \alpha \ge 0$ and $r/2 + \beta \ge 0$. If $f \in \mathbb{B}^r_p(w_{\alpha,\beta})$, then for any $\theta \in (0, 1]$,*

$$E_{k+r}(f)_{\alpha,\beta,p} \le c\omega^{\varphi}_{k,r}(f^{(r)}, \theta)_{\alpha,\beta,p}. \tag{3.16}$$

Proof. Note that $f \in \mathbb{B}^r_p(w_{\alpha,\beta})$ implies that $f^{(r)} \in L_p^{r/2+\alpha,r/2+\beta}$, and so it follows from (3.1) that

$$E_k(f^{(r)})_{r/2+\alpha,r/2+\beta,p} \le c\omega^{\varphi}_{k,0}(f^{(r)}, \theta)_{r/2+\alpha,r/2+\beta,p} = c W_{r,p},$$

where $W_{r,p} := \omega^{\varphi}_{k,r}(f^{(r)}, \theta)_{\alpha,\beta,p}$.

Let $\tilde{P}_k \in \mathbb{P}_k$ be a polynomial such that

$$\left\| w_{r/2+\alpha,r/2+\beta}(f^{(r)} - \tilde{P}_k) \right\|_p < cW_{r,p},$$

and define $P_{k+r} \in \mathbb{P}_{k+r}$ by

$$P_{k+r}(x) := f(0) + \frac{f'(0)}{1!}x + \cdots + \frac{f^{(r-1)}(0)}{(r-1)!}x^{r-1} + \frac{1}{(r-1)!}\int_0^x (x-u)^{r-1}\tilde{P}_k(u)du.$$

Assuming that $x \ge 0$ (for $x < 0$ all estimates are analogous), we have by Hölder's inequality

$$\begin{aligned}
&(r-1)!\,|f(x) - P_{k+r}(x)| \\
&\quad \le \int_0^x (x-u)^{r-1}\left| f^{(r)}(u) - \tilde{P}_k(u)\right| du \\
&\quad = \int_0^x \frac{(x-u)^{r-1}}{w_{r/2+\alpha,r/2+\beta}(u)} w_{r/2+\alpha,r/2+\beta}(u)|f^{(r)}(u) - \tilde{P}_k(u)|du \\
&\quad \le cA_q(x)W_{r,p},
\end{aligned}$$

where $q := p/(p-1)$,

$$A_q(x) := \left(\int_0^x \left(\frac{(x-u)^{r-1}}{w_{r/2+\alpha,r/2+\beta}(u)}\right)^q du\right)^{1/q}, \quad \text{if} \quad q < \infty,$$

and

$$A_\infty(x) := \sup_{u\in[0,x]} \left(\frac{(x-u)^{r-1}}{w_{r/2+\alpha,r/2+\beta}(u)}\right).$$

Now, since

$$\frac{(x-u)^{r-1}}{w_{r/2+\alpha,r/2+\beta}(u)} \le \frac{(x-u)^{r-1}}{(1-u)^{r/2+\alpha}} \le (1-u)^{r/2-\alpha-1},$$

we have

$$A_q^q(x) \le \int_0^x (1-u)^{q(r/2-\alpha-1)}du \quad \text{and} \quad A_\infty(x) \le \sup_{u\in[0,x]} (1-u)^{r/2-\alpha-1}.$$

If $q < \infty$ and $q(r/2-\alpha-1) > -1$, then

$$A_q^q(x) \le \int_0^1 (1-u)^{q(r/2-\alpha-1)}du = c,$$

which yields

$$\|f - P_{k+r}\|_{L_\infty[0,1]} \le cW_{r,p},$$

and hence

$$\|w_{\alpha,\beta}(f - P_{k+r})\|_{L_p[0,1]} \le cW_{r,p}\|w_{\alpha,\beta}\|_{L_p[0,1]} \le cW_{r,p}, \tag{3.17}$$

where we used the fact that $\alpha \in J_p$. Similarly, (3.17) holds if $q = \infty$ $(p = 1)$ and $r/2-\alpha-1 \ge 0$.

If $q < \infty$ and $q(r/2-\alpha-1) < -1$, then

$$A_q^q(x) \le c(1-x)^{q(r/2-\alpha-1)+1},$$

and so, recalling that $x \ge 0$, we have

$$w_{\alpha,\beta}(x)A_q(x) \le c(1-x)^{r/2-1/p}.$$

Hence,

$$\|w_{\alpha,\beta}(f - P_{k+r})\|_{L_p[0,1]} \le c\|w_{\alpha,\beta}A_q\|_{L_p[0,1]}W_{r,p} \le cW_{r,p}. \tag{3.18}$$

Similarly, one shows that (3.18) holds if $q = \infty$ ($p = 1$) and $r/2 - \alpha - 1 < 0$.

It remains to consider the case $q < \infty$ and $q(r/2 - \alpha - 1) = -1$. We have

$$A_q^q(x) \le \int_0^x (1-u)^{-1}du = c|\ln(1-x)|,$$

and so

$$w_{\alpha,\beta}(x)A_q(x) \le c(1-x)^\alpha|\ln(1-x)|^{1/q}.$$

For $p < \infty$, since $\alpha p > -1$, we have

$$\left\|w_{\alpha,\beta}A_q\right\|^p_{L_p[0,1]} \le c\int_0^1 (1-x)^{\alpha p}|\ln(1-x)|^{p-1}dx < c.$$

Finally, if $p = \infty$, then $q = 1$ and $\alpha = r/2 > 0$, and so $\left\|w_{\alpha,\beta}A_1\right\|_{L_\infty[0,1]} < c$. Hence, (3.18) holds in this case as well.

Similarly, one shows that

$$\|w_{\alpha,\beta}(f - P_{k+r})\|_{L_p[-1,0]} \le cW_{r,p},$$

and the proof is complete. □

4. Direct estimates: proof of Theorems 1.4 and 1.5

The following lemma is [10, Corollary 4.4] with $r = 0$.

Lemma 4.1. *Let $k \in \mathbb{N}$, $\alpha \ge 0$, $\beta \ge 0$ and $f \in L_p^{\alpha,\beta}$, $0 < p \le \infty$. Then, there exists $N \in \mathbb{N}$ depending on k, p, α and β, such that for every $n \ge N$ and $0 < \vartheta \le 1$, there is a polynomial $P_n \in \mathbb{P}_n$ satisfying*

$$\left\|w_{\alpha,\beta}(f - P_n)\right\|_p \le c\omega^{*\varphi}_{k,0}(f, \vartheta/n)_{\alpha,\beta,p} \le c\omega^{\varphi}_{k,0}(f, \vartheta/n)_{\alpha,\beta,p},$$

and

$$n^{-k}\left\|w_{\alpha,\beta}\varphi^k P_n^{(k)}\right\|_p \le c\omega^{*\varphi}_{k,0}(f, \vartheta/n)_{\alpha,\beta,p} \le c\omega^{\varphi}_{k,0}(f, \vartheta/n)_{\alpha,\beta,p},$$

where constants c depend only on k, p, α, β and ϑ.

Proof of Theorem 1.4. Estimate (1.5) immediately follows from Lemma 4.1 for $n \ge N$. For $k \le n < N$, (1.5) follows from Corollary 3.2 with $\theta := \vartheta/N$, since

$$E_n(f)_{\alpha,\beta,p} \le E_k(f)_{\alpha,\beta,p} \le c\omega^{\varphi}_{k,0}(f, \vartheta/N)_{\alpha,\beta,p} \le c\omega^{\varphi}_{k,0}(f, \vartheta/n)_{\alpha,\beta,p}. \quad \square$$

Remark 4.2. In the case $1 \le p \le \infty$, it was shown by Ky [11, Theorem 4] (see also Luther and Russo [12]) that if $\alpha, \beta \ge 0$, then

$$E_n(f)_{\alpha,\beta,p} \le c\omega^k_\varphi(f, 1/n)_{w_{\alpha,\beta},p}, \quad n \ge n_0. \tag{4.1}$$

By virtue of [10, Corollary 1.7 and (5.2)], we have, for $1 \le p \le \infty$,

$$\omega^{\varphi}_{k,r}(f^{(r)}, t)_{\alpha,\beta,p} \sim \omega^k_\varphi(f^{(r)}, t)_{w_{\alpha,\beta}\varphi^r,p}, \quad 0 < t \le t_0. \tag{4.2}$$

Thus, in the case $1 \le p \le \infty$, (1.5) with $n \ge n_0$ follows from (4.1). We also remark that, even though (4.1) was stated with $n_0 = k$ in [11], the proof used [3, Theorem 6.1.1] where $0 < t \le t_0$, and so was only justified for sufficiently large n.

Proof of Theorem 1.5. The case $r = 0$ is Theorem 1.4. Thus we may assume that $r \ge 1$. It follows by [3, Theorem 8.2.1 and (6.3.2)] that, for $n \ge n_0$,

$$\begin{aligned} E_n(f)_{\alpha,\beta,p} &\le c \int_0^{1/n} (\Omega_\varphi^{k+r}(f,t)_{w_{\alpha,\beta},p}/t)dt \\ &\le c \int_0^{1/n} t^{r-1} \Omega_\varphi^k(f^{(r)},t)_{w_{\alpha,\beta}\varphi^r,p}\, dt \\ &\le \frac{c}{n^r} \Omega_\varphi^k(f^{(r)},1/n)_{w_{\alpha,\beta}\varphi^r,p} \le \frac{c}{n^r} \omega_\varphi^k(f^{(r)},1/n)_{w_{\alpha,\beta}\varphi^r,p}\,, \end{aligned} \tag{4.3}$$

where the main-part modulus Ω_φ^m is defined in [3, (8.1.2)]. Hence, (1.5) follows by (4.2). For $k + r \le n < n_0$, (1.5) immediately follows from Theorem 3.4 with $\theta := 1/n_0$, as above. This completes the proof. □

5. Inverse theorem: proof of Theorem 1.6

We first prove this theorem in the case $r \ge 1$.

For the proof we need the following fundamental inequality (see [4,14] as well as [3, (8.1.3)]): given $\alpha, \beta \in J_p$, $1 \le p \le \infty$, we have

$$\left\| w_{\alpha,\beta}\varphi^r p_n^{(r)} \right\|_p \le c(r,p,\alpha,\beta) n^r \left\| w_{\alpha,\beta} p_n \right\|_p, \quad p_n \in \mathbb{P}_n. \tag{5.1}$$

Let $f \in L_p^{\alpha,\beta}$ and let $P_n \in \mathbb{P}_n$ be a polynomial of best approximation of f in $L_p^{\alpha,\beta}$. That is, $E_n(f)_{\alpha,\beta,p} = \|w_{\alpha,\beta}(f - P_n)\|_p$, $n \ge 1$.

Throughout the proof, we often use the estimate

$$\begin{aligned} &\sum_{j=l}^{m} (2^j N)^\nu E_{2^j N}(f)_{\alpha,\beta,p} \\ &\le (1 + 2^\nu) \sum_{j=l}^{m-1} (2^j N)^\nu E_{2^j N}(f)_{\alpha,\beta,p} \\ &\le (1 + 2^\nu) 2^\nu \sum_{j=l}^{m-1} \sum_{n=2^{j-1}N+1}^{2^j N} n^{\nu-1} E_n(f)_{\alpha,\beta,p} \\ &= (1 + 2^\nu) 2^\nu \sum_{n=2^{l-1}N+1}^{2^{m-1}N} n^{\nu-1} E_n(f)_{\alpha,\beta,p}, \end{aligned} \tag{5.2}$$

where $\nu \ge 1$ and $1 \le l < m$, which is also valid if $m = \infty$.

We represent f as the telescopic series

$$f = P_{k+r} + (P_N - P_{k+r}) + \sum_{j=0}^{\infty} \left(P_{2^{j+1}N} - P_{2^j N} \right) =: P_{k+r} + Q + \sum_{j=0}^{\infty} Q_j. \tag{5.3}$$

Since

$$\|w_{\alpha,\beta} Q_j\|_p \le \left\| w_{\alpha,\beta}(P_{2^{j+1}N} - f) \right\|_p + \left\| w_{\alpha,\beta}(f - P_{2^j N}) \right\|_p \le c E_{2^j N}(f)_{w_{\alpha,\beta},p}, \tag{5.4}$$

we have by virtue of (5.1) and (1.7), for each $1 \le \nu \le r$,

$$\sum_{j=0}^{\infty} \left\| w_{\alpha,\beta}\varphi^{\nu} Q_j^{(\nu)} \right\|_p \le c \sum_{j=0}^{\infty} (2^{j+1}N)^{\nu} E_{2^j N}(f)_{\alpha,\beta,p}$$

$$\le c N^{\nu} E_N(f)_{\alpha,\beta,p} + c \sum_{n=N+1}^{\infty} n^{\nu-1} E_n(f)_{\alpha,\beta,p} < \infty.$$

By the same argument as in the proof of [9, Theorem 9.1], it follows that almost everywhere $f(x)$ is identical with a function possessing an absolutely continuous derivative of order $(r-1)$ and $f^{(r)} \in L_p[-1+\varepsilon, 1-\varepsilon]$, for any $\varepsilon > 0$. In particular, differentiation of (5.3) is justified, and $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$.

By [10, Lemma 4.1], since $r/2 + \alpha \ge 0$ and $r/2 + \beta \ge 0$, we have

$$\omega_{k,r}^{\varphi}(Q_j^{(r)}, t)_{\alpha,\beta,\,p} \le c \left\| w_{\alpha,\beta}\varphi^{r} Q_j^{(r)} \right\|_p$$

and

$$\omega_{k,r}^{\varphi}(Q_j^{(r)}, t)_{\alpha,\beta,\,p} \le c t^k \left\| w_{\alpha,\beta}\varphi^{r+k} Q_j^{(r+k)} \right\|_p .$$

Hence, by (5.1) and (5.4) we obtain

$$\omega_{k,r}^{\varphi}\left(Q_j^{(r)}, t\right)_{\alpha,\beta,p} \le c(2^{j+1}N)^r \left\| w_{\alpha,\beta} Q_j \right\|_p \le c(2^j N)^r E_{2^j N}(f)_{\alpha,\beta,p}$$

and

$$\omega_{k,r}^{\varphi}\left(Q_j^{(r)}, t\right)_{\alpha,\beta,p} \le c t^k (2^{j+1}N)^{r+k} \left\| w_{\alpha,\beta} Q_j \right\|_p \le c t^k (2^j N)^{r+k} E_{2^j N}(f)_{\alpha,\beta,p}.$$

Denoting $J := \min\{j \in \mathbb{N}_0 : 2^{-j} \le Nt\}$ (note that $2^{-J} \le Nt < 2^{-J+1}$ if $J \ge 1$, and $Nt \ge 1$ if $J = 0$) we now have by (5.2)

$$\omega_{k,r}^{\varphi}\left(\sum_{j=J+1}^{\infty} Q_j^{(r)}, t\right)_{\alpha,\beta,p} \le c \sum_{j=J+1}^{\infty} \omega_{k,r}^{\varphi}\left(Q_j^{(r)}, t\right)_{\alpha,\beta,p} \tag{5.5}$$

$$\le c \sum_{j=J+1}^{\infty} (2^j N)^r E_{2^j N}(f)_{\alpha,\beta,p}$$

$$\le c \sum_{n=2^J N+1}^{\infty} n^{r-1} E_n(f)_{\alpha,\beta,p}$$

$$\le c \sum_{n>\max\{N,1/t\}} n^{r-1} E_n(f)_{\alpha,\beta,p},$$

since $2^J N + 1 > \max\{N, 1/t\}$. Now, if $J \ge 2$, then (5.2) implies

$$\omega_{k,r}^{\varphi}\left(\sum_{j=0}^{J} Q_j^{(r)}, t\right)_{\alpha,\beta,p} \le c t^k \sum_{j=0}^{J} (2^j N)^{r+k} E_{2^j N}(f)_{\alpha,\beta,p} \tag{5.6}$$

$$\le c t^k N^{r+k} E_N(f)_{\alpha,\beta,p} + c t^k \sum_{n=N+1}^{2^{J-1}N} n^{r+k-1} E_n(f)_{\alpha,\beta,p}$$

$$\le c t^k N^{r+k} E_N(f)_{\alpha,\beta,p} + c t^k \sum_{N \le n \le \max\{N,1/t\}} n^{k+r-1} E_n(f)_{\alpha,\beta,p},$$

where we used the fact that $2^{J-1}N \le \max\{N, 1/t\}$. If $J = 0$ or 1, then we have

$$\omega^{\varphi}_{k,r}\left(\sum_{j=0}^{J} Q_j^{(r)}, t\right)_{\alpha,\beta,p} \le ct^k N^{r+k} E_N(f)_{\alpha,\beta,p},$$

and so the last estimate in (5.6) is valid in this case as well.

Finally, if $N \ge k + r$, then

$$\begin{aligned}\omega^{\varphi}_{k,r}(P^{(r)}_{k+r} + Q^{(r)}, t)_{\alpha,\beta,p} = \omega^{\varphi}_{k,r}(Q^{(r)}, t)_{\alpha,\beta,p} &\le ct^k \left\| w_{\alpha,\beta}\varphi^{k+r} Q^{(k+r)}\right\|_p \\ &\le ct^k N^{r+k} \left\| w_{\alpha,\beta} Q\right\|_p \le ct^k N^{r+k} E_{k+r}(f)_{\alpha,\beta,p},\end{aligned} \tag{5.7}$$

and if $N < k + r$, then $\omega^{\varphi}_{k,r}(P^{(r)}_{k+r} + Q^{(r)}, t)_{\alpha,\beta,p} = 0$, so that we do not need (5.7).

Combining (5.5)–(5.7) and using the fact that, if $N \ge k + r$, then $E_N(f)_{\alpha,\beta,p} \le E_{k+r}(f)_{\alpha,\beta,p}$ and, if $N < k+r$, then the first term in the last inequality in (5.6) can be absorbed by the second term in that inequality, we obtain (1.8), and our proof is complete in the case $r \ge 1$.

Suppose now that $r = 0$. We represent f as

$$f = P_k + Q + \sum_{j=0}^{J} Q_j + \left(f - P_{2^{J+1}N}\right), \tag{5.8}$$

where $Q := P_N - P_k$ and $Q_j := P_{2^{j+1}N} - P_{2^j N}$, and estimate the last term. We have

$$\left\| w_{\alpha,\beta}(f - P_{2^{J+1}N})\right\|_p \le cE_{2^{J+1}N}(f)_{\alpha,\beta,p},$$

and in the case $J = 0$ or 1, we use the fact that $Nt \ge c$, to conclude

$$E_{2^{J+1}N}(f)_{\alpha,\beta,p} \le E_N(f)_{\alpha,\beta,p} = N^k t^k (Nt)^{-k} E_N(f)_{\alpha,\beta,p} \le c(N)t^k E_N(f)_{\alpha,\beta,p}.$$

If $J \ge 2$, we recall that $2^{J-1}N < 1/t \le 2^J N$, so that $\max\{N, 1/t\} = 1/t$, and write

$$\begin{aligned}E_{2^{J+1}N}(f)_{\alpha,\beta,p} &\le (2^{J-2}N)^{-1} \sum_{n=2^{J-2}N+1}^{2^{J-1}N} E_n(f)_{\alpha,\beta,p} \\ &\le (2^{J-2}N)^{-k} \sum_{n=2^{J-2}N+1}^{2^{J-1}N} n^{k-1} E_n(f)_{\alpha,\beta,p} \\ &\le 4^k t^k \sum_{N\le n<1/t} n^{k-1} E_n(f)_{\alpha,\beta,p}.\end{aligned}$$

It now remains to apply (5.6) and (5.7) with $r = 0$, in order to complete the proof of (1.8) in the case $r = 0$. □

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