J. Math. Anal. Appl. 474 (2019) 467–479

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Interpolatory estimates for convex piecewise polynomial approximation

K.A. Kopotun [a,*,1], D. Leviatan [b], I.A. Shevchuk [c]

[a] Department of Mathematics, University of Manitoba, Winnipeg, Manitoba, R3T 2N2, Canada
[b] Raymond and Beverly Sackler School of Mathematical Sciences, Tel Aviv University, Tel Aviv 6139001, Israel
[c] Faculty of Mechanics and Mathematics, Taras Shevchenko National University of Kyiv, 01601 Kyiv, Ukraine

ARTICLE INFO

Article history:
Received 2 November 2018
Available online 29 January 2019
Submitted by V. Andrievskii

Keywords:
Convex approximation by polynomials
Degree of approximation
Jackson-type interpolatory estimates

ABSTRACT

In this paper, among other things, we show that, given $r \in \mathbb{N}$, there is a constant $c = c(r)$ such that if $f \in C^r[-1,1]$ is convex, then there is a number $\mathcal{N} = \mathcal{N}(f,r)$, depending on f and r, such that for $n \ge \mathcal{N}$, there are convex piecewise polynomials S of order $r+2$ with knots at the nth Chebyshev partition, satisfying

$$|f(x) - S(x)| \le c(r) \left(\min\left\{1-x^2, n^{-1}\sqrt{1-x^2}\right\}\right)^r \omega_2\left(f^{(r)}, n^{-1}\sqrt{1-x^2}\right),$$

for all $x \in [-1,1]$. Moreover, $\mathcal{N}$ cannot be made independent of f.

1. Introduction, motivation and history

For $r \in \mathbb{N}$, let $C^r[a,b]$, $-1 \le a < b \le 1$, denote the space of r times continuously differentiable functions on $[a,b]$, and set $C^0[a,b] := C[a,b]$, the space of continuous functions on $[a,b]$, equipped with the uniform norm $\|\cdot\|_{[a,b]}$. Let $\mathbb{P}_n$ be the space of algebraic polynomials of degree $\le n$ (that is of order $\le n+1$).

For $f \in C[a,b]$ and any $k \in \mathbb{N}$, set

$$\Delta_u^k(f,x;[a,b]) := \begin{cases} \sum_{i=0}^k (-1)^i \binom{k}{i} f(x+(k/2-i)u), & x \pm (k/2)u \in [a,b], \\ 0, & \text{otherwise}, \end{cases}$$

and denote by

* Corresponding author.
E-mail addresses: kopotunk@cc.umanitoba.ca (K.A. Kopotun), leviatan@tauex.tau.ac.il (D. Leviatan), shevchuk@univ.kiev.ua (I.A. Shevchuk).
[1] Supported by NSERC of Canada Discovery Grant RGPIN 04215-15.

https://doi.org/10.1016/j.jmaa.2019.01.055

$$\omega_k(f,t;[a,b]) := \sup_{0<u\le t} \|\Delta_u^k(f,\cdot;[a,b])\|_{[a,b]},$$

its kth modulus of smoothness. When dealing with $[a,b]=[-1,1]$, we suppress referring to the interval, that is, we denote $\|\cdot\| := \|\cdot\|_{[-1,1]}$, $\omega_k(f,t) := \omega_k(f,t;[-1,1])$, etc.

Finally, let

$$\varphi(x) = \sqrt{1-x^2} \quad \text{and} \quad \rho_n(x) := \frac{\varphi(x)}{n} + \frac{1}{n^2}, \tag{1.1}$$

and note that $\rho_n(x) \sim \varphi(x)/n$, for $x \in [-1+n^{-2}, 1-n^{-2}]$ (we will often use this fact without further discussions).

Pointwise estimates have mostly been investigated for polynomial approximation of continuous functions in $[-1,1]$ and involved usually the quantity $\rho_n(x)$. The first to deal with such estimates was Nikolskii, and he was followed by Timan, Dzyadyk, Freud and Brudnyi. Detailed discussion may be found in the survey paper [4], where an extensive list of references is given. Discussion and references to estimates on pointwise monotone and pointwise convex polynomial approximation involving $\rho_n(x)$ may also be found there. Pointwise estimates of polynomial approximation involving $\varphi(x)$ are due originally to Teljakovskiĭ and Gopengauz, see [1,3] for extensions and many references. Note that for the latter estimates the approximating polynomials must interpolate the function at the endpoints of the interval. We call such estimates interpolatory.

Throughout this paper, we reserve the notation "c" for positive constants that are either absolute or may depend on the parameters k (the order of the modulus of smoothness) and/or r (the order of the derivative). We use the notation "C" and "C_i", $i \in \mathbb{N}_0$, for all other positive constants. We indicate in parentheses the parameters that the constants may depend on. All constants c and C may be different even if they appear in the same line, but the indexed constants C_i are fixed.

The following theorem is an immediate consequence of [3, Corollary 2-3.4].

Theorem 1.1. *Let $r \in \mathbb{N}_0$, $k \in \mathbb{N}$ and $f \in C^r[-1,1]$. Then for any $n \ge \max\{k+r-1, 2r+1\}$, there is a polynomial $P_n \in \mathbb{P}_n$ such that*

$$|f(x) - P_n(x)| \le c(r,k)\rho_n^r(x)\omega_k(f^{(r)}, \rho_n(x)), \quad x \in [-1+n^{-2}, 1-n^{-2}], \tag{1.2}$$

and

$$|f(x) - P_n(x)| \le c(r,k)\varphi^{2r}(x)\omega_k(f^{(r)}, \varphi^{2/k}(x)n^{-2(k-1)/k}), \tag{1.3}$$

for $x \in [-1,-1+n^{-2}] \cup [1-n^{-2},1]$. Moreover, for any $\gamma \in \mathbb{R}$, the quantity $\varphi^{2/k}(x)n^{-2(k-1)/k}$ in (1.3) cannot be replaced by $\varphi^{2\alpha}(x)n^{\gamma}$ with $\alpha > 1/k$.

Remark 1.2. Since $\omega_k(g,\lambda t) \le (\lambda+1)^k\omega_k(g,t)$, $\lambda > 0$, then, for $k \ge 2$ and x such that $\varphi(x) \le c/n$ (*i.e.*, x is near the endpoints of $[-1,1]$),

$$\omega_k(f^{(r)}, \varphi^{2/k}(x)n^{-2(k-1)/k}) \le c(k)[\varphi(x)n]^{2-k}\omega_k(f^{(r)}, \varphi(x)/n),$$

and so estimates (1.2)–(1.3) are stronger than

$$|f(x) - P_n(x)| \le c(r,k)\left(\varphi(x)/n\right)^r \omega_k(f^{(r)}, \varphi(x)/n), \quad x \in [-1,1], \tag{1.4}$$

if $k \le r+2$, and it is known that (1.4) does not hold in general if $k > r+2$ (see, *e.g.*, [3, p. 68] for more discussions).

Remark 1.3. Since $\omega_{k_2}(g,t) \le 2^{k_2-k_1}\omega_{k_1}(g,t)$ if $k_2 > k_1$, estimates (1.2) for "large" k imply those for "small" ones. However, this is not the case for estimates (1.3), and the fact that Theorem 1.1 is valid with some $k_2 \in \mathbb{N}$ does not imply that it is valid with $k_1 \in \mathbb{N}$ such that $k_1 < k_2$. For example, let $f_0(x) := (1+x)^{r+1/2}$. Then, $\omega_k(f_0^{(r)}, t) \sim \min\{1, \sqrt{t}\}$, for all $k \in \mathbb{N}$. Hence, estimate (1.3) becomes, for x "close" to the endpoints of $[-1,1]$,

$$|f_0(x) - P_n(x)| \le c(r,k)\phi_k(x,n), \quad \text{where} \;\; \phi_k(x,n) := \varphi^{2r+1/k}(x) n^{-1+1/k},$$

and

$$\lim_{x\to\pm1} \frac{\phi_{k_2}(x,n)}{\phi_{k_1}(x,n)} = \infty, \quad \text{if} \;\; k_2 > k_1,$$

i.e., this estimate for k_2 is not stronger than that for k_1.

At the same time, it is also rather well known that the estimates (1.2) and (1.3) for $k_1 \in \mathbb{N}$ do not imply those for $k_2 \in \mathbb{N}$ with $k_2 > k_1$. Hence, estimates in Theorem 1.1 for different k's do not follow from one another.

If we approximate monotone functions by monotone polynomials (we call this "monotone approximation" and denote by $\Delta^{(1)}$ the class of all non-decreasing functions on $[-1,1]$), then the situation is drastically different.

In [5], we showed that (1.2) and (1.3) with $k = 2$ are valid for monotone approximation provided that n is sufficiently large depending on the function f that is being approximated. Namely, the following theorem was proved in [5].

Theorem 1.4. *Given $r \in \mathbb{N}$, there is a constant $c = c(r)$ with the property that if $f \in C^r[-1,1] \cap \Delta^{(1)}$, then there exists a number $\mathcal{N} = \mathcal{N}(f,r)$, depending on f and r, such that for every $n \ge \mathcal{N}$, there is $P_n \in \mathbb{P}_n \cap \Delta^{(1)}$ satisfying (1.2) and (1.3) with $k = 2$.*

We note that $\mathcal{N}$ in the statement of Theorem 1.4, in general, cannot be made independent of f. It is still an open question if an analog of this theorem is valid for $k \ge 3$. If $r = 0$, then the situation is slightly different (we refer interested readers to [5] for a more detailed discussion of this).

The proof of Theorem 1.4 was based, in part, on interpolatory estimates for monotone approximation by piecewise polynomials, first obtained by Leviatan and Petrova [7] and [8].

It is a natural question if similar type of estimates/results are valid for convex approximation (*i.e.*, approximation of convex functions by convex polynomials), and the purpose of this manuscript is to begin investigation in this direction.

2. Main results

Given an interval $[a,b]$, let $X = \{x_j\}_{j=0}^n$ denote a partition of $[a,b]$, *i.e.*, $a =: x_0 < x_1 < \cdots < x_{n-1} < x_n := b$, and for $m \in \mathbb{N}$, denote by $S(X,m)$ the set of continuous piecewise polynomials of order m on the partition X, that is, $s \in S(X,m)$ if s is a piecewise polynomial of degree $m-1$ with knots x_j, *i.e.*, on each interval $[x_{j-1}, x_j]$, $1 \le j \le n$, the function s is an algebraic polynomial of degree $\le m-1$.

By the nth Chebyshev partition of $[-1,1]$, we mean the partition $T_n := \{t_j\}_{j=0}^n$, where

$$t_j := t_{j,n} := -\cos(j\pi/n), \quad 0 \le j \le n. \tag{2.1}$$

We refer to t_j's as "Chebyshev knots" and note that t_j, $1 \le j \le n-1$, are the extremum points of the Chebyshev polynomial of the first kind of degree n. It is also convenient to denote $t_j := t_{j,n} := 1$ for $j > n$

and $t_j := t_{j,n} := -1$ for $j < 0$. (We note that Chebyshev knots are sometimes numbered from right to left which is equivalent to defining them as $\tau_j := \cos(j\pi/n)$, $0 \le j \le n$, instead of (2.1).)

Denoting by $\Delta^{(2)}$ the class of all convex functions in $C[-1,1]$, our first result is the following theorem.

Theorem 2.1. *Given $r \in \mathbb{N}$, there is a constant $c = c(r)$ such that if $f \in C^r[-1,1]$ is convex, then there is a number $\mathcal{N} = \mathcal{N}(f,r)$, depending on f and r, such that for $n \ge \mathcal{N}$, there are convex piecewise polynomials S of order $r+2$ with knots at the Chebyshev partition T_n (*i.e., *$S \in S(T_n, r+2) \cap \Delta^{(2)}$), satisfying*

$$|f(x) - S(x)| \le c(r) \left(\frac{\varphi(x)}{n}\right)^r \omega_2\left(f^{(r)}, \frac{\varphi(x)}{n}\right), \quad x \in [-1,1], \tag{2.2}$$

and, moreover, for $x \in [-1, -1+n^{-2}] \cup [1-n^{-2}, 1]$,

$$|f(x) - S(x)| \le c(r)\varphi^{2r}(x)\omega_2\left(f^{(r)}, \frac{\varphi(x)}{n}\right) \tag{2.3}$$

and

$$|f(x) - S(x)| \le c(r)\varphi^{2r}(x)\omega_1\left(f^{(r)}, \varphi^2(x)\right). \tag{2.4}$$

Remark 2.2. As in the case of monotone approximation, $\mathcal{N}$ in the statement of Theorem 2.1, in general, cannot be independent of f (see Theorem 2.5). It is still an open problem if Theorem 2.1 is valid for the moduli of smoothness of order $k \ge 3$ with (2.2) and (2.3)/(2.4) replaced by (1.2) and (1.3).

It is known that an analog of Theorem 2.1 holds for $r = 0$ with $N = 1$ (and so, in the case $r = 0$, we do not have dependence of $\mathcal{N}$ on f). Indeed, the polygonal line, that is, the continuous piecewise linear S, interpolating f at the Chebyshev nodes, is convex and yields (2.2) with $r = 0$ (see, *e.g.*, a similar construction in [6]). Moreover, one can construct a continuous piecewise quadratic polynomial function S interpolating f at the Chebyshev nodes such that S is convex on $[-1,1]$ and the following estimates hold (see [2]):

$$|f(x) - S(x)| \le c\omega_3\left(f, \rho_n(x)\right), \quad x \in [-1+n^{-2}, 1-n^{-2}],$$

and, for $x \in [-1, -1+n^{-2}] \cup [1-n^{-2}, 1]$, in addition to (2.3) and (2.4), we have

$$|f(x) - S(x)| \le c\omega_3(f, n^{-4/3}\varphi^{2/3}(x)).$$

This follows from Lemma 3.1 below with $r = 0$ and $k = 3$ taking into account Remark 3.2(ii).

Below, we show that Theorem 2.1 is a consequence of a more general result on approximation by convex piecewise polynomials on general partitions, Theorem 2.6. However, we first show that, indeed, $\mathcal{N}$ above must depend on f.

We start with the following negative result that shows that an analog of Theorem 1.1 cannot hold for convex polynomial approximation if $r > 0$.

Theorem 2.3. *For any $r \in \mathbb{N}$ and each $n \in \mathbb{N}$, there is a function $f \in C^r[-1,1] \cap \Delta^{(2)}$, such that for every polynomial $P_n \in \mathbb{P}_n \cap \Delta^{(2)}$ and any positive on $(-1,1)$ function ψ such that $\lim_{x\to\pm1}\psi(x) = 0$, either*

$$\limsup_{x\to-1} \frac{|f(x) - P_n(x)|}{\varphi^2(x)\psi(x)} = \infty \quad \text{or} \quad \limsup_{x\to1} \frac{|f(x) - P_n(x)|}{\varphi^2(x)\psi(x)} = \infty. \tag{2.5}$$

Remark 2.4. A similar result is known for monotone approximation (see, *e.g.*, [5, (1.5)]).

Proof. The proof is very similar to that of [1, Theorem 4] (see also [10]), but there are slight variations, and so we give it here for completeness.

Given $n \in \mathbb{N}$ and $r \in \mathbb{N}$, we let $\varepsilon := n^{-2}$ and define

$$f(x) := \begin{cases} 0, & \text{if} \quad -1 \le x \le 1-\varepsilon, \\ (x-1+\varepsilon)^{r+1}, & \text{if} \quad 1-\varepsilon < x \le 1. \end{cases}$$

Then $f \in C^r[-1,1] \cap \Delta^{(2)}$, and suppose to the contrary that (2.5) both fail, *i.e.*, suppose that there exists a polynomial $P_n \in \mathbb{P}_n \cap \Delta^{(2)}$ and a constant A such that

$$|f(x) - P_n(x)| \le A\varphi^2(x)\psi(x),$$

for all x in some small neighborhoods of -1 and 1. This implies that $f(\pm 1) = P_n(\pm 1)$ and $f'(\pm 1) = P_n'(\pm 1)$. Hence, $P_n(-1) = P_n'(-1) = 0$, $P_n(1) = \varepsilon^{r+1}$ and $P_n'(1) = (r+1)\varepsilon^r$. Since $P_n \in \Delta^{(2)}$, the first derivative P_n' is non-decreasing, and so $\|P_n'\| = P_n'(1) = (r+1)\varepsilon^r$. Additionally, since P_n' is non-negative, P_n is non-decreasing, and so $\|P_n\| = P_n(1) = \varepsilon^{r+1}$. Now, Markov's inequality implies that

$$(r+1)\varepsilon^r = \|P_n'\| \le n^2 \|P_n\| = n^2 \varepsilon^{r+1},$$

which is a contradiction (recall that we chose $\varepsilon = n^{-2}$). □

We also have the following analog of Theorem 2.3 for piecewise polynomials that shows that $\mathcal{N}$ in the statement of Theorem 2.1 cannot be made independent of f.

Theorem 2.5. *For any $r, m, n \in \mathbb{N}$, and each partition $X = \{x_j\}_{j=0}^n$ of $[-1,1]$, there is a function $f \in C^r[-1,1] \cap \Delta^{(2)}$ that depends on r, m and n, such that for every $s \in S(X,m) \cap \Delta^{(2)}$ and any positive on $(-1,1)$ function ψ such that $\lim_{x\to\pm 1}\psi(x) = 0$, either*

$$\limsup_{x\to -1} \frac{|f(x)-s(x)|}{\varphi^2(x)\psi(x)} = \infty \quad \text{or} \quad \limsup_{x\to 1} \frac{|f(x)-s(x)|}{\varphi^2(x)\psi(x)} = \infty. \tag{2.6}$$

Proof. We follow, word for word, the proof of Theorem 2.3 except that we apply Markov's inequality on $[x_{n-1}, 1]$ to get

$$(r+1)\varepsilon^r = s'(1) = \|s'\|_{[x_{n-1},1]} \le \frac{2(m-1)^2}{1-x_{n-1}} \|s\|_{[x_{n-1},1]} = \frac{2(m-1)^2}{1-x_{n-1}} \varepsilon^{r+1},$$

and so arrive at a contradiction by choosing ε to be smaller than $\dfrac{(r+1)(1-x_{n-1})}{2(m-1)^2}$. □

We are now ready to state a more general result on approximation by convex piecewise polynomials on general partitions. It is convenient to use the following notation:

$$\Omega_k^L(f,x;[a,b]) := \min_{1\le m\le k} \omega_m(f,(x-a)^{1/m}(b-a)^{(m-1)/m};[a,b]) \tag{2.7}$$

and

$$\Omega_k^R(f,x;[a,b]) := \min_{1\le m\le k} \omega_m(f,(b-x)^{1/m}(b-a)^{(m-1)/m};[a,b]). \tag{2.8}$$

Note that

$$2^{1-k}\omega_k(f, b-a; [a, b]) \le \Omega_k^L(f, b; [a, b]) = \Omega_k^R(f, a; [a, b]) \le \omega_k(f, b-a; [a, b]). \tag{2.9}$$

Theorem 2.6. *For every $r \in \mathbb{N}$ there is a constant $c = c(r)$ with the following property. For each convex function $f \in C^r[a, b]$, there is a number $H > 0$, such that for every partition $X = \{x_j\}_{j=0}^n$ of $[a, b]$, satisfying*

$$x_1 - a \le H \quad \text{and} \quad b - x_{n-1} \le H \tag{2.10}$$

there is a convex piecewise polynomial $s \in S(X, r+2)$, such that

$$|f(x) - s(x)| \le c(x-a)^r\, \Omega_2^L(f^{(r)}, x; [a, x_1]), \quad x \in [a, x_1], \tag{2.11}$$

$$|f(x) - s(x)| \le c(b-x)^r\, \Omega_2^R(f^{(r)}, x; [x_{n-1}, b]), \quad x \in [x_{n-1}, b], \tag{2.12}$$

and, for each $j = 2, \dots, n-1$ and $x \in [x_{j-1}, x_j]$,

$$\begin{aligned} |f(x) - s(x)| &\le c(x_j - x_{j-1})^r \omega_2(f^{(r)}, x_j - x_{j-1}; [x_{j-1}, x_j]) \\ &\quad + c(x_1 - a)^r \omega_2(f^{(r)}, x_1 - a; [a, x_1]) \\ &\quad + c(b - x_{n-1})^r \omega_2(f^{(r)}, b - x_{n-1}; [x_{n-1}, b]). \end{aligned} \tag{2.13}$$

Theorem 2.6 is proved in Section 4 after we discuss some auxiliary results in Section 3, and we now show how it implies Theorem 2.1.

Proof of Theorem 2.1. Suppose that Theorem 2.6 is proved. Then, if we let X be the Chebyshev partition $T_n = \{t_j\}$, where $n \ge \mathcal{N} := 3/\sqrt{H}$, then (2.10) is satisfied since

$$t_1 + 1 = 1 - t_{n-1} = 2\sin^2\left(\frac{\pi}{2n}\right) \le \frac{\pi^2}{2n^2} \le \frac{5}{\mathcal{N}^2} \le H.$$

Now, as is well known and is not difficult to check, $\varphi(x)/n \sim \rho_n(x) \sim t_j - t_{j-1}$, for $x \in [t_{j-1}, t_j]$, $2 \le j \le n-1$. Hence (2.2) follows from (2.11) through (2.13), and (2.3) and (2.4) follow from (2.11) and (2.12). $\square$

3. Auxiliary results

Lemma 3.1. *Let $r \in \mathbb{N}_0$, $k \in \mathbb{N}$, $f \in C^r[a, b]$, $C_0 \ge 1$, and let $\mathcal{P} \in \mathbb{P}_{k+r-1}$ be any polynomial such that $\mathcal{P}^{(\nu)}(a) = f^{(\nu)}(a)$, $0 \le \nu \le r$, and*

$$\|f - \mathcal{P}\|_{[a,b]} \le C_0 (b-a)^r \omega_k(f^{(r)}, b-a; [a, b]). \tag{3.1}$$

Then, for all $x \in [a, b]$ and all $1 \le m \le k$, we have

$$|f(x) - \mathcal{P}(x)| \le cC_0 (x-a)^r \omega_m(f^{(r)}, (x-a)^{1/m}(b-a)^{(m-1)/m}; [a, b]), \tag{3.2}$$

where the constant c depends only on k and r.

Remark 3.2.

(i) In the case $k = 1$, such $\mathcal{P}(f) \in \mathbb{P}_r$ is unique; it is the Taylor polynomial for f at $x = a$, and (3.1) is rather obvious.

(ii) In the case $r = 0$, $\mathcal{P}(f) \in \mathbb{P}_{k-1}$ may be chosen to be any polynomial of degree $\le k-1$ interpolating f at k points in $[a, b]$ that include $x = a$ and such that the distance between any two of them is bounded below by $\lambda(b-a)$ for some constant $\lambda > 0$ (the constant C_0 will depend on λ in this case).

Note that, using the notation (2.7), estimate (3.2) can be restated in the following equivalent form:

$$|f(x) - \mathcal{P}(x)| \le C(x-a)^r \Omega_k^L(f^{(r)}, x; [a, b]), \quad C = C(k, r, C_0).$$

It is also clear that an analog of Lemma 3.1 holds if $\mathcal{P}$ interpolates f and its derivatives at $x = b$ instead of a, *i.e.*, if $P \in \mathbb{P}_{k+r-1}$ satisfies (3.1) and $\mathcal{P}^{(\nu)}(b) = f^{(\nu)}(b)$, $0 \le \nu \le r$, then

$$|f(x) - \mathcal{P}(x)| \le C(k, r, C_0)(b-x)^r \Omega_k^R(f^{(r)}, x; [a, b]), \quad x \in [a, b]. \tag{3.3}$$

Proof of Lemma 3.1. We assume that $a = 0$. It is obvious that we do not lose any generality making this assumption, but it will make some expressions shorter.

Let $x \in (0, b]$ and $1 \le m \le k$ be fixed throughout this proof. Denote $\lambda_x := x^{1/m} b^{(m-1)/m}$ and note that $x \le \lambda_x \le b$. It is also convenient to denote

$$\mathrm{w}(u) := \omega_m(f^{(r)}, u; [a, b]).$$

Now, let $\mathcal{L} \in \mathbb{P}_{m+r-1}$ be such that $\mathcal{L}^{(\nu)}(0) = f^{(\nu)}(0)$, $0 \le \nu \le r-1$, and $\mathcal{L}^{(r)} \in \mathbb{P}_{m-1}$ is any polynomial satisfying Whitney's inequality on $[0, b]$ (*i.e.*, $\left\|f^{(r)} - \mathcal{L}^{(r)}\right\|_{[0,b]} \le c\mathrm{w}(b)$) and interpolating $f^{(r)}$ at $x = 0$. For example, we can define $\mathcal{L}^{(r)}(x) := \mathcal{P}^*(x) - \mathcal{P}^*(0) + f^{(r)}(0)$, where $\mathcal{P}^* \in \mathbb{P}_{m-1}$ is the polynomial of best approximation of $f^{(r)}$ on $[0, b]$.

We first show that the following estimate holds

$$|f(x) - \mathcal{L}(x)| \le cx^r \mathrm{w}(\lambda_x). \tag{3.4}$$

To this end, with $g := f - \mathcal{L}$, since $|g^{(r)}(t)| = |g^{(r)}(t) - g^{(r)}(0)| \le \omega_1(g^{(r)}, x; [0, b])$, $0 \le t \le x$, we have, if $r \ge 1$,

$$|g(x)| \le \frac{1}{(r-1)!} \int_0^x (x-t)^{r-1} |g^{(r)}(t)|\, dt \le x^r \omega_1(g^{(r)}, x; [0, b]). \tag{3.5}$$

Clearly, the same estimate also holds for $r = 0$. We now note that $\omega_m(g^{(r)}, \cdot; [a, b]) = \mathrm{w}(\cdot)$ because $\mathcal{L}^{(r)} \in \mathbb{P}_{m-1}$, and so (3.4) is verified if $m = 1$.

By Whitney's inequality, $\left\|g^{(r)}\right\|_{[0,b]} \le c\mathrm{w}(b)$, and thus if $m \ge 2$, then using (a particular case of) the well known Marchaud inequality: if $F \in C(I)$, then

$$\omega_1(F, t; I) \le c(m)t \left(\int_t^{|I|} \frac{\omega_m(F, u; I)}{u^2}\, du + \frac{\|F\|_I}{|I|} \right),$$

where $|I|$ denotes the length of the interval I, we have from (3.5)

$$|g(x)| \le cx^{r+1} \left(\int_x^b \frac{\omega_m(g^{(r)}, u; [0, b])}{u^2}\, du + \frac{\left\|g^{(r)}\right\|_{[0,b]}}{b} \right)$$

$$\leq cx^{r+1}\left(\int_x^b \frac{\mathrm{w}(u)}{u^2}\,du + \frac{\mathrm{w}(b)}{b}\right) \leq cx^{r+1}\int_x^{2b} \frac{\mathrm{w}(u)}{u^2}\,du.$$

Now, since $u_2^{-m}\mathrm{w}(u_2) \leq 2^m u_1^{-m}\mathrm{w}(u_1)$, for $0 < u_1 < u_2$, we have

$$\begin{aligned}\int_x^{2b} \frac{\mathrm{w}(u)}{u^2}\,du &= \left(\int_x^{\lambda_x} + \int_{\lambda_x}^{2b}\right)\frac{\mathrm{w}(u)}{u^2}\,du \\ &\leq \mathrm{w}(\lambda_x)\int_x^{\infty} u^{-2}\,du + 2^m\lambda_x^{-m}\mathrm{w}(\lambda_x)\int_0^{2b} u^{m-2}\,du \\ &= \frac{\mathrm{w}(\lambda_x)}{x}\left(1 + \frac{2^{2m-1}}{m-1}\right),\end{aligned}$$

and so (3.4) is proved.

Observe now that $Q := \mathcal{P} - \mathcal{L} \in \mathbb{P}_{k+r-1}$ is such that $Q^{(\nu)}(0) = 0$, $0 \leq \nu \leq r$, and so by Markov's inequality and (3.1)

$$\begin{aligned}|Q(x)| &\leq x^{r+1}\left\|Q^{(r+1)}\right\|_{[0,b]} \leq cx^{r+1}b^{-r-1}\|Q\|_{[0,b]} \\ &\leq cx^{r+1}b^{-r-1}\left(\|f-\mathcal{P}\|_{[0,b]} + \|f-\mathcal{L}\|_{[0,b]}\right) \\ &\leq cC_0x^{r+1}b^{-1}\mathrm{w}(b) \leq cC_0x^{r+1}b^{m-1}\lambda_x^{-m}\mathrm{w}(\lambda_x) \\ &\leq cC_0x^r\mathrm{w}(\lambda_x),\end{aligned}$$

which, together with (3.4), immediately implies (3.2). □

Corollary 3.3. *Let $r \in \mathbb{N}_0$ and $f \in C^r[a,b]$, and let $L \in \mathbb{P}_{r+1}$ be the polynomial of degree $\leq r+1$ such that $L^{(\nu)}(a) = f^{(\nu)}(a)$, $0 \leq \nu \leq r$ and $L(b) = f(b)$. Then, for all $x \in [a,b]$, we have*

$$|f(x) - L(x)| \leq c(x-a)^r\Omega_2^L(f^{(r)}, x; [a,b]), \tag{3.6}$$

where the constant c depends only on r.

Remark 3.4. We note that the estimate (3.6) with $\omega_2(f^{(r)}, \sqrt{(x-a)(b-a)}; [a,b])$ instead of $\Omega_2^L(f^{(r)}, x; [a,b])$ appeared, among other places, in [7, Corollary 3.5].

It is clear that an analog of this result holds if interpolation of the derivatives of f takes place at $x = b$ instead of a, *i.e.*, if $L \in \mathbb{P}_{r+1}$ is the polynomial of degree $\leq r+1$ such that $L^{(\nu)}(b) = f^{(\nu)}(b)$, $0 \leq \nu \leq r$ and $L(a) = f(a)$, then

$$|f(x) - L(x)| \leq c(b-x)^r\Omega_2^R(f^{(r)}, x; [a,b]), \quad x \in [a,b]. \tag{3.7}$$

Proof of Corollary 3.3. As in the proof of Lemma 3.1, it is clear that we do not lose generality by assuming $a = 0$. Now, if $g := f - L$, then $g(b) = 0$ and $g^{(\nu)}(0) = 0$, $0 \leq \nu \leq r$, and by Lemma 3.1, it is sufficient to prove that $\|g\|_{[0,b]} \leq cb^r\omega_2(g^{(r)}, b; [0,b])$.

Since $g(x) = \frac{1}{(r-1)!}x^r\int_0^1(1-t)^{r-1}g^{(r)}(xt)dt$, equality $g(b) = 0$ implies that $\int_0^1(1-t)^{r-1}g^{(r)}(bt)dt = 0$, and so

$$\|g\|_{[0,b]} \le cb^r \sup_{0<t\le 1,\ 0\le x\le b} |g^{(r)}(xt) - xg^{(r)}(bt)/b|$$
$$\le cb^r \sup_{0<t\le 1,\ 0\le y\le bt} |g^{(r)}(y) - yg^{(r)}(bt)/(bt)|$$
$$\le cb^r \sup_{0<t\le 1} \omega_2(g^{(r)}, bt; [0, bt]) \le cb^r \omega_2(g^{(r)}, b; [0, b]),$$

as needed. Here, the second last estimate follows from Whitney's inequality using the observation that $l(y) = yg^{(r)}(bt)/(bt)$ is the linear polynomial interpolating $g^{(r)}$ at 0 and bt, and so $\|g^{(r)} - l\|_{[0,bt]} \le c\omega_2(g^{(r)}, bt; [0, bt])$. □

The following lemma was proved in [9].

Lemma 3.5 *([9, Corollary 2.4]). Let $k \in \mathbb{N}$ and let $f \in C^2[a, b]$ be convex. Then there exists a convex polynomial $P \in \mathbb{P}_{k+1}$, satisfying $P(a) = f(a)$ and $P(b) = f(b)$, and either $P'(a) = f'(a)$ and $P'(b) \le f'(b)$, or $P'(a) \ge f'(a)$ and $P'(b) = f'(b)$, such that*

$$\|f - P\|_{[a,b]} \le c(k)(b - a)^2 \omega_k(f'', b - a; [a, b]).$$

We also need the following analog of Lemma 3.5 for $r = 1$.

Lemma 3.6. *Let $f \in C^1[a, b]$ be convex. Then there exists a convex polynomial $P \in \mathbb{P}_2$ (that is a convex parabola), satisfying $P(a) = f(a)$ and $P(b) = f(b)$, and either $P'(a) = f'(a)$ and $P'(b) \le f'(b)$, or $P'(a) \ge f'(a)$ and $P'(b) = f'(b)$, such that*

$$\|f - P\|_{[a,b]} \le c(b - a)\omega_2(f', b - a; [a, b]).$$

Proof. It is clear that it is sufficient to prove this lemma for $[a, b] = [0, 1]$, since we can then apply a linear transformation to a general interval.

Additionally, by subtracting a linear polynomial interpolating f at the endpoints from f we can assume, without loss of generality, that $f(0) = f(1) = 0$. We now define

$$P(x) := \begin{cases} (x - x^2)f'(0), & \text{if } f'(0) + f'(1) \ge 0, \\ (x^2 - x)f'(1), & \text{otherwise.} \end{cases}$$

Clearly, P is convex and satisfies the stated interpolation conditions. In fact, it is a Lagrange–Hermite polynomial interpolating f at 0 and 1, and f' either at 0 or at 1. Hence, we can use, for example, Corollary 3.3 with $r = 1$ or its analog (see (3.7)) to conclude that the needed estimate also holds. Alternatively, we can follow the proof of [9, Lemma 2.3] to verify this estimate. □

An immediate consequence of Lemmas 3.5 and 3.6 is the following result.

Lemma 3.7. *If $r \in \mathbb{N}$ and $f \in C^r[a, b]$ is convex on $[a, b]$, then for each partition $X = \{x_j\}_{j=0}^n$ of $[a, b]$ there is a convex piecewise polynomial $\sigma \in S(X, r + 2)$, satisfying, for each $j = 1, \dots, n$,*

$$\|f - \sigma\|_{[x_{j-1},x_j]} \le c(r)(x_j - x_{j-1})^r \omega_2(f^{(r)}, x_j - x_{j-1}; [x_{j-1}, x_j]), \tag{3.8}$$
$$\sigma'(x_{j-1}+) \ge f'(x_{j-1}), \qquad \sigma'(x_j-) \le f'(x_j), \tag{3.9}$$

and

$$\sigma(x_j) = f(x_j). \tag{3.10}$$

We will now discuss construction of polynomial pieces near the endpoints of $[a, b]$.

For $f \in C^r[a, b]$ and $0 < h \le b - a$, denote by $L_{r,h}(f, x)$ the Lagrange–Hermite polynomial of degree $\le r + 1$ such that

$$L^{(\nu)}_{r,h}(f, a) = f^{(\nu)}(a), \quad 0 \le \nu \le r, \quad \text{and} \quad L_{r,h}(f, a + h) = f(a + h).$$

We also denote

$$\mathbb{L}_{r,h}(f, x) := \int_a^x L_{r-1,h}(f', t)dt + f(a).$$

Lemma 3.8. *Let $r \in \mathbb{N}$ and $h > 0$. If $f \in C^r[a, a + h]$, then*

$$|f(x) - \mathbb{L}_{r,h}(f, x)| \le c(r)(x - a)^r \Omega^L_2(f^{(r)}, x; [a, a + h]), \quad x \in [a, a + h].$$

Proof. It follows from Corollary 3.3 that, for $r \in \mathbb{N}$, $h > 0$ and $g \in C^{r-1}[a, a + h]$, we have

$$|g(x) - L_{r-1,h}(g, x)| \le c(r)(x - a)^{r-1} \Omega^L_2(g^{(r-1)}, x; [a, a + h]), \quad x \in [a, a + h].$$

For any $x \in [a, a + h]$, we have

$$\begin{aligned} |f(x) - \mathbb{L}_{r,h}(f, x)| &= \left| \int_a^x (f'(t) - L_{r-1,h}(f', t)dt \right| \\ &\le c \int_a^x (t - a)^{r-1} \Omega^L_2(f^{(r)}, t; [a, a + h])dt \\ &\le c(x - a)^r \Omega^L_2(f^{(r)}, x; [a, a + h]), \end{aligned}$$

and the proof is complete. □

It was shown in [7, Lemma 3.1] that, for a nondecreasing $g \in C^r[a, b]$, $r \in \mathbb{N}$, the polynomial $L_{r,h}(g, \cdot)$ is also nondecreasing on $[a, a + h]$ provided that $h < b - a$ is sufficiently small depending on f. Note that this statement also is valid (and is trivial) if $r = 0$.

Lemma 3.9 *([7, Lemma 3.1]). Let $r \in \mathbb{N}_0$ and let $g \in C^r[a, b]$ be nondecreasing on $[a, b]$. Then there is a number $H > 0$, such that for all $h \in (0, H)$ the polynomials $L_{r,h}(g, \cdot)$ are nondecreasing on $[a, a + h]$.*

A trivial consequence of Lemma 3.9 is the following result.

Corollary 3.10. *Let $f \in C^r[a, b]$, be convex on $[a, b]$. Then there is a number $H > 0$, such that for all $h \in (0, H)$ the polynomials $\mathbb{L}_{r,h}(f, \cdot)$ are convex on $[a, a + h]$.*

By considering $\widetilde{f}(x) := f(a + b - x)$ instead of f we also get analogous statements and interpolatory estimate near the endpoint b instead of a.

Thus, denoting $\widetilde{\mathbb{L}}_{r,h}(f, x) := \mathbb{L}_{r,h}(g, a + b - x)$, where $g(x) := f(a + b - x)$, we can summarize the above results as follows.

Lemma 3.11. *Let $r \in \mathbb{N}$, and let $f \in C^r[a,b]$ be convex on $[a,b]$. Then there is a number $H > 0$, such that for all $h, \widetilde{h} \in (0, H)$, polynomials $\mathbb{L}_{r,h}(f,\cdot)$ and $\widetilde{\mathbb{L}}_{r,\widetilde{h}}(f,\cdot)$ of degree $\le r+1$ have the following properties:*

(i) $\mathbb{L}_{r,h}(f,\cdot)$ *is convex on* $[a, a+h]$ *and* $\widetilde{\mathbb{L}}_{r,\widetilde{h}}(f,\cdot)$ *is convex on* $[b-\widetilde{h}, b]$,
(ii) $|f(x) - \mathbb{L}_{r,h}(f,x)| \le c(r)(x-a)^r \Omega_2^L(f^{(r)}, x; [a, a+h])$, $x \in [a, a+h]$,
(iii) $|f(x) - \widetilde{\mathbb{L}}_{r,\widetilde{h}}(f,x)| \le c(r)(b-x)^r \Omega_2^R(f^{(r)}, x; [b-\widetilde{h}, b])$, $x \in [b-\widetilde{h}, b]$,
(iv) $\mathbb{L}'_{r,h}(f, a+h) = f'(a+h)$ *and* $\widetilde{\mathbb{L}}'_{r,\widetilde{h}}(f, b-\widetilde{h}) = f'(b-\widetilde{h})$.

4. Proof of Theorem 2.6

It suffices to prove this theorem for $[a,b] = [0,1]$, since we can then get the general result by applying a linear transformation. Additionally, by subtracting a linear polynomial interpolating f at the endpoints we can assume that $f(0) = f(1) = 0$. It is also clear that we can assume that f is not a constant function, and so, because of its convexity, $f(x) < 0$, for all $x \in (0,1)$. Now, denote $M := \|f\|_{[0,1]} > 0$, and let $x_* \in (0,1)$ be such that $f(x_*) = \min_{x \in [0,1]} f(x) = -M$.

Suppose now that a positive number $H_1 < \min\{x_*, 1-x_*\}$ is so small that

$$\max\{-f(H_1), -f(1-H_1)\} < M/2$$

and

$$4c_0 H_1^r \omega_2(f^{(r)}, H_1; [0,1]) < M, \tag{4.1}$$

where c_0 is the maximum of constants $c = c(r)$ from inequalities (ii) and (iii) in Lemma 3.11. Now, let H be the number from Lemma 3.11, and without loss of generality, we assume that $H \le H_1$.

Suppose that a partition $X = \{x_j\}_{j=0}^n$ of $[0,1]$ satisfies (2.10), and set $h := x_1$ and $\widetilde{h} := 1 - x_{n-1}$.

We are now ready to construct the piecewise polynomial $s \in S(X, r+2)$ that yields Theorem 2.6. First, let

$$s(x) := \begin{cases} \mathbb{L}_{r,h}(f,x), & \text{if} \quad x \in [0, x_1), \\ \widetilde{\mathbb{L}}_{r,\widetilde{h}}(f,x), & \text{if} \quad x \in (x_{n-1}, 1], \end{cases}$$

where $\mathbb{L}_{r,h}$ and $\widetilde{\mathbb{L}}_{r,\widetilde{h}}$ are the polynomials from Lemma 3.11, and note that estimates (ii) and (iii) of Lemma 3.11 immediately imply (2.11) and (2.12).

Now, suppose that $\sigma \in S(X, r+2)$ is the piecewise polynomial from Lemma 3.7. Note that we cannot simply define s to be σ on $[x_1, x_{n-1}]$ because s would then be possibly discontinuous at x_1 and x_{n-1}, because polynomials $\mathbb{L}_{r,h}$ and $\widetilde{\mathbb{L}}_{r,\widetilde{h}}$ do not necessarily interpolate f at x_1 and x_{n-1}, respectively.

We are now going to show how to overcome this difficulty.

Set

$$\delta := \mathbb{L}_{r,h}(f, x_1) - f(x_1), \quad \widetilde{\delta} := \widetilde{\mathbb{L}}_{r,\widetilde{h}}(f, x_{n-1}) - f(x_{n-1}), \quad \text{and} \quad \widehat{\delta} := \delta - \widetilde{\delta},$$

and note that by virtue of (4.1) and (2.9) estimates (ii) and (iii) in Lemma 3.11 imply that

$$|\delta| < M/4 \quad \text{and} \quad |\widetilde{\delta}| < M/4,$$

so that $|\widehat{\delta}| < M/2$.

Denote by l the tangent line to f at x_1, and by $\widetilde{l}$ the tangent line to f at x_{n-1}. Then we have

$$\begin{aligned} f(x_{n-1}) - l(x_{n-1}) &= f(x_{n-1}) - f(x_*) + f(x_*) - l(x_*) + l(x_*) - l(x_{n-1}) \\ &> f(x_{n-1}) - f(x_*) \ge f(1 - H_1) - f(x_*) > M/2, \end{aligned} \tag{4.2}$$

and similarly

$$\begin{aligned} f(x_1) - \widetilde{l}(x_1) &= f(x_1) - f(x_*) + f(x_*) - \widetilde{l}(x_*) + \widetilde{l}(x_*) - \widetilde{l}(x_1) \\ &> f(x_1) - f(x_*) \ge f(H_1) - f(x_*) > M/2. \end{aligned} \tag{4.3}$$

To define s on $[x_1, x_{n-1}]$ we consider two cases: $\widehat{\delta} \ge 0$ and $\widehat{\delta} < 0$.

Case 1: $\widehat{\delta} \ge 0$.

In this case, we define

$$s(x) := \lambda(\sigma(x) - l(x)) + l(x) + \delta, \quad x \in [x_1, x_{n-1}],$$

where

$$\lambda := 1 - \frac{\widehat{\delta}}{f(x_{n-1}) - l(x_{n-1})}.$$

It is straightforward to check that

$$s(x_1) = \mathbb{L}_{r,h}(f, x_1) \quad \text{and} \quad s(x_{n-1}) = \widetilde{\mathbb{L}}_{r,\widetilde{h}}(f, x_{n-1}),$$

and so s is continuous on $[0, 1]$.

Now, since by (4.2), $0 < \lambda \le 1$, it follows from (iv) of Lemma 3.11 that

$$s'(x_1+) = \lambda(\sigma'(x_1+) - f'(x_1)) + f'(x_1) \ge f'(x_1) = s'(x_1-)$$

and, since f' is nondecreasing,

$$s'(x_{n-1}-) = \lambda\sigma'(x_{n-1}-) + (1 - \lambda)f'(x_1) \le f'(x_{n-1}) = s'(x_{n-1}+),$$

and so s is a convex function on $[0, 1]$.

Since $\sigma - l$ is a nondecreasing nonnegative function on $[x_1, x_{n-1}]$, we also have, for $x \in [x_1, x_{n-1}]$,

$$\begin{aligned} \sigma(x) - s(x) &= \sigma(x) - \lambda(\sigma(x) - l(x)) - l(x) - \delta \\ &= (1 - \lambda)(\sigma(x) - l(x)) - \delta \ge -\delta \end{aligned}$$

and

$$\begin{aligned} \sigma(x) - s(x) &= (1 - \lambda)(\sigma(x) - l(x)) - \delta \\ &\le (1 - \lambda)(\sigma(x_{n-1}) - l(x_{n-1})) - \delta \\ &= \frac{\widehat{\delta}}{f(x_{n-1}) - l(x_{n-1})}(\sigma(x_{n-1}) - l(x_{n-1})) - \delta \\ &= -\widetilde{\delta}. \end{aligned}$$

Hence,

$$|f(x) - s(x)| \le |f(x) - \sigma(x)| + |\delta| + |\widetilde{\delta}|, \quad x \in [x_1, x_{n-1}],$$

and, together with estimates (ii) and (iii) of Lemma 3.11 and (3.8), this proves (2.13).

Case 2: $\widehat{\delta} < 0$.

In this case, we define

$$s(x) := \widetilde{\lambda}(\sigma(x) - \widetilde{l}(x)) + \widetilde{l}(x) + \widetilde{\delta}, \quad x \in [x_1, x_{n-1}],$$

where

$$\widetilde{\lambda} := 1 + \frac{\widehat{\delta}}{f(x_1) - \widetilde{l}(x_1)},$$

and we proceed as above using (4.3) instead of (4.2). This completes the proof. $\square$

References

[1] H.H. Gonska, D. Leviatan, I.A. Shevchuk, H.-J. Wenz, Interpolatory pointwise estimates for polynomial approximations, Constr. Approx. 16 (2000) 603–629.

[2] K.A. Kopotun, Pointwise and uniform estimates for convex approximation of functions by algebraic polynomials, Constr. Approx. 10 (2) (1994) 153–178.

[3] K.A. Kopotun, Simultaneous approximation by algebraic polynomials, Constr. Approx. 12 (1) (1996) 67–94.

[4] K.A. Kopotun, D. Leviatan, A. Prymak, I.A. Shevchuk, Uniform and pointwise shape preserving approximation by algebraic polynomials, Surv. Approx. Theory 6 (2011) 24–74.

[5] K.A. Kopotun, D. Leviatan, I.A. Shevchuk, Interpolatory pointwise estimates for monotone polynomial approximation, J. Math. Anal. Appl. 459 (2) (2018) 1260–1295.

[6] D. Leviatan, Pointwise estimates for convex polynomial approximation, Proc. Amer. Math. Soc. 98 (1986) 471–474.

[7] D. Leviatan, I.L. Petrova, Interpolatory estimates in monotone piecewise polynomial approximation, J. Approx. Theory 223 (2017) 1–8.

[8] D. Leviatan, I.L. Petrova, Corrigendum, J. Approx. Theory 228 (2018) 79–80.

[9] D. Leviatan, I.A. Shevchuk, Coconvex polynomial approximation, J. Approx. Theory 121 (2003) 100–118.

[10] T.O. Petrova, A counterexample to convex interpolation approximation, in: Theory of the Approximation of Functions and Related Problems, in: Pr. Inst. Mat. Nats. Akad. Nauk Ukr. Mat. Zastos., vol. 35, Natsional. Akad. Nauk Ukraïny, Inst. Mat., Kyiv, 2002, pp. 107–112 (in Ukrainian, with Ukrainian summary).