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Interpolatory pointwise estimates for monotone polynomial approximation

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A B S T R A C T

Given a nondecreasing function f on $[-1,1]$, we investigate how well it can be approximated by nondecreasing algebraic polynomials that interpolate it at ± 1. We establish pointwise estimates of the approximation error by such polynomials that yield interpolation at the endpoints (*i.e.,* the estimates become zero at ± 1). We call such estimates "interpolatory estimates". In 1985, DeVore and Yu were the first to obtain this kind of results for monotone polynomial approximation. Their estimates involved the second modulus of smoothness $\omega_2(f,\cdot)$ of f evaluated at $\sqrt{1-x^2}/n$ and were valid for all $n \ge 1$. The current paper is devoted to proving that if $f \in C^r[-1,1]$, $r \ge 1$, then the interpolatory estimates are valid for the second modulus of smoothness of $f^{(r)}$, however, only for $n \ge \mathcal{N}$ with $\mathcal{N} = \mathcal{N}(f,r)$, since it is known that such estimates are in general invalid with $\mathcal{N}$ independent of f. Given a number $\alpha > 0$, we write $\alpha = r + \beta$ where r is a nonnegative integer and $0 < \beta \le 1$, and denote by $\mathrm{Lip}^*\,\alpha$ the class of all functions f on $[-1,1]$ such that $\omega_2(f^{(r)},t) = O(t^\beta)$. Then, one important corollary of the main theorem in this paper is the following result that has been an open problem for $\alpha \ge 2$ since 1985:

> If $\alpha > 0$, then a function f is nondecreasing and in $\mathrm{Lip}^*\,\alpha$, if and only if, there exists a constant C such that, for all sufficiently large n, there are nondecreasing polynomials P_n, of degree n, such that
>
> $$|f(x) - P_n(x)| \le C\left(\frac{\sqrt{1-x^2}}{n}\right)^\alpha, \quad x \in [-1,1].$$

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1. Introduction and main results

Given a nondecreasing function f on $[-1,1]$ and a set $\Xi := \{\xi_i\}_{i=1}^m \subset [-1,1]$ ($\xi_i \neq \xi_j$ if $i \neq j$), is there a nondecreasing algebraic polynomial that not only approximates f well but also interpolates f at the points in Ξ? For a general set Ξ, the answer is clearly “no”. If $m \geq 3$, then the nondecreasing interpolating polynomial may not exist at all (consider f which is constant on $[\xi_1, \xi_2]$ and such that $f(\xi_3) > f(\xi_2)$).

If $m = 1$, then the case for interpolation at either -1 or 1 (but not both) was considered in [4], and we leave the discussion of the case when $-1 < \xi_1 < 1$ for another time.

Finally, if $m = 2$, then the nondecreasing polynomial interpolating f at ξ_1 and ξ_2 exists, but it does not approximate f well at all if $[\xi_1, \xi_2] \neq [-1,1]$ (again, consider f which is constant on $[\xi_1, \xi_2]$ and is strictly increasing outside this interval). Hence, for $m = 2$, the only non-trivial case that remains is when the nondecreasing polynomial interpolates f at the endpoints of $[-1,1]$. We call the pointwise estimates of the degree of approximation of f by such polynomials that yield interpolation at the endpoints (*i.e.,* the estimates become zero at ± 1) “interpolatory estimates in monotone polynomial approximation”.

We also note that the situation with strictly increasing functions is rather different (see *e.g.* [5,13] and the references therein), since for any strictly increasing function f and any collection of points Ξ, there exists a strictly increasing polynomial of a sufficiently large degree that interpolates f at all points in Ξ. How well this polynomial approximates f is an interesting problem but we do not consider it in this manuscript.

More discussions of various related results on monotone approximation can be found in our survey paper [8].

For $r \in \mathbb{N}$, let $C^r[a,b]$, $-1 \leq a < b \leq 1$, denote the space of r times continuously differentiable functions on $[a,b]$, and let $C^0[a,b] = C[a,b]$ denote the space of continuous functions on $[a,b]$, equipped with the uniform norm $\|\cdot\|_{[a,b]}$.

For $f \in C[a,b]$ and any $k \in \mathbb{N}$, set

$$\Delta_u^k(f,x;[a,b]) := \begin{cases} \displaystyle\sum_{i=0}^{k} (-1)^i \binom{k}{i} f(x + (k/2 - i)u), & x \pm (k/2)u \in [a,b], \\ 0, & \text{otherwise}, \end{cases}$$

and denote by

$$\omega_k(f,t;[a,b]) := \sup_{0<u\leq t} \|\Delta_u^k(f,\cdot;[a,b])\|_{[a,b]}$$

its kth modulus of smoothness. When dealing with $[a,b] = [-1,1]$, we suppress referring to the interval, that is, we denote $\|\cdot\| := \|\cdot\|_{[-1,1]}$ and $\omega_k(f,t) := \omega_k(f,t;[-1,1])$.

Finally, let

$$\varphi(x) := \sqrt{1-x^2} \quad \text{and} \quad \rho_n(x) := \frac{\varphi(x)}{n} + \frac{1}{n^2}, \tag{1.1}$$

and denote by $\Delta^{(1)}$ the class of all nondecreasing functions on $[-1,1]$, and by Π_n the space of algebraic polynomials of degree $\leq n$.

In 1985, DeVore and Yu [1, Theorem 1] proved that, for $f \in C[-1,1] \cap \Delta^{(1)}$ and any $n \in \mathbb{N}$, there exists a polynomial $P_n \in \Pi_n \cap \Delta^{(1)}$ such that

$$|f(x) - P_n(x)| \leq c\omega_2\left(f, \frac{\varphi(x)}{n}\right), \quad x \in [-1,1], \tag{1.2}$$

where c is an absolute constant.

In 1998, it was proved in [11, Theorem 4] that there exists $f \in C[-1,1] \cap \Delta^{(1)}$ such that

$$\limsup_{n\to\infty} \inf_{P_n \in \Pi_n \cap \Delta^{(1)}} \max_{x\in[-1,1]} \frac{|f(x) - P_n(x)|}{\omega_3(f, \rho_n(x))} = \infty, \tag{1.3}$$

which implies that ω_2 in (1.2) cannot be replaced by ω_3 even if the constant c and how large n is are allowed to depend on the function f.

If the function f is smoother, then the following is valid (see [14]):

For any $k, r \in \mathbb{N}$ and $f \in C^r[-1,1] \cap \Delta^{(1)}$, there exists a sequence of polynomials $P_n \in \Pi_n \cap \Delta^{(1)}$ such that, for every $n \ge k + r - 1$ and each $x \in [-1,1]$, we have

$$|f(x) - P_n(x)| \le c(k,r)\rho_n^r(x)\omega_k(f^{(r)}, \rho_n(x)).$$

A natural question now is whether (1.2) may be strengthened for functions having higher smoothness. More precisely, the following problem needs to be resolved: find all values of $k \in \mathbb{N}$ and $r \in \mathbb{N}_0$ such that the following statement is true, and investigate whether or not the number $\mathcal{N}$ in this statement has to depend on f.

Statement 1.1. *For every $f \in C^r[-1,1] \cap \Delta^{(1)}$, $r \ge 1$, there exist a number $\mathcal{N} \in \mathbb{N}$ and a sequence $\{P_n\}_{n=\mathcal{N}}^\infty$ of polynomials $P_n \in \Pi_n \cap \Delta^{(1)}$ such that, for every $n \ge \mathcal{N}$ and each $x \in [-1,1]$, we have*

$$|f(x) - P_n(x)| \le c(k,r)\left(\frac{\varphi(x)}{n}\right)^r \omega_k\left(f^{(r)}, \frac{\varphi(x)}{n}\right). \tag{1.4}$$

In view of (1.2) and (1.3), Statement 1.1 is true if $k + r \le 2$ (with $\mathcal{N} = 1$) and is not true for $r = 0$ and $k \ge 3$.

Using the same method as was used to prove [4, Theorem 4] one can show that, for any $r \in \mathbb{N}$ and each $n \in \mathbb{N}$, there is a function $f \in C^r[-1,1] \cap \Delta^{(1)}$, such that for every polynomial $P_n \in \Pi_n \cap \Delta^{(1)}$ and any function ψ, positive on $(-1,1)$, such that $\lim_{x\to\pm1} \psi(x) = 0$, either

$$\limsup_{x\to-1} \frac{|f(x) - P_n(x)|}{\varphi^2(x)\psi(x)} = \infty \quad \text{or} \quad \limsup_{x\to1} \frac{|f(x) - P_n(x)|}{\varphi^2(x)\psi(x)} = \infty. \tag{1.5}$$

In particular, this implies that Statement 1.1 is not valid with $\mathcal{N}$ independent of f if $k + r \ge 3$. However, in this paper, we show that this statement is valid for $k = 2$ and any $r \in \mathbb{N}$ provided that $\mathcal{N}$ depends on f. Namely, the following theorem is the main result in this manuscript.

Theorem 1.2. *Given $r \in \mathbb{N}$, there is a constant $c = c(r)$ with the property that if $f \in C^r[-1,1] \cap \Delta^{(1)}$, then there exists a number $\mathcal{N} = \mathcal{N}(f, r)$, depending on f and r, such that for every $n \ge \mathcal{N}$, there is $P_n \in \Pi_n \cap \Delta^{(1)}$ satisfying*

$$|f(x) - P_n(x)| \le c(r)\left(\frac{\varphi(x)}{n}\right)^r \omega_2\left(f^{(r)}, \frac{\varphi(x)}{n}\right), \quad x \in [-1,1]. \tag{1.6}$$

Moreover, for $x \in \left[-1, -1 + n^{-2}\right] \cup \left[1 - n^{-2}, 1\right]$ the following stronger estimate is valid:

$$|f(x) - P_n(x)| \le c(r)\varphi^{2r}(x)\omega_2\left(f^{(r)}, \frac{\varphi(x)}{n}\right). \tag{1.7}$$

Given a number $\alpha > 0$, we write $\alpha = r + \beta$ where r is a nonnegative integer and $0 < \beta \le 1$. Denote by $\mathrm{Lip}^* \alpha$ the class of all functions f on $[-1,1]$ such that $\omega_2(f^{(r)}, t) = O(t^\beta)$.

An immediate corollary of Theorem 1.2 and the classical (Dzyadyk) converse theorems for approximation by algebraic polynomials is the following result on characterization of $\mathrm{Lip}^* \alpha$.

Corollary 1.3. *If $\alpha > 0$, then a function f is nondecreasing and in $\mathrm{Lip}^* \alpha$, if and only if, there exists a constant C such that, for sufficiently large n, there are nondecreasing polynomials P_n of degree n such that*

$$|f(x) - P_n(x)| \le C \left(\frac{\sqrt{1-x^2}}{n} \right)^\alpha, \quad x \in [-1,1].$$

Note that, for $0 < \alpha < 2$, Corollary 1.3 follows from (1.2) (and was stated in [1]).

In order to state another corollary of Theorem 1.2 we recall that W^r denotes the space of $(r-1)$ times continuously differentiable functions on $[-1,1]$ such that $f^{(r-1)}$ is absolutely continuous in $(-1,1)$ and $\left\| f^{(r)} \right\|_\infty < \infty$, where $\|\cdot\|_\infty$ denotes the essential supremum on $[-1,1]$.

Corollary 1.4. *For any $f \in W^r \cap \Delta^{(1)}$, $r \in \mathbb{N}$, there exists a number $\mathcal{N} = \mathcal{N}(f,r)$, such that for every $n \ge \mathcal{N}$,*

$$\inf_{P_n \in \Pi_n \cap \Delta^{(1)}} \left\| \frac{f - P_n}{\varphi^r} \right\|_\infty \le \frac{c(r)}{n^r} \left\| f^{(r)} \right\|_\infty .$$

Note that, for $r \le 2$, Corollary 1.4 follows from (1.2) with $\mathcal{N} = 1$.

The paper is organized as follows. In Section 2, we introduce various notations that are used throughout the paper. Several inequalities for the Chebyshev partition are discussed in Section 3, and Section 4 is devoted to a discussion of polynomial approximation of indicator functions. In Section 5, we prove several auxiliary results on various properties of piecewise polynomials. We need those since our proof of Theorem 1.2 will be based on approximating f by certain monotone piecewise polynomial functions, and then approximating these functions by monotone polynomials. In Section 6, we discuss approximation of monotone piecewise polynomials with “small” first derivatives by monotone polynomials. Section 7 is devoted to constructing a certain partition of unity. Simultaneous polynomial approximation of piecewise polynomials and their derivatives is discussed in Section 8 and, in Section 9, we construct one particular polynomial with controlled first derivative. Finally, in Section 10, we use all these auxiliary results to prove a lemma on monotone polynomial approximation of piecewise polynomials that is then used in Section 11 to prove Theorem 1.2.

We conclude this section by stating the following open problem.

Open Problem 1.5. Find all pairs (r,k) with $r \in \mathbb{N}$ and $k \ge 3$ for which Statement 1.1 is valid (with $\mathcal{N}$ dependent on f).

2. Notations

Recall that the Chebyshev partition of $[-1,1]$ is the ordered set $X_n := (x_j)_{j=0}^n$, where

$$x_j := x_{j,n} := \cos(j\pi/n), \quad 0 \le j \le n.$$

We refer to x_j's as “Chebyshev knots” and note that x_j's are the extremum points of the Chebyshev polynomial of the first kind of degree n. It is also convenient to denote $x_j := x_{j,n} := 1$ for $j < 0$ and $x_j := x_{j,n} := -1$ for $j > n$. Also, let $I_j := [x_j, x_{j-1}]$, $h_j := |I_j| := x_{j-1} - x_j$, and

$$\chi_j(x) := \chi_{[x_j,1]}(x) = \begin{cases} 1, & \text{if } x_j \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Denote by $\Sigma_k := \Sigma_{k,n}$ the set of all right continuous piecewise polynomials of degree $\le k-1$ with knots at x_j, $1 \le j \le n-1$. That is,

$$S \in \Sigma_k \quad \text{if and only if} \quad S|_{[x_j,x_{j-1})} \in \Pi_{k-1},\ 2 \le j \le n, \quad \text{and} \quad S|_{[x_1,1]} \in \Pi_{k-1}.$$

Throughout this paper, for $S \in \Sigma_k$, we denote the polynomial piece of S inside the interval I_j by p_j, *i.e.*,

$$p_j := p_j(S) := S|_{[x_j,x_{j-1})}, \quad 2 \le j \le n, \quad \text{and} \quad p_1 := p_1(S) := S|_{[x_1,1]}.$$

For $k \in \mathbb{N}$, let Φ^k be the class of all "k-majorants", *i.e.*, continuous nondecreasing functions ψ on $[0,\infty)$ such that $\psi(0)=0$ and $t^{-k}\psi(t)$ is nonincreasing on $[0,\infty)$. In other words,

$$\Phi^k = \left\{\psi \in C[0,\infty) \;\middle|\; \psi\uparrow,\ \psi(0)=0,\ \text{and } t_2^{-k}\psi(t_2) \le t_1^{-k}\psi(t_1) \text{ for } 0 < t_1 \le t_2\right\}.$$

Note that, given $f \in C^r[-1,1]$, while the function $\phi(t) := t^r\omega_k(f^{(r)},t)$ does not have to be in Φ^{k+r}, it is equivalent to a function from Φ^{k+r}. Namely, $\phi(t) \le \phi^*(t) \le 2^k\phi(t)$, where $\phi^*(t) := \sup_{u>t} t^{k+r}u^{-k-r}\phi(u) \in \Phi^{k+r}$ (see, *e.g.*, [2, p. 202]).

For $1 \le i,j \le n$, let

$$I_{i,j} := \bigcup_{k=\min\{i,j\}}^{\max\{i,j\}} I_k = \left[x_{\max\{i,j\}}, x_{\min\{i,j\}-1}\right]$$

and

$$h_{i,j} := |I_{i,j}| = \sum_{k=\min\{i,j\}}^{\max\{i,j\}} h_k = x_{\min\{i,j\}-1} - x_{\max\{i,j\}}.$$

In other words, $I_{i,j}$ is the smallest interval that contains both I_i and I_j, and $h_{i,j}$ is its length.

For $\phi \in \Phi^k$, which is not identically zero (otherwise everything is either trivial or of no value), and $S \in \Sigma_k$, denote

$$b_{i,j}(S,\phi) := \frac{\|p_i - p_j\|_{I_i}}{\phi(h_j)} \left(\frac{h_j}{h_{i,j}}\right)^k, \quad 1 \le i,j \le n. \tag{2.1}$$

(Note that $b_{i,j}(S,\phi) = a_{i,j}(S)/\phi(h_j)$ with $a_{i,j}$ defined in [12, (6.1)].)

Also, for $S \in \Sigma_k$ and an interval $A \subseteq [-1,1]$ containing at least one interval I_ν, denote

$$b_k(S,\phi,A) := \max_{1\le i,j\le n}\left\{b_{i,j}(S,\phi) \;\middle|\; I_i \subset A \quad \text{and} \quad I_j \subset A\right\},$$

and

$$b_k(S,\phi) := b_k(S,\phi,[-1,1]) = \max_{1\le i,j\le n} b_{i,j}(S,\phi).$$

Throughout this paper, we reserve the notation "c" for positive constants that are either absolute or may only depend on the parameter k (and eventually will depend on r). We use the notation "C" and "C_i" (the latter only in Section 10) for all other positive constants and indicate in each section the parameters that they may depend on. All constants c and C may be different on different occurrences (even if they appear in the same line), but the indexed constants C_i are fixed throughout Section 10.

3. Inequalities for the Chebyshev partition

In this section, we collect all the facts and inequalities for the Chebyshev partition that we need throughout this paper.

It is rather well known (see, *e.g.*, [2, pp. 382–383, 408]) and not too difficult to verify that

$$\frac{\varphi(x)}{n} < \rho_n(x) < h_j < 5\rho_n(x), \quad x \in I_j, \quad 1 \le j \le n, \qquad (3.1)$$
$$h_{j\pm 1} < 3h_j, \quad 1 \le j \le n,$$

and

$$\rho_n^2(y) < 4\rho_n(x)(|x-y| + \rho_n(x)) \quad \text{and} \qquad (3.2)$$
$$(|x-y| + \rho_n(x))/2 < |x-y| + \rho_n(y) < 2(|x-y| + \rho_n(x)), \quad x, y \in [-1,1].$$

(We remark that the inequalities on the second line in (3.2) immediately follow from the estimate on the first line.)

Also, we observe that

$$\rho_n(x) \le |x - x_j|, \quad \text{for any } \ 0 \le j \le n \quad \text{and} \quad x \notin [x_{j+1}, x_{j-1}]. \qquad (3.3)$$

Indeed, (3.3) holds for $x = x_{j\pm 1}$ (excluding x_{-1} and x_{n+1}) by (3.1), and for all other $x \notin [x_{j+1}, x_{j-1}]$, it follows from the inequalities $x + \rho_n(x) \le x_{j+1} + \rho_n(x_{j+1})$ if $x < x_{j+1}$, and $x - \rho_n(x) \ge x_{j-1} - \rho_n(x_{j-1})$ if $x > x_{j-1}$, that can be verified directly or using the fact that $x + \rho_n(x)$ increases on $\left[-1, n/\sqrt{n^2+1}\right] \supset [-1, x_1]$ and $x - \rho_n(x)$ increases on $\left[-n/\sqrt{n^2+1}, 1\right] \supset [x_{n-1}, 1]$.

Now, denote

$$\psi_j := \psi_j(x) := \frac{|I_j|}{|x - x_j| + |I_j|} \quad \text{and} \quad \delta_n(x) := \min\{1, n\varphi(x)\}, \quad x \in [-1,1],$$

and note that

$$\delta_n(x) = 1 \quad \text{if} \quad x \in [x_{n-1}, x_1]$$

and

$$\delta_n(x) \le n\varphi(x) < \pi\delta_n(x) \quad \text{if} \quad x \in [-1, x_{n-1}] \cup [x_1, 1].$$

It follows from (3.1) and (3.2) that

$$\rho_n^2(x) < 4\rho_n(x_j)\left(|x - x_j| + \rho_n(x_j)\right) < 8h_j\left(|x - x_j| + \rho_n(x)\right), \qquad (3.4)$$

and thus

$$\left(\frac{\rho_n(x)}{\rho_n(x) + |x - x_j|}\right)^2 < \frac{8h_j}{\rho_n(x) + |x - x_j|} < c\psi_j(x). \qquad (3.5)$$

Similarly, (3.1) and (3.2) imply (see, *e.g.*, [7, (26)]) that

$$c\psi_j^2(x)\rho_n(x) \le \rho_n(x_j) \le c\psi_j^{-1}(x)\rho_n(x), \quad 1 \le j \le n \quad \text{and} \quad x \in [-1,1], \qquad (3.6)$$

where c are some absolute constants.

It is not difficult to see that, for all $1 \le j \le n$ and $x \in [-1, 1]$,

$$\rho_n(x) + \operatorname{dist}(x, I_j) \le \rho_n(x) + |x - x_j| \le 16\,(\rho_n(x) + \operatorname{dist}(x, I_j))\,. \tag{3.7}$$

Indeed, the first inequality in (3.7) is obvious, and the second follows from

$$|x - x_j| \le 4\operatorname{dist}(x, I_j) + 15\rho_n(x),$$

which is verified using (3.1) and separately considering the cases $x \in I_{j-1} \cup I_j \cup I_{j+1}$ and $x \notin I_{j-1} \cup I_j \cup I_{j+1}$ (in the latter case, there is at least one interval I_i, $i \ne j$, between x and I_j, so that $|x-x_j| \le h_j+\operatorname{dist}(x, I_j) \le 4\operatorname{dist}(x, I_j)$).

Also, it is straightforward to check that

$$\sum_{j=1}^{n} \psi_j^2(x) \le c, \quad x \in [-1, 1], \tag{3.8}$$

and so, by virtue of (3.7) and (3.5),

$$\sum_{j=1}^{n} \left(\frac{\rho_n(x)}{\rho_n(x) + \operatorname{dist}(x, I_j)}\right)^4 \le c. \tag{3.9}$$

In order to quote several results from [7] in the form used in this paper we need the following observation. First, it is known (see, *e.g.*, [7, Proposition 4]) that

$$\frac{1 - x^2}{(1 + x_{j-1})(1 - x_j)} \le 2\psi_j^{-2}(x), \quad 1 \le j \le n \quad \text{and} \quad -1 \le x \le 1.$$

Now, since

$$\min_{1\le j\le n} \{(1 + x_{j-1})(1 - x_j)\} \ge 1 - x_1 \ge 2/n^2,$$

we have

$$\frac{1 - x^2}{(1 + x_{j-1})(1 - x_j)} \le \frac{n^2\varphi^2(x)}{2},$$

and hence, for all $1 \le j \le n$ and $x \in [-1, 1]$,

$$\frac{1 - x^2}{(1 + x_{j-1})(1 - x_j)} \le 2\min\{1, n^2\varphi^2(x)\}\psi_j^{-2}(x) = 2\delta_n^2(x)\psi_j^{-2}(x). \tag{3.10}$$

Conversely, by (3.6)

$$\frac{1 - x^2}{(1 + x_{j-1})(1 - x_j)} \ge \frac{c\varphi^2(x)}{n^2\rho_n^2(x_j)} \ge c\psi_j^2(x)\frac{n^2\varphi^2(x)}{(n\varphi(x) + 1)^2} \ge c\psi_j^2(x)\delta_n^2(x), \tag{3.11}$$

where the first inequality is valid since

$$(1 + x_{j-1})(1 - x_j) = 1 - x_j^2 + h_j(1 - x_j) \le n^2\rho_n^2(x_j) + \rho_n(x_j) \le 2n^2\rho_n^2(x_j).$$

4. Auxiliary results on polynomial approximation of indicator functions

All constants C in this section depend on α and β.

Lemma 4.1. *Given $\alpha, \beta \ge 1$, there exist polynomials τ_j, $1 \le j \le n-1$, of degree $\le Cn$ satisfying, for all $x \in [-1,1]$,*

$$\tau_j'(x) \ge C|I_j|^{-1}\delta_n^{8\alpha}(x)\psi_j^{30(\alpha+\beta)}(x), \tag{4.1}$$

$$\left|\tau_j^{(q)}(x)\right| \le C|I_j|^{-q}\delta_n^{\alpha}(x)\psi_j^{\beta}(x), \quad 1 \le q \le \alpha, \tag{4.2}$$

and

$$|\chi_j(x) - \tau_j(x)| \le C\delta_n^{\alpha}(x)\psi_j^{\beta}(x). \tag{4.3}$$

Proof. First, estimates (4.2) and (4.3) immediately follow from [7, Lemma 6] taking into account (3.10) and setting $\mu := \lceil 10\alpha + 10\beta \rceil$ and $\xi := \lceil 3\alpha \rceil$ in that lemma. Estimate (4.1) was not proved in [7], and so, even though its proof is very similar to that of (4.2) and (4.3), we adduce it here for the sake of completeness.

Recall the definition of polynomials τ_j:

$$\tau_j(x) = d_j^{-1} \int_{-1}^{x} (1-y^2)^{\xi} t_j^{\mu}(y)\, dy, \tag{4.4}$$

where

$$t_j(x) := \left(\frac{\cos 2n \arccos x}{x - x_j^0}\right)^2 + \left(\frac{\sin 2n \arccos x}{x - \bar{x}_j}\right)^2, \tag{4.5}$$

$\bar{x}_j := \cos((j-1/2)\pi/n)$ for $1 \le j \le n$, $x_j^0 := \cos((j-1/4)\pi/n)$ for $1 \le j < n/2$, $x_j^0 := \cos((j-3/4)\pi/n)$ for $n/2 \le j \le n$, and the normalizing constants d_j are chosen so that $\tau_j(1) = 1$.

It is known (see, *e.g.*, [7, (22), Proposition 5]) and is not difficult to prove that

$$t_j(x) \sim (|x - x_j| + h_j)^{-2}, \quad x \in [-1,1] \quad \text{and} \quad 1 \le j \le n, \tag{4.6}$$

and

$$d_j \sim (1 + x_{j-1})^{\xi}(1 - x_j)^{\xi} h_j^{-2\mu+1}, \quad \text{if } \mu \ge \xi + 1.$$

Here and later, by $X \sim Y$ we mean that there exists a positive constant c (independent of the important parameters) such that $c^{-1}X \le Y \le cX$.

Hence, using (3.11), we have

$$\begin{aligned}
\tau_j'(x) &= d_j^{-1}(1-x^2)^{\xi} t_j^{\mu}(x) \\
&\ge C \frac{h_j^{2\mu-1}}{(1+x_{j-1})^{\xi}(1-x_j)^{\xi}} (1-x^2)^{\xi} (|x-x_j| + h_j)^{-2\mu} \\
&\ge C h_j^{-1} \delta_n^{2\xi}(x) \psi_j^{2\mu+2\xi}(x) \\
&\ge C h_j^{-1} \delta_n^{8\alpha}(x) \psi_j^{30(\alpha+\beta)}(x). \quad \square
\end{aligned}$$

Lemma 4.2. *Given $\alpha, \beta > 0$, there exist polynomials $\widetilde{\tau}_j$, $1 \le j \le n-1$, of degree $\le Cn$ satisfying*

$$\widetilde{\tau}_j'(x) \le 0, \quad \text{for } x \in [-1, x_j] \cup [x_{j-1}, 1], \tag{4.7}$$

and, for all $x \in [-1, 1]$,

$$\left|\widetilde{\tau}_j'(x)\right| \le C|I_j|^{-1}\delta_n^\alpha(x)\psi_j^\beta(x) \tag{4.8}$$

and

$$|\chi_j(x) - \widetilde{\tau}_j(x)| \le C\delta_n^\alpha(x)\psi_j^\beta(x). \tag{4.9}$$

Proof. We let

$$\widetilde{\tau}_j(x) := \widetilde{d}_j^{-1} \int_{-1}^{x} (y - x_j)(x_{j-1} - y)(1 - y^2)^\xi t_j^\mu(y)\, dy$$

with t_j defined in (4.5) and $\widetilde{d}_j$ is so chosen that $\widetilde{\tau}_j(1) = 1$, and where ξ and μ are sufficiently large and will be prescribed later. Clearly, (4.7) is satisfied.

It is possible to show (see, *e.g.*, [6, Proposition 4] with $m = k = \xi + 1$, $a_1 = \dots = a_{m-1} = -1$, $b_1 = \dots = b_{k-1} = 1$, $a_m = x_j$, $b_k = x_{j-1}$) that

$$\widetilde{d}_j \sim (1 + x_{j-1})^\xi (1 - x_j)^\xi h_j^{-2\mu+3}, \quad \text{if } \mu \ge 10\xi + 15.$$

Hence, using (4.6) we have

$$\left|\widetilde{\tau}_j'(x)\right| = \widetilde{d}_j^{-1}(1 - x^2)^\xi |x - x_j||x_{j-1} - x||t_j^\mu(x)| \le C\left(\frac{1 - x^2}{(1 + x_{j-1})(1 - x_j)}\right)^\xi h_j^{-1}\psi_j^{2\mu-2}(x).$$

We note (cf. [7, (25)]) that, for all $x \in [-1, 1]$,

$$\frac{1 + x}{1 + x_{j-1}} \le c\psi_j^{-1}(x) \quad \text{and} \quad \frac{1 - x}{1 - x_j} \le c\psi_j^{-1}(x).$$

Now, if $x < x_j$, then

$$\begin{aligned}
|\chi_j(x) - \widetilde{\tau}_j(x)| &= |\widetilde{\tau}_j(x)| = \left|\int_{-1}^{x} \widetilde{\tau}_j'(y) dy\right| \\
&\le Ch_j^{-1} \int_{-1}^{x} \left(\frac{1 + y}{1 + x_{j-1}}\right)^\xi \left(\frac{h_j}{|y - x_j| + h_j}\right)^{2\mu-\xi-2} dy \\
&\le C\left(\frac{1 + x}{1 + x_{j-1}}\right)^\xi h_j^{2\mu-\xi-3} \int_{-\infty}^{x} (x_j - y + h_j)^{-2\mu+\xi+2} dy \\
&\le C\left(\frac{1 - x^2}{(1 + x_{j-1})(1 - x_j)}\right)^\xi \psi_j^{2\mu-\xi-3}.
\end{aligned}$$

Similarly, for $x \ge x_j$, we write

$$
\begin{aligned}
|\chi_j(x) - \widetilde{\tau}_j(x)| &= |1 - \widetilde{\tau}_j(x)| = \left| \int_x^1 \widetilde{\tau}_j'(y)dy \right| \\
&\le C h_j^{-1} \int_x^1 \left(\frac{1-y}{1-x_j} \right)^{\xi} \left(\frac{h_j}{|y-x_j|+h_j} \right)^{2\mu-\xi-2} dy \\
&\le C \left(\frac{1-x}{1-x_j} \right)^{\xi} h_j^{2\mu-\xi-3} \int_x^{\infty} (y - x_j + h_j)^{-2\mu+\xi+2} dy \\
&\le C \left(\frac{1-x^2}{(1+x_{j-1})(1-x_j)} \right)^{\xi} \psi_j^{2\mu-\xi-3}.
\end{aligned}
$$

Finally, using (3.10), we conclude that

$$
\left| \widetilde{\tau}_j'(x) \right| \le C \delta_n^{2\xi} h_j^{-1} \psi_j^{2\mu-2\xi-2}(x)
$$

and

$$
|\chi_j(x) - \widetilde{\tau}_j(x)| \le C \delta_n^{2\xi} h_j^{-1} \psi_j^{2\mu-3\xi-3}(x),
$$

and it is enough to set $\xi := \lceil \alpha/2 \rceil$ and $\mu := \lceil \beta + 5\alpha \rceil + 25$ in order to complete the proof. □

5. Auxiliary results on properties of piecewise polynomials

All constants c in this section depend only on k.

The following lemma is valid (compare with [3, Lemma 1.4]).

Lemma 5.1. *Let $k \in \mathbb{N}$, $\phi \in \Phi^k$, $f \in C[-1,1]$ and $S \in \Sigma_{k,n}$. If*

$$
\omega_k(f,t) \le \phi(t)
$$

and

$$
|f(x) - S(x)| \le \phi(\rho_n(x)), \quad x \in [-1,1], \tag{5.1}
$$

then

$$
b_k(S,\phi) \le c.
$$

Proof. Recall that ϕ is not identically zero, so that $\phi(x) > 0$, $x > 0$. For $1 \le i,j \le n$, we have

$$
b_{i,j}(S,\phi) \le \frac{\|p_i - f\|_{I_i}}{\phi(h_j)} \left(\frac{h_j}{h_{i,j}} \right)^k + \frac{\|f - p_j\|_{I_i}}{\phi(h_j)} \left(\frac{h_j}{h_{i,j}} \right)^k =: \sigma_1 + \sigma_2.
$$

Now, we note that, for any $1 \le \nu \le n$, inequalities (5.1) and (3.1) imply

$$
\|p_\nu - f\|_{I_\nu} \le \|\phi(\rho_n)\|_{I_\nu} \le \phi(h_\nu).
$$

Hence, $\sigma_1 \le 1$, where we used the fact that if $h_i \le h_j$, then $\phi(h_i) \le \phi(h_j)$, and if $h_i > h_j$, then $\phi(h_i)/\phi(h_j) \le h_i^k/h_j^k$.

In order to estimate σ_2, we first recall the following estimate (see [2, (6.17), p. 235]). For any $g \in C[-1,1]$, $k \in \mathbb{N}$, $a \in [-1,1]$ and $h > 0$ such that $a + (k-1)h \in [-1,1]$,

$$|g(x)| \le c\left(1 + \frac{|x-a|}{h}\right)^k \left(\omega_k(g,h) + \|g\|_{[a,a+(k-1)h]}\right), \quad x \in [-1,1].$$

Setting $g := f - p_j$, $a := x_j$ and $h := h_j / \max\{1, k-1\}$, and observing that $\omega_k(g,h) = \omega_k(f - p_j, h) = \omega_k(f,h) \le \phi(h)$, we get

$$|f(x) - p_j(x)| \le c\left(1 + \frac{|x-x_j|}{h_j}\right)^k \left(\phi(h_j) + \|f - p_j\|_{I_j}\right), \quad x \in [-1,1],$$

and so

$$\|f - p_j\|_{I_i} \le c\left(\frac{h_{i,j}}{h_j}\right)^k \phi(h_j).$$

Hence, $\sigma_2 \le c$, and the proof is complete. □

The next lemma, although claims a different inequality than [3, Lemma 2.1], is proved along the same lines. We bring its proof for the sake of completeness.

Lemma 5.2. *Let $k \in \mathbb{N}$, $\phi \in \Phi^k$ and $S \in \Sigma_{k,n} \cap C[-1,1]$. Then*

$$b_k(S,\phi) \le c\left\|\frac{\rho_n S'}{\phi(\rho_n)}\right\|_\infty. \tag{5.2}$$

Proof. We note that in the case $k = 1$, the statement of the lemma is trivial since $\Sigma_{1,n} \cap C[-1,1] = \Pi_0$, and so both sides of (5.2) are identically zero. Hence, we assume that $k \ge 2$, and we may also assume that

$$\left\|\frac{\rho_n S'}{\phi(\rho_n)}\right\|_\infty = 1. \tag{5.3}$$

Since

$$p_j(x) = S(-1) + \int_{-1}^{x_j} S'(u)du + \int_{x_j}^{x} p_j'(u)du, \quad 1 \le j \le n,$$

it follows that

$$p_j(x) - p_i(x) = \int_{x_i}^{x_j} S'(u)du + \int_{x_j}^{x} p_j'(u)du - \int_{x_i}^{x} p_i'(u)du,$$

and hence,

$$\|p_j - p_i\|_{I_i} \le \left|\int_{x_i}^{x_j} |S'(u)|du\right| + \int_{I_{i,j}} |p_j'(u)|du + \int_{I_i} |p_i'(u)|du \le 2h_{i,j}\|S'\|_{I_{i,j}} + h_{i,j}\|p_j'\|_{I_{i,j}} =: \sigma_1 + \sigma_2.$$

We first estimate σ_2. If $v \in I_j$, then it follows by (5.3) that

$$|p_j'(v)| = |S'(v)| \le \frac{\phi(\rho_n(v))}{\rho_n(v)} \le c\frac{\phi(h_j)}{h_j},$$

and since p_j is a polynomial of degree $\le k-1$, this, in turn, implies that

$$\sigma_2 = h_{i,j}\|p_j'\|_{I_{i,j}} \le ch_{i,j}\frac{\phi(h_j)}{h_j}\left(\frac{h_{i,j}}{h_j}\right)^{k-2} \le c\phi(h_j)\left(\frac{h_{i,j}}{h_j}\right)^k. \tag{5.4}$$

We now estimate σ_1. First, note that it follows from (3.2) (with $y := x_j$ and any $u \in I_{i,j}$) that $h_j^2 \le ch_{i,j}\rho_n(u)$. If $\rho_n(u) < h_j$, this implies

$$\frac{\phi(\rho_n(u))}{\rho_n(u)} \le c\frac{\phi(h_j)}{h_j^2}h_{i,j} \le c\frac{\phi(h_j)}{h_j^k}h_{i,j}^{k-1}, \quad u \in I_{i,j}.$$

If $\rho_n(u) \ge h_j$, then

$$\frac{\phi(\rho_n(u))}{\rho_n(u)} \le \frac{\phi(h_j)}{h_j^k}\rho_n^{k-1}(u) \le \frac{\phi(h_j)}{h_j^k}h_{i,j}^{k-1}, \quad u \in I_{i,j},$$

and so using (5.3) again we have

$$\sigma_1 = 2h_{i,j}\|S'\|_{I_{i,j}} \le 2h_{i,j}\left\|\frac{\phi(\rho_n)}{\rho_n}\right\|_{I_{i,j}} \le c\frac{\phi(h_j)}{h_j^k}h_{i,j}^k.$$

Combining this with (5.4), we obtain

$$\|p_j - p_i\|_{I_i} \le c\,\phi(h_j)\left(\frac{h_{i,j}}{h_j}\right)^k,$$

and the proof is complete. □

6. Monotone polynomial approximation of piecewise polynomials with "small" derivatives

All constants C in this section may depend on k and α.

Lemma 6.1. *Let $\alpha > 0$, $k \in \mathbb{N}$ and $\phi \in \Phi^k$, be given. If $S \in \Sigma_{k,n} \cap \Delta^{(1)}$ is such that*

$$|S'(x)| \le \frac{\phi(\rho_n(x))}{\rho_n(x)}, \quad x \in [x_{n-1}, x_1] \setminus \{x_j\}_{j=1}^{n-1}, \tag{6.1}$$

$$0 \le S(x_j+) - S(x_j-) \le \phi(\rho_n(x_j)), \quad 1 \le j \le n-1, \tag{6.2}$$

and

$$S'(x) = 0, \quad x \in [-1, x_{n-1}) \cup (x_1, 1], \tag{6.3}$$

then there is a polynomial $P \in \Delta^{(1)} \cap \Pi_{Cn}$ such that

$$|S(x) - P(x)| \le C\delta_n^\alpha(x)\,\phi\left(\rho_n(x)\right), \quad x \in [-1, 1]. \tag{6.4}$$

Note that, clearly, condition (6.2) is automatically satisfied at all knots x_j where S is continuous.

Proof. Let

$$S_1(x) := \begin{cases} S(x_j), & x \in [x_j, x_{j-1}), \quad 2 \le j \le n, \\ S(x_1), & x \in [x_1, 1]. \end{cases}$$

Clearly, (6.1) through (6.3) imply

$$|S(x) - S_1(x)| \le c\phi(\rho_n(x)), \quad x \in [-1, 1], \tag{6.5}$$

and (6.3) yields (recall that S is right continuous)

$$S_1(x) = S(x), \quad x \in I_1 \cup I_n. \tag{6.6}$$

We may write,

$$\begin{aligned} S_1(x) &= \sum_{j=2}^{n} S(x_j)\big(\chi_j(x) - \chi_{j-1}(x)\big) + S(x_1)\chi_1(x) \\ &= S(-1) + \sum_{j=1}^{n-1} \big(S(x_j) - S(x_{j+1})\big)\chi_j(x), \quad x \in [-1, 1]. \end{aligned}$$

Let

$$P(x) := S(-1) + \sum_{j=1}^{n-1} \big(S(x_j) - S(x_{j+1})\big)\tau_j(x), \quad x \in [-1, 1],$$

where τ_j are the polynomials from Lemma 4.1 with the same α and $\beta = k + 2$.

Then, P is a nondecreasing polynomial of degree $\le Cn$ and, in view of (6.5) and (6.6), we only need to estimate $|S_1(x) - P(x)|$. First, note that (3.6) implies, for all $1 \le j \le n$ and $x \in [-1, 1]$,

$$\phi(h_j) \le \phi\left(c\psi_j^{-1}(x)\rho_n(x)\right) \le C\,\psi_j^{-k}(x)\phi\left(\rho_n(x)\right).$$

Now, since (6.1) and (6.2) imply that

$$|S(x_j) - S(x_{j+1})| \le C\phi(h_j), \quad 1 \le j \le n-1,$$

using (4.3), we conclude that, for all $1 \le j \le n-1$ and $x \in [-1, 1]$,

$$|S(x_j) - S(x_{j+1})||\chi_j(x) - \tau_j(x)| \le C\phi(h_j)\delta_n^\alpha(x)\psi_j^{k+2}(x) \le C\phi\left(\rho_n(x)\right)\delta_n^\alpha(x)\psi_j^2(x).$$

Therefore, by (3.8), we have

$$\begin{aligned} |S_1(x) - P(x)| &\le \sum_{j=1}^{n-1} |S(x_j) - S(x_{j+1})||\chi_j(x) - \tau_j(x)| \\ &\le C\phi\left(\rho_n(x)\right)\delta_n^\alpha(x) \sum_{j=1}^{n-1} \psi_j^2(x) \\ &\le C\phi\left(\rho_n(x)\right)\delta_n^\alpha(x). \end{aligned}$$

Combined with (6.5) and (6.6), our proof is complete. □

7. On one partition of unity

Lemma 7.1. *Let $\alpha_1, \beta_1 > 0$, and let $n, n_1 \in \mathbb{N}$, $n_1 > n$, be such that n_1 is divisible by n. Then, there is a collection $\{\widetilde{T}_{j,n_1}\}_{j=1}^n$ of polynomials $\widetilde{T}_{j,n_1} \in \Pi_{C(\alpha_1,\beta_1)n_1}$, such that the following relations hold:*

$$\sum_{j=1}^{n} \widetilde{T}_{j,n_1}(x) \equiv 1, \quad x \in [-1,1], \tag{7.1}$$

$$\widetilde{T}'_{1,n_1}(x) \geq 0 \quad \text{and} \quad \widetilde{T}'_{n,n_1}(x) \leq 0, \quad x \in [-1,1], \tag{7.2}$$

$$|\widetilde{T}_{j,n_1}(x)| \leq C\delta_{n_1}^{\alpha_1}(x) \left(\frac{\rho_{n_1}(x)}{\rho_{n_1}(x) + \operatorname{dist}(x, I_j)}\right)^{\beta_1}, \tag{7.3}$$

for all

$$x \in \mathcal{D}_j := \begin{cases} [-1, x_1], & \text{if} \quad j = 1, \\ [x_{n-1}, 1], & \text{if} \quad j = n, \\ [-1, 1], & \text{if} \quad 2 \leq j \leq n-1, \end{cases}$$

and

$$|\widetilde{T}^{(q)}_{j,n_1}(x)| \leq C\frac{\delta_{n_1}^{\alpha_1}(x)}{\rho_{n_1}^q(x)} \left(\frac{\rho_{n_1}(x)}{\rho_{n_1}(x) + \operatorname{dist}(x, I_j)}\right)^{\beta_1}, \quad 1 \leq q \leq \alpha_1 \ \text{and} \ x \in [-1,1], \tag{7.4}$$

where all constants C depend only on α_1, β_1 and are independent of the ratio n_1/n.

Proof. Let τ_{i,n_1}, $1 \leq i \leq n_1 - 1$, be the polynomials from Lemma 4.1 with α and β to be prescribed, and denote $\tau_{0,n_1} \equiv 0$ and $\tau_{n_1,n_1} \equiv 1$.

Set

$$T_{i,n_1} := \tau_{i,n_1} - \tau_{i-1,n_1}, \quad 1 \leq i \leq n_1,$$

and note that

$$\sum_{i=1}^{n_1} T_{i,n_1} \equiv 1. \tag{7.5}$$

Let $d := n_1/n$ and define

$$\widetilde{T}_{j,n_1} := \sum_{i=d(j-1)+1}^{dj} T_{i,n_1} = \sum_{I_{i,n_1} \subset I_j} T_{i,n_1}, \quad 1 \leq j \leq n. \tag{7.6}$$

Then (7.1) readily follows by (7.5), and (7.2) is evident.

We now note that, for all $x \in [-1,1]$,

$$\psi_{i\pm1,n_1}(x) < 4\psi_{i,n_1}(x), \quad 1 \leq i \leq n_1,$$

(recall that $\psi_{0,n_1}(x) \equiv 0$ and $\psi_{n_1+1,n_1}(x) \equiv 0$) and

$$\chi_{i,n_1}(x) - \chi_{i-1,n_1}(x) = \chi_{[x_{i,n_1},x_{i-1,n_1})}(x) = \delta^\alpha_{n_1}(x)\chi_{[x_{i,n_1},x_{i-1,n_1})}(x) \le C\delta^\alpha_{n_1}(x)\psi^\beta_{i,n_1}(x),$$

for $2 \le i \le n_1 - 1$.

Hence, by (4.3),

$$\begin{aligned}|T_{i,n_1}(x)| &\le |\tau_{i,n_1}(x) - \chi_{i,n_1}(x)| + |\chi_{i,n_1}(x) - \chi_{i-1,n_1}(x)| + |\chi_{i-1,n_1}(x) - \tau_{i-1,n_1}(x)| \\ &\le C\delta^\alpha_{n_1}(x)\psi^\beta_{i,n_1}(x), \quad 2 \le i \le n_1 - 1, \quad x \in [-1, 1].\end{aligned}$$

If $i = 1$, then, for $x \in [-1, x_{1,n_1}) \supset [-1, x_{1,n}]$,

$$|T_{1,n_1}(x)| = |\tau_{1,n_1}(x)| = |\tau_{1,n_1}(x) - \chi_{1,n_1}(x)| \le C\delta^\alpha_{n_1}(x)\psi^\beta_{i,n_1}(x),$$

and similarly, for $i = n_1$ and $x \in [x_{n_1-1,n_1}, 1] \supset [x_{n_1-1,n}, 1]$,

$$|T_{n_1,n_1}(x)| = |1 - \tau_{n_1-1,n_1}(x)| = |\chi_{n_1-1,n_1}(x) - \tau_{n_1-1,n_1}(x)| \le C\delta^\alpha_{n_1}(x)\psi^\beta_{i,n_1}(x).$$

Hence, for $x \in \mathfrak{D}_j$,

$$|\widetilde{T}_{j,n_1}(x)| \le C\delta^\alpha_{n_1}(x) \sum_{I_{i,n_1}\subset I_j} \psi^\beta_{i,n_1}(x). \tag{7.7}$$

Similarly, it follows from (4.2) that, for all $x \in [-1, 1]$,

$$|\widetilde{T}^{(q)}_{j,n_1}(x)| \le C\delta^\alpha_{n_1}(x) \sum_{I_{i,n_1}\subset I_j} h^{-q}_{i,n_1}\psi^\beta_{i,n_1}(x), \quad 1 \le q \le \alpha. \tag{7.8}$$

Therefore, we may treat (7.7) as a particular case of (7.8) for $q = 0$, keeping in mind that x is assumed to be in $\mathfrak{D}_j$ in that case.

We are now ready to prove (7.3) and (7.4). First, we note that (3.1) and (3.2) imply that

$$\psi^2_{i,n_1}(x) = \left(\frac{h_{i,n_1}}{|x - x_{i,n_1}| + h_{i,n_1}}\right)^2 \le c\frac{\rho_{n_1}(x)}{|x - x_{i,n_1}| + \rho_{n_1}(x)}.$$

Hence,

$$\begin{aligned}|\widetilde{T}^{(q)}_{j,n_1}(x)| &\le C\delta^\alpha_{n_1}(x) \sum_{I_{i,n_1}\subset I_j} h^{-q}_{i,n_1}\left(\frac{h_{i,n_1}}{|x - x_{i,n_1}| + \rho_{n_1}(x)}\right)^{q+1}\left(\frac{\rho_{n_1}(x)}{|x - x_{i,n_1}| + \rho_{n_1}(x)}\right)^{(\beta-q-1)/2} \\ &= C\delta^\alpha_{n_1}(x)\rho^{(\beta-q-1)/2}_{n_1}(x) \sum_{I_{i,n_1}\subset I_j} \frac{h_{i,n_1}}{(|x - x_{i,n_1}| + \rho_{n_1}(x))^{(\beta+q+1)/2}} \\ &\le C\delta^\alpha_{n_1}(x)\rho^{(\beta-q-1)/2}_{n_1}(x) \int_{\operatorname{dist}(x,I_j)}^{\infty} \frac{du}{(u + \rho_{n_1}(x))^{(\beta+q+1)/2}} \\ &= C\frac{\delta^\alpha_{n_1}(x)}{\rho^q_{n_1}(x)}\left(\frac{\rho_{n_1}(x)}{\operatorname{dist}(x, I_j) + \rho_{n_1}(x)}\right)^{(\beta-q-1)/2}.\end{aligned}$$

It remains to set $\alpha := \lceil\alpha_1\rceil$ and $\beta := 2\beta_1 + \alpha_1 + 1$, and the proof is complete. $\square$

In the proof below, we need estimates (7.3) and (7.4) for $x \in \mathfrak{D}_j$ with $\delta_{n_1}(x)$ replaced by $\delta_n(x)$ which is smaller near the endpoints of $[-1, 1]$.

Corollary 7.2. *Let $\alpha_2, \beta_2 > 0$, and let $n, n_1 \in \mathbb{N}$, $n_1 > n$, be such that n_1 is divisible by n. Then, there is a collection $\{\widetilde{T}_{j,n_1}\}_{j=1}^n$ of polynomials $\widetilde{T}_{j,n_1} \in \Pi_{C(\alpha_2,\beta_2)n_1}$, such that (7.1) and (7.2) are valid, and*

$$|\widetilde{T}_{j,n_1}^{(q)}(x)| \le C \frac{\delta_n^{\alpha_2}(x)}{\rho_{n_1}^q(x)} \left(\frac{\rho_{n_1}(x)}{\rho_{n_1}(x) + \operatorname{dist}(x, I_j)} \right)^{\beta_2}, \quad 0 \le q \le \alpha_2, \tag{7.9}$$

for all $x \in \mathfrak{D}_j$, where all constants C depend only on α_2, β_2 and are independent of the ratio n_1/n.

Proof. Since

$$\frac{\delta_{n_1}(x)}{\delta_n(x)} \le \begin{cases} \dfrac{n_1}{n}, & \text{if} \quad \varphi(x) \le 1/n, \\ 1, & \text{if} \quad \varphi(x) > 1/n, \end{cases}$$

we only need to prove (7.9) for $x \in \widetilde{\mathfrak{D}}_j := \mathfrak{D}_j \cap \{x \mid \varphi(x) \le 1/n\}$ (for all other $x \in \mathfrak{D}_j$ it is an immediate corollary of (7.3) and (7.4) with $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$). Note that for all $1 \le j \le n$ and $x \in \widetilde{\mathfrak{D}}_j$,

$$\operatorname{dist}(x, I_j) \ge \operatorname{dist}(\widetilde{\mathfrak{D}}_j, I_j) \ge \operatorname{dist}(x_1, \sqrt{1 - n^{-2}}) \ge n^{-2}.$$

Hence, it follows from (7.3) and (7.4), that for all $0 \le q \le \alpha_1$, $1 \le j \le n$ and $x \in \widetilde{\mathfrak{D}}_j$, we have

$$\begin{aligned} |\widetilde{T}_{j,n_1}^{(q)}(x)| &\le C \frac{\delta_n^{\alpha_1}(x)}{\rho_{n_1}^q(x)} \left(\frac{n_1}{n}\right)^{\alpha_1} \left(\frac{\rho_{n_1}(x)}{\rho_{n_1}(x) + \operatorname{dist}(x, I_j)} \right)^{\beta_1} \\ &\le C \frac{\delta_n^{\alpha_1}(x)}{\rho_{n_1}^q(x)} \left(\frac{n_1}{n}\right)^{\alpha_1} \left(\frac{\rho_{n_1}(x)}{\rho_{n_1}(x) + n^{-2}} \right)^{\alpha_1} \left(\frac{\rho_{n_1}(x)}{\rho_{n_1}(x) + \operatorname{dist}(x, I_j)} \right)^{\beta_1 - \alpha_1} \\ &\le C \frac{\delta_n^{\alpha_1}(x)}{\rho_{n_1}^q(x)} \left(\frac{\rho_{n_1}(x)}{\rho_{n_1}(x) + \operatorname{dist}(x, I_j)} \right)^{\beta_1 - \alpha_1}, \end{aligned}$$

since

$$\frac{n_1 \rho_{n_1}(x)}{n \rho_{n_1}(x) + 1/n} \le \frac{\varphi(x) + 1/n_1}{n\varphi(x)/n_1 + 1/n} \le 1,$$

for $n_1 \ge n$ and $\varphi(x) \le 1/n$. It remains to set $\alpha_1 := \alpha_2$ and $\beta_1 := \alpha_2 + \beta_2$. $\square$

8. Simultaneous polynomial approximation of piecewise polynomials and their derivatives

All constants C in this section depend on k and γ.

We need the following result which is similar to [12, Lemma 18] and which is proved in a similar way.

Lemma 8.1. *Let $\gamma > 0$, $k \in \mathbb{N}$, $\phi \in \Phi^k$, and let $n, n_1 \in \mathbb{N}$ be such that n_1 is divisible by n. If $S \in \Sigma_{k,n}$, then there exists a polynomial $D_{n_1}(\cdot, S)$ of degree $\le Cn_1$ such that*

$$|S(x) - D_{n_1}(x, S)| \le C \delta_n^\gamma(x) \phi(\rho_n(x)) b_k(S, \phi). \tag{8.1}$$

Moreover, if $S \in C^{r-1}[-1, 1]$ for some $r \in \mathbb{N}$, $r \le k$, and $A := [x_{\mu^*}, x_{\mu_*}]$, $0 \le \mu_* < \mu^* \le n$, then for all $x \in A \setminus \{x_j\}_{j=1}^{n-1}$ and $0 \le q \le r$, we have

$$\left| S^{(q)}(x) - D_{n_1}^{(q)}(x, S) \right| \le C \delta_n^\gamma(x) \frac{\phi(\rho_n(x))}{\rho_n^q(x)} \left(b_k(S, \phi, A) + b_k(S, \phi) \frac{n}{n_1} \left(\frac{\rho_n(x)}{\operatorname{dist}(x, [-1, 1] \setminus A)} \right)^{\gamma + 1} \right). \tag{8.2}$$

The constants C above are independent of the ratio n_1/n.

Proof. We denote

$$D_{n_1}(x,S) := \sum_{j=1}^{n} p_j(x)\widetilde{T}_{j,n_1}(x), \tag{8.3}$$

where $\widetilde{T}_{j,n_1}$ are polynomials of degree $\le C(\alpha_2,\beta_2)n_1$ from the statement of Corollary 7.2. Note that $D_{n_1}(\cdot,S)$ is a polynomial of degree $< k + C(\alpha_2,\beta_2)n_1$. The parameters α_2 and β_2 depend on γ and k are chosen to be sufficiently large. For example, $\alpha_2 = \gamma$ and $\beta_2 = \gamma + 4k + 5$ will do.

For the sake of brevity, we will use the notation $\rho := \rho_n(x)$, $\delta := \delta_n(x)$, $\rho_1 := \rho_{n_1}(x)$ and $\widetilde{T}_j := \widetilde{T}_{j,n_1}$. Recall that $I_{i,j}$ is the smallest interval containing both I_i and I_j, and $h_{i,j} := |I_{i,j}|$. Suppose now that x is fixed and let $1 \le \nu \le n$ be the smallest number such that $x \in I_\nu$ (*i.e.*, if $x = x_\eta$, then x belongs to both I_η and $I_{\eta+1}$, and we pick $\nu = \eta$).

We now observe that (3.4) and (3.7) imply

$$\frac{h_\nu}{h_j} < 5\frac{\rho}{h_j} < 40\frac{|x-x_j|+\rho}{\rho} \sim \frac{\rho + \operatorname{dist}(x,I_j)}{\rho}, \quad 1 \le j \le n. \tag{8.4}$$

Also,

$$\frac{h_{\nu,j}}{h_\nu} \le c\frac{\rho + \operatorname{dist}(x,I_j)}{\rho}, \quad 1 \le j \le n. \tag{8.5}$$

Indeed, if $|j-\nu| \le 1$, then it is enough to note that (3.1) implies that $h_{\nu,j} \sim h_\nu$. If $|j-\nu| \ge 2$, then we use the fact that there is at least one interval I_i between I_ν and I_j, and so (3.1) implies

$$h_{\nu,j} = h_\nu + h_j + \operatorname{dist}(I_\nu,I_j) \le h_\nu + 4\operatorname{dist}(I_\nu,I_j) \le h_\nu + 4\operatorname{dist}(x,I_j),$$

and (8.5) follows.

Since $S(x) = p_\nu(x)$, (7.1) implies

$$S(x) - D_{n_1}(x,S) = S(x)\sum_{j=1}^{n}\widetilde{T}_j(x) - \sum_{j=1}^{n} p_j(x)\widetilde{T}_j(x) = \sum_{1\le j\le n, j\ne\nu}(p_\nu(x)-p_j(x))\widetilde{T}_j(x),$$

and so

$$\begin{aligned} S^{(q)}(x) - D_{n_1}^{(q)}(x,S) &= \sum_{1\le j\le n, j\ne\nu}\Big((p_\nu(x)-p_j(x))\,\widetilde{T}_j(x)\Big)^{(q)} \\ &= \sum_{1\le j\le n, j\ne\nu}\sum_{i=0}^{q}\binom{q}{i}\Big(p_\nu^{(i)}(x)-p_j^{(i)}(x)\Big)\,\widetilde{T}_j^{(q-i)}(x), \end{aligned}$$

with the assumption that $x \notin \{x_j\}_{j=1}^{n-1}$ if $q \ge 1$, since $S^{(q)}$ may not exist at those points. Note also that $x \in \mathcal{D}_j$ for all $1 \le j \le n$, $j \ne \nu$, and so (7.9) can be used for all polynomials $\widetilde{T}_j$ appearing in the above sum.

Now, since

$$\phi(h_j) \le \phi(h_{\nu,j}) \le \phi(h_\nu)\left(\frac{h_{\nu,j}}{h_\nu}\right)^k \le c\phi(\rho)\left(\frac{h_{\nu,j}}{h_\nu}\right)^k,$$

it follows from (2.1), (8.4) and (8.5) that, for all $i \ge 0$ (of course, the inequality is trivial if $i \ge k$),

$$\|p_\nu^{(i)} - p_j^{(i)}\|_{I_\nu} \le c h_\nu^{-i} \|p_\nu - p_j\|_{I_\nu} \le c b_{\nu,j}(S,\phi) \frac{\phi(h_j)}{h_\nu^i} \left(\frac{h_{\nu,j}}{h_j}\right)^k \tag{8.6}$$
$$\le c b_{\nu,j}(S,\phi) \frac{\phi(\rho)}{\rho^i} \left(\frac{h_{\nu,j}^2}{h_\nu h_j}\right)^k \le c b_{\nu,j}(S,\phi) \frac{\phi(\rho)}{\rho^i} \left(\frac{\rho + \operatorname{dist}(x, I_j)}{\rho}\right)^{3k}.$$

Observing that

$$\frac{\rho_1}{\rho_1 + \operatorname{dist}(x, I_j)} \le \frac{\rho}{\rho + \operatorname{dist}(x, I_j)} \tag{8.7}$$

and using (7.9) we now conclude that, for all $0 \le i \le q$ and $1 \le j \le n$, $j \ne \nu$,

$$\left| \left(p_\nu^{(i)}(x) - p_j^{(i)}(x)\right) \widetilde{T}_j^{(q-i)}(x) \right| \le C b_{\nu,j}(S,\phi) \delta^{\alpha_2} \frac{\phi(\rho)}{\rho^i \rho_1^{q-i}} \left(\frac{\rho_1}{\rho_1 + \operatorname{dist}(x, I_j)}\right)^{\beta_2 - 3k}.$$

If $i = q$, this becomes

$$\left| \left(p_\nu^{(q)}(x) - p_j^{(q)}(x)\right) \widetilde{T}_j(x) \right| \le C b_{\nu,j}(S,\phi) \delta^{\alpha_2} \frac{\phi(\rho)}{\rho^q} \left(\frac{\rho_1}{\rho_1 + \operatorname{dist}(x, I_j)}\right)^{\beta_2 - 3k}, \tag{8.8}$$

and, in particular, if $i = q = 0$, then

$$\left| \left(p_\nu(x) - p_j(x)\right) \widetilde{T}_j(x) \right| \le C b_{\nu,j}(S,\phi) \delta^{\alpha_2} \phi(\rho) \left(\frac{\rho_1}{\rho_1 + \operatorname{dist}(x, I_j)}\right)^{\beta_2 - 3k}. \tag{8.9}$$

Now, with an additional assumption that $j \ne \nu \pm 1$ (which implies that $\operatorname{dist}(x, I_j) > \rho/3$), and using $\rho_1/\rho \le n/n_1$, we have

$$\left| \left(p_\nu^{(i)}(x) - p_j^{(i)}(x)\right) \widetilde{T}_j^{(q-i)}(x) \right| \tag{8.10}$$
$$\le C b_{\nu,j}(S,\phi) \delta^{\alpha_2} \frac{\phi(\rho)}{\rho^q} \frac{\rho_1}{\rho} \left(\frac{\rho}{\rho_1 + \operatorname{dist}(x, I_j)}\right)^{q-i+1} \left(\frac{\rho_1}{\rho_1 + \operatorname{dist}(x, I_j)}\right)^{\beta_2 - 3k - q + i - 1}$$
$$\le C b_{\nu,j}(S,\phi) \delta^{\alpha_2} \frac{\phi(\rho)}{\rho^q} \frac{n}{n_1} \left(\frac{\rho_1}{\rho_1 + \operatorname{dist}(x, I_j)}\right)^{\beta_2 - 3k - q - 1}.$$

It remains to consider the case $q \ge 1$, $i \le q-1$ and $j = \nu \pm 1$. We only consider the case $j = \nu + 1$, the case $j = \nu - 1$ being completely analogous.

We now have to use the fact that S is assumed to be sufficiently smooth. Indeed, if $S \in C^{q-1}[-1,1]$, we have $p_\nu^{(l)}(x_\nu) = p_{\nu+1}^{(l)}(x_\nu)$, $0 \le l \le q-1$, and so by (8.6),

$$\left| p_\nu^{(i)}(x) - p_{\nu+1}^{(i)}(x) \right| = \frac{1}{(q-i-1)!} \left| \int_{x_\nu}^{x} (x-u)^{q-i-1} \left(p_\nu^{(q)}(u) - p_{\nu+1}^{(q)}(u)\right) du \right|$$
$$\le |x - x_\nu|^{q-i} \left\| p_\nu^{(q)} - p_{\nu+1}^{(q)} \right\|_{I_\nu}$$
$$\le c |x - x_\nu|^{q-i} b_{\nu,\nu+1}(S,\phi) \frac{\phi(\rho)}{\rho^q} \left(\frac{\rho + |x - x_\nu|}{\rho}\right)^{3k}.$$

Therefore,

$$\left|\left(p_\nu^{(i)}(x)-p_{\nu+1}^{(i)}(x)\right)\widetilde{T}_{\nu+1}^{(q-i)}(x)\right| \le Cb_{\nu,\nu+1}(S,\phi)\delta^{\alpha_2}\frac{\phi(\rho)|x-x_\nu|^{q-i}}{\rho^q\rho_1^{q-i}}\left(\frac{\rho_1}{\rho_1+|x-x_\nu|}\right)^{\beta_2-3k}$$
$$\le Cb_{\nu,\nu+1}(S,\phi)\delta^{\alpha_2}\frac{\phi(\rho)}{\rho^q}\left(\frac{\rho_1}{\rho_1+|x-x_\nu|}\right)^{\beta_2-3k-q+i}.$$

In summary, the estimate

$$\left|\left(p_\nu^{(i)}(x)-p_{\nu\pm1}^{(i)}(x)\right)\widetilde{T}_{\nu\pm1}^{(q-i)}(x)\right| \le Cb_{\nu,\nu\pm1}(S,\phi)\delta^{\alpha_2}\frac{\phi(\rho)}{\rho^q}\left(\frac{\rho_1}{\rho_1+\operatorname{dist}(x,I_{\nu\pm1})}\right)^{\beta_2-3k-q}, \tag{8.11}$$

is valid for all $0\le i\le q$ provided that $S\in C^{q-1}[-1,1]$ (for $i=q$ it follows from (8.8)).

Using (8.9), (8.7), (3.9) and the estimate $b_{\nu,j}(S,\phi)\le b_k(S,\phi)$, we have

$$|S(x)-D_{n_1}(x,S)| \le Cb_k(S,\phi)\delta^{\alpha_2}\phi(\rho)\sum_{1\le j\le n, j\ne\nu}\left(\frac{\rho}{\rho+\operatorname{dist}(x,I_j)}\right)^{\beta_2-3k} \tag{8.12}$$
$$\le Cb_k(S,\phi)\delta^\gamma\phi(\rho),$$

and (8.1) is proved.

We will now prove (8.2). Suppose that $S\in C^{r-1}[-1,1]$ and $0\le q\le r$. We write

$$S^{(q)}(x)-D_{n_1}^{(q)}(x,S) = \sum_{1\le j\le n, j\ne\nu}\Big((p_\nu(x)-p_j(x))\,\widetilde{T}_j(x)\Big)^{(q)}$$
$$=\left(\sum_{j\in\mathfrak{J}_1}+\sum_{j\in\mathfrak{J}_2}+\sum_{j\in\mathfrak{J}_3}+\sum_{j\in\mathfrak{J}_4}\right)\Big((p_\nu(x)-p_j(x))\,\widetilde{T}_j(x)\Big)^{(q)}$$
$$=: \sigma_1(x)+\sigma_2(x)+\sigma_3(x)+\sigma_4(x),$$

where

$$\mathfrak{J}_1 := \left\{j \;\middle|\; 1\le j\le n, I_j\subset A, j\ne\nu,\nu\pm1\right\},$$
$$\mathfrak{J}_2 := \left\{j \;\middle|\; 1\le j\le n, I_j\not\subset A, j\ne\nu,\nu\pm1\right\},$$
$$\mathfrak{J}_3 := \left\{j \;\middle|\; 1\le j\le n, j=\nu+1\right\},$$
$$\mathfrak{J}_4 := \left\{j \;\middle|\; 1\le j\le n, j=\nu-1\right\}.$$

Note that some of the sets $\mathfrak{J}_l$ may be empty (making the corresponding functions $\sigma_l\equiv 0$). For example, if $\nu=1$, then $\mathfrak{J}_4=\emptyset$ and $\sigma_4\equiv0$; if $A\subset I_{\nu+1}\cup I_\nu\cup I_{\nu+1}$, then $\mathfrak{J}_1=\emptyset$ and $\sigma_1\equiv0$, etc.

In order to estimate σ_1, using (8.10), (8.7), (3.9) and the estimate $b_{\nu,j}(S,\phi)\le b_k(S,\phi,A)$, $j\in\mathfrak{J}_1$, we have

$$|\sigma_1(x)| \le Cb_k(S,\phi,A)\delta^{\alpha_2}\frac{\phi(\rho)}{\rho^q}\frac{n}{n_1}\sum_{j\in\mathfrak{J}_1}\left(\frac{\rho_1}{\rho_1+\operatorname{dist}(x,I_j)}\right)^{\beta_2-3k-q-1} \le Cb_k(S,\phi,A)\delta^\gamma\frac{\phi(\rho)}{\rho^q}.$$

To estimate σ_2, we use (8.10), (8.7), (3.7), (3.4) and $b_{\nu,j}(S,\phi)\le b_k(S,\phi)$, $j\in\mathfrak{J}_2$, and write

$$|\sigma_2(x)| \le Cb_k(S,\phi)\delta^{\alpha_2}\frac{\phi(\rho)}{\rho^q}\frac{n}{n_1}\sum_{j\in\mathfrak{J}_2}\left(\frac{\rho_1}{\rho_1+\operatorname{dist}(x,I_j)}\right)^{\beta_2-3k-q-1}$$

$$
\begin{aligned}
&\le Cb_k(S,\phi)\delta^{\alpha_2}\frac{\phi(\rho)}{\rho^q}\frac{n}{n_1}\sum_{j\in\mathcal{J}_2}\frac{h_j}{\rho}\left(\frac{\rho}{\rho+|x-x_j|}\right)^{\beta_2-3k-q-2}\\
&\le Cb_k(S,\phi)\delta^{\gamma}\frac{\phi(\rho)}{\rho^q}\frac{n}{n_1}\sum_{j\in\mathcal{J}_2}\frac{h_j}{\rho}\left(\frac{\rho}{\rho+|x-x_j|}\right)^{\gamma+2}\\
&\le Cb_k(S,\phi)\delta^{\gamma}\frac{\phi(\rho)}{\rho^q}\frac{n}{n_1}\rho^{\gamma+1}\sum_{j\in\mathcal{J}_2}\frac{h_j}{(\rho+|x-x_j|)^{\gamma+2}}\\
&\le Cb_k(S,\phi)\delta^{\gamma}\frac{\phi(\rho)}{\rho^q}\frac{n}{n_1}\rho^{\gamma+1}\int\limits_{\operatorname{dist}(x,[-1,1]\setminus A)}^{\infty}\frac{du}{(\rho+u)^{\gamma+2}}\\
&\le Cb_k(S,\phi)\delta^{\gamma}\frac{\phi(\rho)}{\rho^q}\frac{n}{n_1}\left(\frac{\rho}{\rho+\operatorname{dist}(x,[-1,1]\setminus A)}\right)^{\gamma+1}.
\end{aligned}
$$

Finally, we will estimate σ_3 (the proof for σ_4 is completely analogous). First, if $I_{\nu+1}\subset A$, then $b_{\nu,\nu+1}(S,\phi)\le b_k(S,\phi,A)$ and so (8.11) yields

$$
|\sigma_3(x)|\le Cb_k(S,\phi,A)\delta^{\gamma}\frac{\phi(\rho)}{\rho^q}.
$$

If $I_{\nu+1}\not\subset A$, then $\nu=\mu^*$ (and so $\operatorname{dist}(x,[-1,1]\setminus A)\le|x-x_\nu|=\operatorname{dist}(x,I_{\nu+1})$), $b_{\nu,\nu+1}(S,\phi)\le b_k(S,\phi)$, and again using (8.11), we have

$$
\begin{aligned}
|\sigma_3(x)|&\le Cb_k(S,\phi)\delta^{\gamma}\frac{\phi(\rho)}{\rho^q}\left(\frac{\rho_1}{\rho_1+\operatorname{dist}(x,I_{\nu+1})}\right)^{\gamma+1}\\
&\le Cb_k(S,\phi)\delta^{\gamma}\frac{\phi(\rho)}{\rho^q}\frac{\rho_1}{\rho}\left(\frac{\rho}{\rho_1+\operatorname{dist}(x,[-1,1]\setminus A)}\right)^{\gamma+1}\\
&\le Cb_k(S,\phi)\delta^{\gamma}\frac{\phi(\rho)}{\rho^q}\frac{n}{n_1}\left(\frac{\rho}{\operatorname{dist}(x,[-1,1]\setminus A)}\right)^{\gamma+1}.
\end{aligned}
$$

The proof is now complete. $\square$

9. One particular polynomial with controlled first derivative

All constants C in this section depend on α and β.

The following lemma is a modification of [12, Lemma 10].

Lemma 9.1. *Let $\alpha,\beta>0$, $k\in\mathbb{N}$ and $\phi\in\Phi^k$. Also, let $E\subset[-1,1]$ be a closed interval which is the union of $m_E\ge 100$ of the intervals I_j, and let a set $J\subset E$ consist of m_J intervals I_j, where $1\le m_J<m_E/4$. Then there exists a polynomial $Q_n(x)=Q_n(x,E,J)$ of degree $\le Cn$, satisfying*

$$
Q_n'(x)\ge C\frac{m_E}{m_J}\delta_n^{8\alpha}(x)\frac{\phi(\rho_n(x))}{\rho_n(x)}\left(\frac{\rho_n(x)}{\max\{\rho_n(x),\operatorname{dist}(x,E)\}}\right)^{60(\alpha+\beta)+4k+2},\quad x\in J\cup([-1,1]\setminus E),\tag{9.1}
$$

$$
Q_n'(x)\ge-\delta_n^{\alpha}(x)\frac{\phi(\rho_n(x))}{\rho_n(x)},\quad x\in E\setminus J,\tag{9.2}
$$

and

$$|Q_n(x)| \le C\, m_E^{k+3} \delta_n^\alpha(x) \rho_n(x)\, \phi(\rho_n(x)) \sum_{j:\, I_j \subset E} \frac{h_j}{(|x-x_j| + \rho_n(x))^2}, \qquad (9.3)$$
$$x \in [-1,1].$$

Proof. First, it will be shown that we may assume that $I_n \not\subset E$ provided that the condition $m_J < m_E/4$ is replaced by a slightly weaker $m_J \le m_E/4$.

Suppose that the lemma is proved for all E_1 such that $I_n \not\subset E_1$, let E be such that $I_n \subset E$, set $E_1 := (E \setminus I_n)^{cl}$ and $Q_n(x,E,J) := Q_n(x,E_1,J_1)$ (with J_1 to be prescribed), and consider the following three cases noting that, if the inequality in (9.1) holds for a particular x, then the inequality in (9.2) holds for that x as well, and that $\max\{\rho_n(x), \operatorname{dist}(x,E_1)\} \sim \max\{\rho_n(x), \operatorname{dist}(x,E)\}$.

Case (i): If $I_n \subset J$ and $m_J \ge 2$, then we define $J_1 := (J \setminus I_n)^{cl}$, and note that $E_1 \setminus J_1 = E \setminus J$ (and so $J_1 \cup ([-1,1] \setminus E_1) = J \cup ([-1,1] \setminus E)$), $1 \le m_{J_1} < m_{E_1}/4$, and $m_{E_1}/m_{J_1} < 2m_E/m_J$.

Case (ii): If $m_J = 1$ and $J = I_n$, then we define $J_1 := I_{n-1}$, and note that $E_1 \setminus J_1 \subset E \setminus J$ (and so $J_1 \cup ([-1,1] \setminus E_1) \supset J \cup ([-1,1] \setminus E)$), $1 = m_{J_1} < m_{E_1}/4$, and $m_{E_1}/m_{J_1} < m_E/m_J$.

Case (iii): If $I_n \not\subset J$, then $J \subset E_1$ and we define $J_1 := J$. Then, $E_1 \setminus J_1 \subset E \setminus J$, $1 \le m_{J_1} \le m_{E_1}/4$ (since $4m_J < m_E$ implies that $4m_J \le m_E - 1 = m_{E_1}$), and $m_{E_1}/m_{J_1} < m_E/m_J$.

Hence, in the rest of the proof, we assume that $I_n \not\subset E$ and $m_J \le m_E/4$.

It is convenient to use the notation $\rho := \rho_n(x)$, $\delta := \delta_n(x)$ and $\psi_j := \psi_j(x)$. It is also convenient to denote

$$\begin{aligned}
&\mathcal{E} := \left\{1 \le j \le n \;\middle|\; I_j \subset E\right\}, \quad \mathcal{J} := \left\{1 \le j \le n \;\middle|\; I_j \subset J\right\}, \\
&j_* := \min\left\{j \;\middle|\; j \in \mathcal{E}\right\}, \quad j^* := \max\left\{j \;\middle|\; j \in \mathcal{E}\right\}, \\
&\mathcal{A} := \mathcal{J} \cup \{j_*, j^*\} \quad \text{and} \quad \mathcal{B} := \mathcal{E} \setminus \mathcal{A}.
\end{aligned}$$

Note that $j^* = j_* + m_E - 1$, $E = [x_{j^*}, x_{j_*-1}]$, $\#\mathcal{E} = m_E$, $m_J = \#\mathcal{J} \sim \#\mathcal{A}$, and $\#\mathcal{B} \sim m_E$.

Note that (3.6) implies $c\psi_j^2 \rho \le h_j \le c\psi_j^{-1}\rho$, and so

$$\phi(h_j) \le \max\{1, h_j^k \rho^{-k}\} \phi(\rho) \le c\psi_j^{-k} \phi(\rho). \qquad (9.4)$$

Similarly,

$$\phi(h_j) \ge \min\{1, h_j^k \rho^{-k}\} \phi(\rho) \ge c\psi_j^{2k} \phi(\rho). \qquad (9.5)$$

Let

$$Q_n(x) := \kappa \left(\frac{m_E}{m_J} \sum_{j \in \mathcal{A}} \tau_j(x) \phi(h_j) - \lambda \sum_{j \in \mathcal{B}} \widetilde{\tau}_j(x) \phi(h_j) \right),$$

where τ_j and $\widetilde{\tau}_j$ are polynomials of degree $\le Cn$ from Lemmas 4.1 and 4.2, respectively, λ is chosen so that

$$Q_n(1) = \frac{m_E}{m_J} \sum_{j \in \mathcal{A}} \phi(h_j) - \lambda \sum_{j \in \mathcal{B}} \phi(h_j) = 0, \qquad (9.6)$$

and κ is to be prescribed.

We will now show that λ is bounded by a constant independent of m_E/m_J.

Let $\widetilde{E} \subset E$ be the subinterval of E such that

(i) $\widetilde{E}$ is a union of $\lfloor m_E/3\rfloor$ intervals I_j, and
(ii) $\widetilde{E}$ is centered at 0 as much as E allows it, *i.e.*, among all subintervals of E consisting of $\lfloor m_E/3\rfloor$ intervals I_j, the center of $\widetilde{E}$ is closest to 0.

Then, using the fact that the lengths of $|I_i|$ in the Chebyshev partition are increasing toward the middle of $[-1,1]$ and are decreasing toward the endpoints, we conclude that every interval I_j inside $\widetilde{E}$ is not smaller than any interval I_i in $E\setminus\widetilde{E}$, *i.e.*,

$$\text{if}\quad I_j\subset\widetilde{E}\quad\text{and}\quad I_i\subset E\setminus\widetilde{E},\quad\text{then}\quad |I_j|\ge|I_i|. \tag{9.7}$$

Moreover, we will now show that all intervals I_j inside $\widetilde{E}$ have about the same lengths.

We use the following result (see [12, Lemma 5] which, unfortunately, contains an inadvertent omission in the conditions for [12, (4.6)]):

If $0\le j_1\le i<j_2\le n$, then

$$\frac{j_2-j_1}{2}\le\frac{x_{j_1}-x_{j_2}}{x_i-x_{i+1}}\le(j_2-j_1)^2. \tag{9.8}$$

Moreover, if, in addition, either $2i+1\le j_2+j_1$ and $j_2\le 3j_1$, or $2i+1>j_2+j_1$ and $n-j_1\le 3(n-j_2)$, then

$$\frac{j_2-j_1}{2}\le\frac{x_{j_1}-x_{j_2}}{x_i-x_{i+1}}\le 2(j_2-j_1). \tag{9.9}$$

In particular, if both inequalities

$$j_2\le 3j_1\quad\text{and}\quad n-j_1\le 3(n-j_2) \tag{9.10}$$

are satisfied, then (9.9) holds.

Suppose that $\widetilde{E}=[x_{i^*},x_{i_*}]$. Then $i^*-i_*=\lfloor m_E/3\rfloor$. Now, if $0\in\widetilde{E}$, then $i_*\le n/2\le i^*$, and so $3i_*-i^*=2i^*-3\lfloor m_E/3\rfloor\ge n-3\lfloor m_E/3\rfloor\ge 0$ and $3(n-i^*)-(n-i_*)=2n-3\lfloor m_E/3\rfloor-2i_*\ge 0$. Therefore, conditions (9.10) are satisfied.

If $0\notin\widetilde{E}$, then either $E\subset(0,1]$ or $E\subset[-1,0)$ and so, in particular, $m_E\le\lfloor n/2\rfloor$. Suppose that $E\subset(0,1]$ (the other case can be dealt with by symmetry). Then $i^*=j^*<n/2$ and $i_*=j^*-\lfloor m_E/3\rfloor=j_*+m_E-1-\lfloor m_E/3\rfloor\ge m_E-\lfloor m_E/3\rfloor\ge 2m_E/3$. Hence, $3i_*-i^*=2i_*-\lfloor m_E/3\rfloor\ge m_E/3\ge 0$ and $3(n-i^*)-(n-i_*)=2n-3i^*+i_*>n/2>0$. Hence, conditions (9.10) are satisfied in this case as well.

Using (9.9) we now conclude that

$$|I_j|\sim\frac{|\widetilde{E}|}{m_E},\quad\text{for all}\ \ I_j\subset\widetilde{E}.$$

Now, denote $\widetilde{\mathcal{E}}:=\left\{1\le j\le n\ \middle|\ I_j\subset\widetilde{E}\right\}$. Since $\#\widetilde{\mathcal{E}}=\lfloor m_E/3\rfloor$, for $\widetilde{\mathcal{B}}:=\mathcal{B}\cap\widetilde{\mathcal{E}}=\widetilde{\mathcal{E}}\setminus\mathcal{A}$, we have

$$\#\widetilde{\mathcal{B}}\ge\#\widetilde{\mathcal{E}}-\#\mathcal{A}\ge\lfloor m_E/3\rfloor-m_J-2\ge m_E/3-m_E/4-3\ge m_E/20.$$

Therefore,

$$\sum_{j\in\mathcal{B}}\phi(h_j)\ge\sum_{j\in\widetilde{\mathcal{B}}}\phi(h_j)\sim\#\widetilde{\mathcal{B}}\cdot\phi\left(|\widetilde{E}|/m_E\right)\sim m_E\cdot\phi\left(|\widetilde{E}|/m_E\right),$$

and since by (9.7),

$$\sum_{j\in\mathcal{A}} \phi(h_j) \le c\#\mathcal{A}\cdot\phi\left(|\widetilde{E}|/m_E\right) \sim m_J\cdot\phi\left(|\widetilde{E}|/m_E\right),$$

we conclude that

$$0<\lambda\le c\frac{m_E}{m_J}\cdot\frac{m_J\cdot\phi\left(|\widetilde{E}|/m_E\right)}{m_E\cdot\phi\left(|\widetilde{E}|/m_E\right)}\sim 1,$$

i.e., λ is bounded by a constant independent of m_E/m_J.

Now, for any $x\in J\cup([-1,1]\setminus E)$ (as well as for any $x\in I_{j_*}\cup I_{j^*}$), taking into account that $\widetilde{\tau}_j'(x)\le 0$ for all $j\in\mathcal{B}$, and using Lemma 4.1, (9.5) and (3.5) we have

$$\begin{aligned}
Q_n'(x) &\ge \kappa\frac{m_E}{m_J}\sum_{j\in\mathcal{A}}\tau_j'(x)\phi(h_j)\\
&\ge C\kappa\delta^{8\alpha}(x)\frac{m_E}{m_J}\sum_{j\in\mathcal{A}}\phi(h_j)h_j^{-1}\psi_j^{30(\alpha+\beta)}\\
&\ge C\kappa\delta^{8\alpha}(x)\frac{m_E}{m_J}\frac{\phi(\rho)}{\rho}\sum_{j\in\mathcal{A}}\psi_j^{30(\alpha+\beta)+2k+1}\\
&\ge C\kappa\delta^{8\alpha}(x)\frac{m_E}{m_J}\frac{\phi(\rho)}{\rho}\sum_{j\in\mathcal{A}}\left(\frac{\rho}{\rho+|x-x_j|}\right)^{60(\alpha+\beta)+4k+2}\\
&\ge C\kappa\delta^{8\alpha}(x)\frac{m_E}{m_J}\frac{\phi(\rho)}{\rho}\left(\frac{\rho}{\max\{\rho,\operatorname{dist}(x,E)\}}\right)^{60(\alpha+\beta)+4k+2},
\end{aligned}$$

since, for $x\notin E$, $\max\{\rho,\operatorname{dist}(x,E)\}\sim\min\{|x-x_{j^*}|,|x-x_{j_*}|\}+\rho$, and, for $x\in J$, $x\in I_j$ for some $j\in A$, and so $\rho/(|x-x_j|+\rho)\sim 1$ for that j.

If $x\in E\setminus J$ and $x\notin I_{j_*}\cup I_{j^*}$, then there exists $j_0\in\mathcal{B}$ such that $x\in I_{j_0}$. Hence,

$$Q_n'(x)\ge -\kappa\lambda\widetilde{\tau}_{j_0}'(x)\phi(h_{j_0})\ge -C\kappa h_{j_0}^{-1}\delta^\alpha\psi_{j_0}^\beta\phi(h_{j_0})\ge -C\kappa\frac{\phi(\rho)}{\rho}\delta^\alpha\ge -\frac{\phi(\rho)}{\rho}\delta^\alpha,$$

for sufficiently small κ.

We now estimate $|Q_n(x)|$. Let

$$L(x):=\kappa\left(\frac{m_E}{m_J}\sum_{j\in\mathcal{A}}\chi_j(x)\phi(h_j)-\lambda\sum_{j\in\mathcal{B}}\chi_j(x)\phi(h_j)\right).$$

Then, by virtue of (4.3), (4.9), (9.4) and $\psi_j^2\le c\rho(|x-x_j|+\rho)^{-1}$, we have

$$\begin{aligned}
|Q_n(x)-L(x)| &= \kappa\left|\frac{m_E}{m_J}\sum_{j\in\mathcal{A}}(\tau_j(x)-\chi_j(x))\,\phi(h_j)-\lambda\sum_{j\in\mathcal{B}}(\widetilde{\tau}_j(x)-\chi_j(x))\,\phi(h_j)\right|\\
&\le Cm_E\delta^\alpha\sum_{j\in\mathcal{E}}\phi(h_j)\psi_j^\beta\le Cm_E\delta^\alpha\phi(\rho)\sum_{j\in\mathcal{E}}\psi_j^{\beta-k}
\end{aligned}$$

$$\le Cm_E\delta^\alpha\phi(\rho)\sum_{j\in\mathcal{E}}\frac{h_j}{\rho}\psi_j^{\beta-k-2}$$
$$\le Cm_E\delta^\alpha\phi(\rho)\sum_{j\in\mathcal{E}}\frac{h_j}{\rho}\left(\frac{\rho}{|x-x_j|+\rho}\right)^{(\beta-k-2)/2}$$
$$\le Cm_E\delta^\alpha\phi(\rho)\sum_{j\in\mathcal{E}}\frac{h_j\rho}{(|x-x_j|+\rho)^2},$$

provided $(\beta-k-2)/2\ge 2$.

Hence, it remains to estimate $|L(x)|$. First assume that $x\notin E$. If $x\le x_{j^*}$, then $\chi_j(x)=0$, $j\in\mathcal{A}\cup\mathcal{B}$, and $L(x)=0$. If, on the other hand, $x>x_{j_*}$, then $\chi_j(x)=1$, $j\in\mathcal{A}\cup\mathcal{B}$, so that (9.6) implies that $L(x)=0$. Hence, in particular, $L(x)=0$ for $x\in I_1\cup I_n$.

Suppose now that $x\in E\setminus I_1$ (recall that we already assumed that E does not contain I_n). Then, (9.8) implies that, for all $j\in\mathcal{E}$, $h_j\le c|E|/m_E\le c\rho m_E$ (since, again by (9.8), it follows that $|E|\le c\rho m_E^2$), and so $\phi(h_j)\le cm_E^k\phi(\rho)$.

Hence, since $\delta=1$ on $[x_{n-1},x_1]$,

$$|L(x)|\le C\left(\frac{m_E}{m_J}\sum_{j\in\mathcal{A}}\phi(h_j)+\lambda\sum_{j\in\mathcal{B}}\phi(h_j)\right)\le Cm_E^{k+1}\delta^\alpha\phi(\rho).$$

It remains to note that

$$1=|E|\sum_{j\in\mathcal{E}}\frac{h_j}{|E|^2}\le c|E|\sum_{j\in\mathcal{E}}\frac{h_j}{(|x-x_j|+\rho)^2}\le cm_E^2\sum_{j\in\mathcal{E}}\frac{\rho h_j}{(|x-x_j|+\rho)^2},$$

and the proof is complete. □

10. Monotone polynomial approximation of piecewise polynomials

All constants C and C_i in this section depend only on k and α.

First, we need the following auxiliary result, the proof of which is similar to that of [12, Lemma 12].

Lemma 10.1. *Let $k\in\mathbb{N}$, $\phi\in\Phi^k$ and $S\in\Sigma_{k,n}$ be such that*

$$b_k(S,\phi)\le 1. \tag{10.1}$$

If $1\le\mu,\nu\le n$ are such that the interval $I_{\mu,\nu}$ contains at least $2k-3$ intervals I_i and points $x_i^*\in(x_i,x_{i-1})$ so that

$$\rho_n(x_i^*)\phi^{-1}(\rho_n(x_i^*))|S'(x_i^*)|\le 1, \tag{10.2}$$

then, for every $1\le j\le n$, we have

$$\left\|\rho_n\phi^{-1}(\rho_n)S'\right\|_{L_\infty(I_j)}\le c(k)\left[(j-\mu)^{4k}+(j-\nu)^{4k}\right]. \tag{10.3}$$

Proof. Clearly, it is enough to prove the lemma for $k\ge 2$ since (10.3) is trivial if $k=1$. Fix $1\le j\le n$. Since every polynomial piece of S has degree $\le k-1$, it follows from (10.1) that, for every $1\le i\le n$,

$$\left\|p_i'-p_j'\right\|_{I_i}\le ch_i^{-1}\left\|p_i-p_j\right\|_{I_i}\le ch_i^{-1}\phi(h_j)\left(\frac{h_{i,j}}{h_j}\right)^k.$$

Thus, using $h_j^2 \le c h_i h_{i,j}$, that follows from (3.2), and $\phi(h_i) \le \phi(h_j)\,(h_{i,j}/h_j)^k$, we have, for $x_i^* \in (x_i, x_{i-1})$ for which (10.2) holds,

$$\begin{aligned}|p_j'(x_i^*)| &\le c h_i^{-1}\phi(h_j)\left(\frac{h_{i,j}}{h_j}\right)^k + \rho_n^{-1}(x_i^*)\phi(\rho_n(x_i^*)) \\ &\le c h_i^{-1}\left(\phi(h_j)\left(\frac{h_{i,j}}{h_j}\right)^k + \phi(h_i)\right) \le c h_i^{-1}\phi(h_j)\left(\frac{h_{i,j}}{h_j}\right)^k \\ &\le c h_j^{-1}\phi(h_j)\left(\frac{h_{i,j}}{h_j}\right)^{k+1}.\end{aligned}$$

Since (9.8) implies that

$$\frac{h_{i,j}}{h_j} \le c\,(|i-j|+1)^2\,,$$

we conclude that

$$|p_j'(x_i^*)| \le c h_j^{-1}\phi(h_j)\,(|i-j|+1)^{2k+2}\,.$$

We now use the fact that there are $k-1$ points $(x_{i_l}^*)_{l=1}^{k-1}$ with any two of them separated by at least one interval $I_i \subset I_{\mu,\nu}$.

For any $x \in (x_j, x_{j-1})$, we represent p_j' (which is a polynomial of degree $\le k-2$) as

$$p_j'(x) = \sum_{l=1}^{k-1} p_j'(x_{i_l}^*) \prod_{1\le m\le k-1, m\ne l} \frac{x - x_{i_m}^*}{x_{i_l}^* - x_{i_m}^*},$$

estimate

$$\left|\frac{x - x_{i_m}^*}{x_{i_l}^* - x_{i_m}^*}\right| \le c\frac{h_{j,i_m}}{h_{i_m}} \le c\,(|j-i_m|+1)^2 \le c\left((j-\mu)^2 + (j-\nu)^2\right),$$

and obtain

$$\begin{aligned}\rho_n(x)\phi^{-1}(\rho_n(x))|S'(x)| &\le c h_j \phi^{-1}(h_j))|p_j'(x)| \\ &\le c\sum_{l=1}^{k-1}(|j-i_l|+1)^{2k+2}\left((j-\mu)^2+(j-\nu)^2\right)^{k-2} \\ &\le c\left((j-\mu)^2+(j-\nu)^2\right)^{2k-1},\end{aligned}$$

which implies (10.3). $\square$

Theorem 10.2. *Let $k, r \in \mathbb{N}$, $k \ge r+1$, and let $\phi \in \Phi^k$ be of the form $\phi(t) := t^r\psi(t)$, $\psi \in \Phi^{k-r}$. Also, let $d_+ \ge 0$, $d_- \ge 0$ and $\alpha \ge 0$ be given. Then there is a number $\mathcal{N} = \mathcal{N}(k, r, \phi, d_+, d_-, \alpha)$ satisfying the following assertion. If $n \ge \mathcal{N}$ and $S \in \Sigma_{k,n} \cap C[-1,1] \cap \Delta^{(1)}$ is such that*

$$b_k(S, \phi) \le 1, \tag{10.4}$$

and, additionally,

$$\text{if } d_+ > 0, \text{ then } \quad d_+|I_2|^{r-1} \le \min_{x\in I_2} S'(x), \tag{10.5}$$

$$\text{if } d_+ = 0, \text{ then } \quad S^{(i)}(1) = 0, \text{ for all } \ 1 \le i \le k-2, \tag{10.6}$$

$$\text{if } d_- > 0, \text{ then } \quad d_-|I_{n-1}|^{r-1} \le \min_{x\in I_{n-1}} S'(x), \tag{10.7}$$

$$\text{if } d_- = 0, \text{ then } \quad S^{(i)}(-1) = 0, \text{ for all } \ 1 \le i \le k-2, \tag{10.8}$$

then there exists a polynomial $P \in \Delta^{(1)} \cap \Pi_{Cn}$ *satisfying, for all* $x \in [-1, 1]$,

$$|S(x) - P(x)| \le C\, \delta_n^{\alpha}(x)\phi(\rho_n(x)), \qquad \text{if } d_+ > 0 \text{ and } d_- > 0, \tag{10.9}$$

$$|S(x) - P(x)| \le C\, \delta_n^{\min\{\alpha,2k-2\}}(x)\phi(\rho_n(x)), \text{ if } \min\{d_+, d_-\} = 0. \tag{10.10}$$

Proof. Throughout the proof, we fix $\beta := k+6$ and $\gamma := 60(\alpha+\beta)+4k+1$. Hence, the constants $C_1, \dots, C_6$ (defined below) as well as the constants C, may depend only on k and α. Note that S does not have to be differentiable at the Chebyshev knots x_j. Hence, when we write $S'(x)$ (or $S_i'(x)$, $1 \le i \le 4$) everywhere in this proof, we implicitly assume that $x \ne x_j$, $1 \le j \le n-1$. Also, recall that $\rho := \rho_n(x)$ and $\delta := \delta_n(x)$.

Let $C_1 := C$, where the constant C is taken from (9.1) (without loss of generality we assume that $C_1 \le 1$), and let $C_2 := C$ with C taken from (8.2) with $q = 1$. We also fix an integer C_3 such that

$$C_3 \ge 8k/C_1. \tag{10.11}$$

Without loss of generality, we may assume that n is divisible by C_3, and put $n_0 := n/C_3$.

We divide $[-1, 1]$ into n_0 intervals

$$E_q := [x_{qC_3}, x_{(q-1)C_3}] = I_{qC_3} \cup \dots \cup I_{(q-1)C_3+1}, \quad 1 \le q \le n_0,$$

consisting of C_3 intervals I_i each (*i.e.*, $m_{E_q} = C_3$, for all $1 \le q \le n_0$).

We write "$j \in UC$" (where "UC" stands for "Under Control)" if there is $x_j^* \in (x_j, x_{j-1})$ such that

$$S'(x_j^*) \le \frac{5C_2\phi(\rho_n(x_j^*))}{\rho_n(x_j^*)}. \tag{10.12}$$

We say that $q \in G$ (for "Good), if the interval E_q contains at least $2k-3$ intervals I_j with $j \in UC$. Then, (10.12) and Lemma 10.1 imply that,

$$S'(x) \le \frac{C\phi(\rho)}{\rho}, \quad x \in E_q,\ q \in G. \tag{10.13}$$

Set

$$E := \cup_{q\notin G} E_q,$$

and decompose S into a "small" part and a "big" one, by setting

$$s_1(x) := \begin{cases} S'(x), & \text{if } \ x \notin E, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$s_2(x) := S'(x) - s_1(x) = \begin{cases} 0, & \text{if} \quad x \notin E, \\ S'(x), & \text{otherwise}, \end{cases}$$

and putting

$$S_1(x) := S(-1) + \int_{-1}^{x} s_1(u)du \quad \text{and} \quad S_2(x) := \int_{-1}^{x} s_2(u)du.$$

(Note that s_1 and s_2 are well defined for $x \neq x_j$, $1 \le j \le n-1$, so that S_1 and S_2 are well defined everywhere and possess derivatives for $x \neq x_j$, $1 \le j \le n-1$.)

Evidently,

$$S_1, S_2 \in \Sigma_{k,n},$$

and

$$S_1'(x) \ge 0 \text{ and } S_2'(x) \ge 0, \quad x \in [-1,1].$$

Now, (10.13) implies that

$$S_1'(x) \le \frac{C\phi(\rho)}{\rho}, \quad x \in [-1,1],$$

which, in turn, yields by Lemma 5.2,

$$b_k(S_1, \phi) \le C.$$

Together with (10.4), we obtain

$$b_k(S_2, \phi) \le b_k(S_1, \phi) + b_k(S, \phi) \le C + 1 \le \lceil C + 1 \rceil =: C_4. \tag{10.14}$$

The set E is a union of disjoint intervals $F_p = [a_p, b_p]$, between any two of which, all intervals E_q are with $q \in G$. We may assume that $n > C_3C_4$, and write $p \in AG$ (for "Almost Good"), if F_p consists of no more than C_4 intervals E_q, that is, it consists of no more than C_3C_4 intervals I_j. Hence, by Lemma 10.1 (with μ and ν chosen so that $I_{\mu,\nu}$ is the union of such an interval F_p, $p \in AG$, and one of the adjacent intervals E_q with $q \in G$),

$$S_2'(x) \le \frac{C\,\phi(\rho)}{\rho}, \quad x \in F_p,\ p \in AG. \tag{10.15}$$

One may think of intervals F_p, $p \notin AG$, as "long" intervals where S' is "large" on many subintervals I_i and rarely dips down to 0. Intervals F_p, $p \in AG$, as well as all intervals E_q which are not contained in any F_p's (*i.e.*, all "good" intervals E_q) are where S' is "small" in the sense that the inequality $S'(x) \le C\phi(\rho)/\rho$ is valid there.

Set

$$F := \cup_{p \notin AG} F_p,$$

note that $E = \cup_{p \in AG} F_p \cup F$, and decompose S again by setting

$$s_4 := \begin{cases} S'(x), & \text{if} \quad x \in F, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$s_3(x) := S'(x) - s_4(x) = \begin{cases} 0, & \text{if} \quad x \in F, \\ S'(x), & \text{otherwise,} \end{cases}$$

and putting

$$S_3(x) := S(-1) + \int_{-1}^{x} s_3(u)du \quad \text{and} \quad S_4(x) := \int_{-1}^{x} s_4(u)du. \tag{10.16}$$

Then, evidently,

$$S_3, S_4 \in \Sigma_{k,n}, \quad S_3 + S_4 = S, \tag{10.17}$$

and

$$S_3'(x) \ge 0 \ \text{ and } \ S_4'(x) \ge 0, \quad x \in [-1,1]. \tag{10.18}$$

We remark that, if $x \notin \cup_{p\in AG} F_p$, then $s_1(x) = s_3(x)$ and $s_2(x) = s_4(x)$. If $x \in \cup_{p\in AG} F_p$, then $s_1(x) = s_4(x) = 0$ and $s_2(x) = s_3(x) = S'(x)$.

For $x \in \cup_{p\in AG} F_p$, (10.15) implies that

$$S_3'(x) = S_2'(x) \le \frac{C\,\phi(\rho)}{\rho}.$$

For all other x's,

$$S_3'(x) = S_1'(x) \le \frac{C\,\phi(\rho)}{\rho}.$$

We conclude that

$$S_3'(x) \le \frac{C_5\,\phi(\rho)}{\rho}, \quad x \in [-1,1], \tag{10.19}$$

which by virtue of Lemma 5.2, yields that $b_k(S_3, \phi) \le C$. As above, we obtain

$$b_k(S_4, \phi) \le b_k(S_3, \phi) + b_k(S, \phi) \le C + 1 \le \lceil C + 1 \rceil =: C_6. \tag{10.20}$$

We will approximate S_3 and S_4 by nondecreasing polynomials that achieve the required degree of pointwise approximation.

Approximation of S_3:

If $d_+ > 0$, then there exists $\mathcal{N}^* \in \mathbb{N}$, $\mathcal{N}^* = \mathcal{N}^*(d_+, \psi)$, such that, for $n > \mathcal{N}^*$,

$$\frac{\phi(\rho)}{\rho} = \rho^{r-1}\psi(\rho) < \frac{d_+|I_2|^{r-1}}{C_5} \le C_5^{-1}\,S'(x), \quad x \in I_2,$$

where the first inequality follows since $\psi(\rho) \le \psi(2/n) \to 0$ as $n \to \infty$, and the second inequality follows by (10.5). Hence, by (10.19), if $n > \mathcal{N}^*$, then $s_3(x) \ne S'(x)$ for $x \in I_2$. Therefore, since $s_3(x) = S'(x)$, for all $x \notin F$, we conclude that $I_2 \subset F$, and so $E_1 \subset F$, and $s_3(x) = 0$, $x \in E_1$. In particular, $s_3(x) \equiv 0$, $x \in I_1$.

Similarly, if $d_- > 0$, then using (10.7) we conclude that there exists $\mathcal{N}^{**} \in \mathbb{N}$, $\mathcal{N}^{**} = \mathcal{N}^{**}(d_-, \psi)$, such that, if $n > \mathcal{N}^{**}$, then $s_3(x) \equiv 0$ for all $x \in I_n$.

Thus, when both d_+ and d_- are strictly positive, we conclude that for $n \ge \max\{\mathcal{N}^*, \mathcal{N}^{**}\}$, we have

$$s_3(x) = 0, \quad \text{for all } \ x \in I_1 \cup I_n. \tag{10.21}$$

Therefore, in view of (10.17) and (10.18), it follows by Lemma 6.1 combined with (10.19) that, in the case $d_+ > 0$ and $d_- > 0$, there exists a nondecreasing polynomial $r_n \in \Pi_{Cn}$ such that

$$|S_3(x) - r_n(x)| \le C\,\delta^\alpha \phi(\rho), \quad x \in [-1, 1]. \tag{10.22}$$

Suppose now that $d_+ = 0$ and $d_- > 0$. First, proceeding as above, we conclude that $s_3 \equiv 0$ on I_n. Additionally, if $E_1 \subset F$, then, as above, $s_3 \equiv 0$ on I_1 as well. Hence, (10.21) holds which, in turn, implies (10.22).

If $E_1 \not\subset F$, then $s_3(x) = S'(x)$, $x \in I_1$, and so it follows from (10.6) that, for some constant $c_* \ge 0$,

$$s_3(x) = S'(x) = c_*(1-x)^{k-2}, \quad x \in I_1.$$

By (10.19) we conclude that

$$c_* \le C_5 \frac{\phi(\rho_n(x_1))}{(1-x_1)^{k-2}\rho_n(x_1)} \sim n^{2k-2}\phi(n^{-2}).$$

Hence, for $x \in I_1$,

$$S_3'(x) = s_3(x) \le C\,(n\varphi(x))^{2k-4}\, n^2 \phi(n^{-2}) \le C\delta^{2k-4}\frac{\phi(\rho)}{\rho}$$

and

$$0 \le S_3(1) - S_3(x) = \int\limits_x^1 s_3(u)\,du \le c(1-x)^{k-1} n^{2k-2}\phi(n^{-2}) \le C\delta^{2k-2}\phi(\rho).$$

We now define

$$\widetilde{S}_3(x) := \begin{cases} S_3(x), & \text{if } x < x_1, \\ S_3(1), & \text{if } x \in [x_1, 1]. \end{cases}$$

Then $\widetilde{S}_3 \in \Sigma_{k,n} \cap \Delta^{(1)}$, $\widetilde{S}_3'(x) \le C\rho^{-1}\phi(\rho)$, $x \notin \{x_j\}_{j=1}^{n-1}$, and $\widetilde{S}_3' \equiv 0$ on $I_1 \cup I_n$. Note also that $\widetilde{S}_3$ may be discontinuous at x_1 but the jump is bounded by $\phi(\rho_n(x_1))$ there. Hence, Lemma 6.1 implies that there exists a nondecreasing polynomial $r_n \in \Pi_{Cn}$ such that

$$|\widetilde{S}_3(x) - r_n(x)| \le C\,\delta^\alpha \phi(\rho), \quad x \in [-1, 1].$$

Now, since

$$\left|\widetilde{S}_3(x) - S_3(x)\right| \le C\delta^{2k-2}\phi(\rho), \quad x \in [-1, 1],$$

we conclude that

$$|S_3(x) - r_n(x)| \le C\,\delta^{\min\{\alpha,2k-2\}}\phi(\rho), \quad x \in [-1,1]. \tag{10.23}$$

Finally, if $d_- = 0$ and $d_+ > 0$, then the considerations are completely analogous and, if $d_- = 0$ and $d_+ = 0$, then $\widetilde{S}_3$ can be modified further on I_n using (10.8) and the above argument.

Hence, we've constructed a nondecreasing polynomial $r_n \in \Pi_{Cn}$ such that, in the case when both d_+ and d_- are strictly positive, (10.22) holds, and (10.23) is valid if at least one of these numbers is 0.

Approximation of S_4:

Given a set $A \subset [-1,1]$, denote

$$A^e := \cup_{I_j \cap A \neq \emptyset} I_j \quad \text{and} \quad A^{2e} := (A^e)^e,$$

where $I_0 = \emptyset$ and $I_{n+1} = \emptyset$. For example, $[x_7, x_3]^e = [x_8, x_2]$, $I_1^e = I_1 \cup I_2$, etc.

Also, given subinterval $I \subset [-1,1]$ with its endpoints at the Chebyshev knots, we refer to the right-most and the left-most intervals I_i contained in I as $EP_+(I)$ and $EP_-(I)$, respectively (for the "End Point" intervals). More precisely, if $1 \le \mu < \nu \le n$ and

$$I = \bigcup_{i=\mu}^{\nu} I_i,$$

then $EP_+(I) := I_\mu$, $EP_-(I) := I_\nu$ and $EP(I) := EP_+(I) \cup EP_-(I) = I_\mu \cup I_\nu$. For example, $EP_+[-1,1] := I_1$, $EP_-[-1,1] := I_n$, $EP_+[x_7, x_3] = [x_4, x_3] = I_4$, $EP_-[x_7, x_3] = [x_7, x_6] = I_7$, $EP[x_7, x_3] = I_4 \cup I_7$, etc. Here, we simplified the notation by using $EP_\pm[a,b] := EP_\pm([a,b])$ and $EP[a,b] := EP([a,b])$.

In order to approximate S_4, we observe that for $p \notin AG$,

$$S_4'(x) = S_2'(x), \quad x \in F_p^{2e},$$

so that by virtue of (10.14), we conclude that

$$b_k(S_4, \phi, F_p^{2e}) = b_k(S_2, \phi, F_p^{2e}) \le b_k(S_2, \phi) \le C_4. \tag{10.24}$$

(Note that, for $p \in AG$, S_4 is constant in F_p^{2e} and so $b_k(S_4, \phi, F_p^{2e}) = 0$.)

We will approximate S_4 using the polynomial $D_{n_1}(\cdot, S_4) \in \Pi_{Cn_1}$ defined in Lemma 8.1 (with $n_1 := C_6 n$), and then we construct two "correcting" polynomials $\overline{Q}_n, M_n \in \Pi_{Cn}$ (using Lemma 9.1) in order to make sure that the resulting approximating polynomial is nondecreasing.

We begin with $\overline{Q}_n$. For each q for which $E_q \subset F$, let J_q be the union of all intervals $I_j \subset E_q$ with $j \in UC$ with the union of both intervals $I_j \subset E_q$ at the endpoints of E_q. In other words,

$$J_q := \bigcup_j \left\{ I_j \;\middle|\; j \in UC \quad \text{and} \quad I_j \subset E_q \right\} \,\cup\, EP(E_q).$$

Since $E_q \subset F$, then $q \notin G$ and so the number of intervals $I_j \subset E_q$ with $j \in UC$ is at most $2k-4$. Hence, by (10.11),

$$m_{J_q} \le 2k - 2 < 2k \le \frac{C_1 C_3}{4} \le \frac{C_3}{4}.$$

Recalling that the total number m_{E_q} of intervals I_j in E_q is C_3 we conclude that Lemma 9.1 can be used with $E := E_q$ and $J := J_q$. Thus, set

$$\overline{Q}_n := \sum_{q\,:\,E_q\subset F} Q_n(\cdot, E_q, J_q),$$

where Q_n are polynomials from Lemma 9.1, and denote

$$J := \bigcup_{q\,:\,E_q\subset F} J_q.$$

Then, (9.1) through (9.3) imply that $\overline{Q}_n$ satisfies

$$\begin{aligned} &\text{(a)} && \overline{Q}_n'(x) \ge 0, \quad x \in [-1,1]\setminus F, \\ &\text{(b)} && \overline{Q}_n'(x) \ge -\frac{\phi(\rho)}{\rho} \quad x \in F\setminus J, \\ &\text{(c)} && \overline{Q}_n'(x) \ge 4\frac{\phi(\rho)}{\rho}\delta^{8\alpha}, \quad x \in J. \end{aligned} \tag{10.25}$$

Note that the inequalities in (10.25) are valid since, for any given x, all relevant $Q_n'(x, E_q, J_q)$, except perhaps one, are nonnegative, and

$$C_1\frac{m_{E_q}}{m_{J_q}} \ge \frac{C_1C_3}{2k} \ge 4.$$

Also, it follows from (9.3) that, for any $x \in [-1,1]$,

$$\begin{aligned} |\overline{Q}_n(x)| &\le C\delta^\alpha\rho\phi(\rho)\sum_{q\,:\,E_q\subset F}\sum_{j:\,I_j\subset E_q}\frac{h_j}{(|x-x_j|+\rho)^2} \\ &\le C\delta^\alpha\rho\phi(\rho)\sum_{j=1}^{n}\frac{h_j}{(|x-x_j|+\rho)^2} \\ &\le C\delta^\alpha\rho\phi(\rho)\int_0^\infty\frac{du}{(u+\rho)^2} \\ &= C\delta^\alpha\phi(\rho). \end{aligned} \tag{10.26}$$

Next, we define the polynomial M_n. For each F_p with $p \notin AG$, let J_p^- denote the union of the two intervals on the left side of F_p^e (or just the interval I_n if $-1 \in F_p$), and let J_p^+ denote the union of the two intervals on the right side of F_p^e (or just one interval I_1 if $1 \in F_p$), *i.e.*,

$$J_p^- = EP_-(F_p^e) \cup EP_-(F_p) \quad \text{and} \quad J_p^+ = EP_+(F_p^e) \cup EP_+(F_p).$$

Also, let F_p^- and F_p^+ be the closed intervals each consisting of $m_{F_p^\pm} := C_3C_4$ intervals I_j and such that $J_p^- \subset F_p^- \subset F_p^e$ and $J_p^+ \subset F_p^+ \subset F_p^e$, and put

$$J_p^* := J_p^- \cup J_p^+ \quad \text{and} \quad J^* := \cup_{p\notin AG} J_p^*.$$

Now, we set

$$M_n := \sum_{p\notin AG}\left(Q_n(\cdot, F_p^+, J_p^+) + Q_n(\cdot, F_p^-, J_p^-)\right).$$

Since $m_{F_p^+} = m_{F_p^-} = C_3C_4$ and $m_{J_p^+}, m_{J_p^-} \le 2$, it follows from (10.11) that

$$\min\left\{\frac{m_{F_p^+}}{m_{J_p^+}}, \frac{m_{F_p^-}}{m_{J_p^-}}\right\} \ge \frac{C_1C_3C_4}{2} \ge 2C_4.$$

Then Lemma 9.1 implies

$$|M_n(x)| \le C\,\delta^\alpha \phi(\rho) \tag{10.27}$$

(this follows from (9.3) using the same sequence of inequalities that was used to prove (10.26) above), and

$$\begin{aligned}
&\text{(a)} \quad M_n'(x) \ge -2\frac{\phi(\rho)}{\rho}, \quad x \in F \setminus J^*,\\
&\text{(b)} \quad M_n'(x) \ge 2C_4\,\delta^{8\alpha}\frac{\phi(\rho)}{\rho}, \quad x \in J^*,\\
&\text{(c)} \quad M_n'(x) \ge 2C_4\,\delta^{8\alpha}\frac{\phi(\rho)}{\rho}\left(\frac{\rho}{\operatorname{dist}(x,F)}\right)^{\gamma+1}, \ x \in [-1,1]\setminus F^e,
\end{aligned} \tag{10.28}$$

where in the last inequality we used the fact that

$$\max\{\rho, \operatorname{dist}(x, F^e)\} \le \operatorname{dist}(x, F), \quad x \in [-1,1]\setminus F^e,$$

which follows from (3.3).

The third auxiliary polynomial is $D_{n_1} := D_{n_1}(\cdot, S_4)$ with $n_1 = C_6 n$ constructed in Lemma 8.1. By (10.20), (8.1) yields

$$|S_4(x) - D_{n_1}(x)| \le C\,\delta^\gamma \phi(\rho) \le C\,\delta^\alpha \phi(\rho), \quad x \in [-1,1], \tag{10.29}$$

since $\gamma > \alpha$, and (8.2) implies that, for any interval $A \subset [-1,1]$ having Chebyshev knots as endpoints,

$$\begin{aligned}
|S_4'(x) - D_{n_1}'(x)| &\le C_2\,\delta^\gamma \frac{\phi(\rho)}{\rho} b_k(S_4, \phi, A)\\
&\quad + C_2C_6\,\delta^\gamma \frac{\phi(\rho)}{\rho}\frac{n}{n_1}\left(\frac{\rho}{\operatorname{dist}(x,[-1,1]\setminus A)}\right)^{\gamma+1}, \quad x \in A.
\end{aligned} \tag{10.30}$$

We now define

$$R_n := D_{n_1} + C_2\overline{Q}_n + C_2M_n. \tag{10.31}$$

By virtue of (10.26), (10.27), and (10.29) we obtain

$$|S_4(x) - R_n(x)| \le C\,\delta^\alpha \phi(\rho), \quad x \in [-1,1],$$

which combined with (10.22) and (10.23), proves (10.9) and (10.10) for $P := R_n + r_n$.

Thus, in order to conclude the proof of Theorem 10.2, we should prove that P is nondecreasing. We recall that r_n is nondecreasing, so it is sufficient to show that R_n is nondecreasing as well.

Note that (10.31) implies

$$R_n'(x) \ge C_2\overline{Q}_n'(x) + C_2M_n'(x) - |S_4'(x) - D_{n_1}'(x)| + S_4'(x), \quad x \in [-1,1],$$

(this inequality is extensively used in the three cases below), and that (10.30) holds for *any* interval A with Chebyshev knots as the endpoints, and so we can use different intervals A for different points $x \in [-1,1]$. We consider three cases depending on whether (i) $x \in F \setminus J^*$, or (ii) $x \in J^*$, or (iii) $x \in [-1,1] \setminus F^e$.

Case (i): If $x \in F \setminus J^*$, then, for some $p \notin AG$, $x \in F_p \setminus J_p^*$, and so we take $A := F_p$. Then, the quotient inside the parentheses in (10.30) is bounded above by 1 (this follows from (3.3)). Also, since $s_4(x) = S'(x)$, $x \in F$, it follows that $b_k(S_4, \phi, F_p) = b_k(S, \phi, F_p) \le 1$. Hence,

$$|S_4'(x) - D_{n_1}'(x)| \le C_2 \frac{\phi(\rho)}{\rho} b_k(S_4, \phi, F_p) + C_2 C_6 \frac{\phi(\rho)}{\rho} \frac{n}{n_1} \le 2C_2 \frac{\phi(\rho)}{\rho}, \quad x \in F \setminus J^*. \tag{10.32}$$

Note that $x \notin I_1 \cup I_n$ (since $F \setminus J^*$ does not contain any intervals in $EP(F_p)$, $p \notin AG$), and so $\delta = 1$.

It now follows by (10.25)(c), (10.28)(a), (10.32) and (10.18), that

$$R_n'(x) \ge C_2 \frac{\phi(\rho)}{\rho}(4 - 2 - 2) = 0, \quad x \in J \setminus J^*.$$

If $x \in F \setminus (J \cup J^*)$, then (10.12) is violated and so

$$S_4'(x) = S'(x) > \frac{5C_2\phi(\rho)}{\rho}.$$

Hence, by virtue of (10.25)(b), (10.28)(a) and (10.32), we get

$$R_n'(x) \ge C_2 \frac{\phi(\rho)}{\rho}(-1 - 2 - 2 + 5) = 0, \quad x \in F \setminus (J \cup J^*).$$

Case (ii): If $x \in J^*$, then, $x \in J_p^*$, for some $p \notin AG$, and we take $A := F_p^{2e}$. Then, (10.24) and (10.30) imply (again, (3.3) is used to estimate the quotient inside the parentheses in (10.30)),

$$\begin{aligned} |S_4'(x) - D_{n_1}'(x)| &\le C_2 \, \delta^\gamma \frac{\phi(\rho)}{\rho} b_k(S_4, \phi, F_p^{2e}) + C_2 C_6 \, \delta^\gamma \frac{\phi(\rho)}{\rho} \frac{n}{n_1} \\ &\le C_2 C_4 \, \delta^\gamma \frac{\phi(\rho)}{\rho}, \quad x \in J^*. \end{aligned} \tag{10.33}$$

Now, we note that $EP(F_p) \subset J$, for all $p \notin AG$, and so $F \cap J^* \subset J$. Hence, using (10.25)(a,c), (10.28)(b), (10.33) and (10.18), we obtain

$$R_n'(x) \ge 2C_2 C_4 \, \delta^{8\alpha} \frac{\phi(\rho)}{\rho} - 2C_2 C_4 \, \delta^\gamma \frac{\phi(\rho)}{\rho} \ge 0,$$

since $\gamma > 8\alpha$, and so $\delta^\gamma \le \delta^{8\alpha}$.

Case (iii): If $x \in [-1,1] \setminus F^e$, then we take A to be the connected component of $[-1,1] \setminus F$ that contains x. Then by (10.30),

$$\begin{aligned} |S_4'(x) - D_{n_1}'(x)| &\le C_2 \, \delta^\gamma \frac{\phi(\rho)}{\rho} b_k(S_4, \phi, A) + C_2 C_6 \, \delta^\gamma \frac{\phi(\rho)}{\rho} \frac{n}{n_1} \left(\frac{\rho}{\operatorname{dist}(x, [-1,1] \setminus A)} \right)^{\gamma+1} \\ &= C_2 \, \delta^\gamma \frac{\phi(\rho)}{\rho} \left(\frac{\rho}{\operatorname{dist}(x, F)} \right)^{\gamma+1}, \quad x \in [-1,1] \setminus F^e, \end{aligned} \tag{10.34}$$

where we used the fact that S_4 is constant in A, and so $b_k(S_4, \phi, A) = 0$.

Now, (10.25)(a), (10.28)(c), (10.34) and (10.18) imply,

$$R_n'(x) \ge \frac{\phi(\rho)}{\rho}\left(\frac{\rho}{\operatorname{dist}(x,F)}\right)^{\gamma+1}\left(2C_2C_4\delta^{8\alpha} - C_2\,\delta^{\gamma}\right) \ge 0,$$

since $C_4 \ge 1$ and $\gamma > 8\alpha$.

Thus, $R_n'(x) \ge 0$ for all $x \in [-1,1]$, and so we have constructed a nondecreasing polynomial P, satisfying (10.9) and (10.10), for each $n \ge \mathcal{N}$. This completes the proof. $\square$

11. Proof of Theorem 1.2

In order to prove Theorem 1.2, we first approximate f by appropriate piecewise polynomials. To this end we make use, among other things, of the following result on pointwise monotone piecewise polynomial approximation (see [9]).

Theorem 11.1. *Given $r \in \mathbb{N}$, there is a constant $c = c(r)$ with the property that if $f \in C^r[-1,1] \cap \Delta^{(1)}$, then there is a number $\widetilde{\mathcal{N}} = \widetilde{\mathcal{N}}(f,r)$, depending on f and r, such that for $n \ge \widetilde{\mathcal{N}}$, there are nondecreasing continuous piecewise polynomials $S \in \Sigma_{r+2,n}$ satisfying*

$$|f(x) - S(x)| \le c(r)\left(\frac{\varphi(x)}{n}\right)^r \omega_2\left(f^{(r)}, \frac{\varphi(x)}{n}\right), \quad x \in [-1,1]. \tag{11.1}$$

Moreover,

$$|f(x) - S(x)| \le c(r)\varphi^{2r}(x)\omega_2\left(f^{(r)}, \frac{\varphi(x)}{n}\right), \quad x \in [-1, x_{n-1}] \cup [x_1, 1]. \tag{11.2}$$

As was shown in [9], near ± 1, polynomial pieces of the spline S from the statement of Theorem 11.1 can be taken to be Lagrange–Hermite polynomials of degree $\le r+1$. Namely,

$$S\big|_{[x_2,1]}(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \cdots + \frac{f^{(r)}(1)}{r!}(x-1)^r + a_+(n;f)(x-1)^{r+1}$$

and

$$S\big|_{[-1,x_{n-2}]}(x) = f(-1) + \frac{f'(-1)}{1!}(x+1) + \cdots + \frac{f^{(r)}(-1)}{r!}(x+1)^r + a_-(n;f)(x+1)^{r+1},$$

where constants $a_+(n,f)$ and $a_-(n,f)$ depend only on n and f, and are chosen so that $S(x_2) = f(x_2)$ and $S(x_{n-2}) = f(x_{n-2})$. It was shown in [9, (3.1)] that

$$|a_+(n,f)| \le \frac{1}{r!\,(|I_1| + |I_2|)}\omega_1(f^{(r)}, |I_1| + |I_2|, I_1 \cup I_2)$$

and

$$|a_-(n,f)| \le \frac{1}{r!\,(|I_{n-1}| + |I_n|)}\omega_1(f^{(r)}, |I_{n-1}| + |I_n|, I_{n-1} \cup I_n).$$

On I_j's with $j \ne 1, 2, n-1, n$, polynomial pieces p_j of S were constructed using [10, Lemma 2, p. 58].

For $f \in C^r[-1,1]$, let $i_+ \ge 1$, be the smallest integer $1 \le i \le r$, if it exists, such that $f^{(i)}(1) \ne 0$, and denote

$$D_+(r,f) := \begin{cases} (2r!)^{-1}|f^{(i_+)}(1)|, & \text{if } i_+ \text{ exists,} \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, let $i_- \geq 1$, be the smallest integer $1 \leq i \leq r$, if it exists, such that $f^{(i)}(-1) \neq 0$, and denote

$$D_-(r,f) := \begin{cases} (2r!)^{-1}|f^{(i_-)}(-1)|, & \text{if } i_- \text{ exists,} \\ 0, & \text{otherwise.} \end{cases}$$

Using the above as well as the observation that $|I_1| + |I_2| \to 0$ and $|I_{n-1}| + |I_n| \to 0$ as $n \to \infty$, we can strengthen Theorem 11.1 as follows.

Lemma 11.2. *Given $r \in \mathbb{N}$, there is a constant $c = c(r)$ with the property that if a function $f \in C^r[-1,1] \cap \Delta^{(1)}$, then there is an integer $\mathcal{N} = \mathcal{N}(f,r)$ depending on f and r, such that for $n \geq \mathcal{N}$, there are nondecreasing continuous piecewise polynomials $S \in \Sigma_{r+2,n}$ satisfying (11.1), (11.2),*

$$S^{(i)}(-1) = f^{(i)}(-1) \quad \text{and} \quad S^{(i)}(1) = f^{(i)}(1), \quad \text{for all } 1 \leq i \leq r, \tag{11.3}$$

$$S'(x) \geq D_+(r,f)(1-x)^{r-1}, \quad x \in (x_2, 1], \tag{11.4}$$

and

$$S'(x) \geq D_-(r,f)(x+1)^{r-1}, \quad x \in [-1, x_{n-2}). \tag{11.5}$$

Proof of Theorem 1.2. Given $r \in \mathbb{N}$ and a nondecreasing $f \in C^{(r)}[-1,1]$, let $\psi \in \Phi^2$ be such that $\omega_2(f^{(r)}, t) \sim \psi(t)$, denote $\phi(t) := t^r\psi(t)$, and note that $\phi \in \Phi^{r+2}$.

For each $n \geq \mathcal{N}$, we take the piecewise polynomial $S \in \Sigma_{r+2,n}$ of Lemma 11.2 and we observe that

$$\omega_{r+2}(f,t) \leq t^r \omega_2(f^{(r)}, t) \sim \phi(t),$$

so that by Lemma 5.1 with $k = r+2$, we conclude that

$$b_{r+2}(S, \phi) \leq c := \varsigma.$$

Now, it follows from (11.4) and (3.1) that

$$\min_{x \in I_2} S'(x) \geq D_+(r,f)|I_1|^{r-1} \geq 3^{-r+1} D_+(r,f)|I_2|^{r-1}$$

and, similarly, (11.5) yields

$$\min_{x \in I_{n-1}} S'(x) \geq 3^{-r+1} D_-(r,f)|I_{n-1}|^{r-1}.$$

Hence, using Theorem 10.2 with $k = r+2$, $d_+ := \varsigma^{-1}3^{-r+1}D_+(r,f)$, $d_- := \varsigma^{-1}3^{-r+1}D_-(r,f)$ and $\alpha = 2k - 2 = 2r+2$, and observing that $b_{r+2}(\varsigma^{-1}S, \phi) \leq 1$, we conclude that there exists a polynomial $P \in \Pi_{cn} \cap \Delta^{(1)}$ such that

$$|S(x) - P(x)| \leq c\delta_n^{2r+2}(x)\rho_n^r(x)\psi(\rho_n(x)), \quad x \in [-1,1]. \tag{11.6}$$

In particular, for $x \in I_1 \cup I_n$, $x \neq -1, 1$, using the fact that $\rho_n(x) \sim n^{-2}$ for these x, and $t^{-2}\psi(t)$ is nonincreasing we have

$$
\begin{aligned}
|S(x) - P(x)| &\le c(n\varphi(x))^{2r+2}\rho_n^r(x)\psi(\rho_n(x)) \\
&\le cn^2\varphi^{2r+2}(x)\left(\frac{n\rho_n(x)}{\varphi(x)}\right)^2\psi\left(\frac{\varphi(x)}{n}\right) \\
&\le c\varphi^{2r}(x)\omega_2\left(f^{(r)}, \frac{\varphi(x)}{n}\right).
\end{aligned} \tag{11.7}
$$

In turn, this implies for $x \in I_1 \cup I_n$, that

$$
|S(x) - P(x)| \le c\left(\frac{\varphi(x)}{n}\right)^r \omega_2\left(f^{(r)}, \frac{\varphi(x)}{n}\right),
$$

which combined with (11.6) implies

$$
|S(x) - P(x)| \le c\left(\frac{\varphi(x)}{n}\right)^r \omega_2\left(f^{(r)}, \frac{\varphi(x)}{n}\right), \quad x \in [-1, 1]. \tag{11.8}
$$

Finally, (11.8) together with (11.1) yield (1.6), and (11.7) together with (11.2) yield (1.7). The proof of Theorem 1.2 is complete. $\square$

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