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ЗМІСТ

All articles in this issue are dedicated to the memory of Victor N. Konovalov (04.04.1946–01.11.2008)

* Всі статті в цьому номері присвячено пам'яті Віктора Миколайовича Коновалова (04.04.1946 – 01.11.2008).

Dedicated to the memory of our friend and colleague
Victor N. Konovalov 4.4.1946–1.11.2008

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ARE THE DEGREES OF THE BEST (CO)CONVEX AND UNCONSTRAINED POLYNOMIAL APPROXIMATIONS THE SAME? II

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In Part I of the paper, we have proved that, for every $\alpha > 0$ and a continuous function f, which is either convex $(s = 0)$ or changes convexity at a finite collection $Y_s = \{y_i\}_{i=1}^s$ of points $y_i \in (-1, 1)$,

$$\sup\{n^\alpha E_n^{(2)}(f, Y_s) \colon n \geq \mathcal{N}^*\} \leq c(\alpha, s) \sup\{n^\alpha E_n(f) \colon n \geq 1\},$$

where $E_n(f)$ and $E_n^{(2)}(f, Y_s)$ denote, respectively, the degrees of the best unconstrained and (co)convex approximations and $c(\alpha, s)$ is a constant depending only on α and s. Moreover, it has been shown that $\mathcal{N}^*$ may be chosen to be 1 for $s = 0$ or $s = 1$, $\alpha \neq 4$, and that it must depend on Y_s and α for $s = 1$, $\alpha = 4$ or $s \geq 2$.

In Part II of the paper, we show that a more general inequality

$$\sup\{n^\alpha E_n^{(2)}(f, Y_s) \colon n \geq \mathcal{N}^*\} \leq c(\alpha, \mathcal{N}, s) \sup\{n^\alpha E_n(f) \colon n \geq \mathcal{N}\},$$

is valid, where, depending on the triple $(\alpha, \mathcal{N}, s)$, the number $\mathcal{N}^*$ may depend on α, $\mathcal{N}$, Y_s, and f or be independent of these parameters.

1. Introduction and Main Results

Let $\mathbb{C}[-1, 1]$ be the space of continuous functions on $[-1, 1]$ equipped with the uniform norm $\|\cdot\|$ and let $\mathbb{Y}_s$, $s \in \mathbb{N}$, be the set of all collections $Y_s := \{y_i\}_{i=1}^s$ of points y_i, such that $y_{s+1} := -1 < y_s < \ldots < y_1 < 1 =: y_0$. For $Y_s \in \mathbb{Y}_s$ by $\Delta^2(Y_s)$ we denote the set of all piecewise convex functions $f \in \mathbb{C}[-1, 1]$ that change convexity at the points Y_s, and are convex on $[y_1, 1]$. In particular, $\mathbb{Y}_0 = \{\varnothing\}$ and $\Delta^2 = \Delta^2(Y_0)$ denotes the set of all convex continuous functions. If f is twice continuously differentiable in $(-1, 1)$, then $f \in \Delta^2(Y_s)$ if and only if

$$f''(x)\Pi(x; Y_s) \geq 0, x \in (-1, 1), \qquad \text{where} \qquad \Pi(x; Y_s) := \prod_{i=1}^{s}(x - y_i) \qquad (\Pi(x, Y_0) :\equiv 1).$$

Further, by

$$E_n(f) := \inf\left\{\|f - P_n\| \colon\ P_n \in \mathbb{P}_n\right\}$$

and

$$E_n^{(2)}(f, Y_s) := \inf\left\{\|f - P_n\| \colon\ P_n \in \mathbb{P}_n \cap \Delta^2(Y_s)\right\}$$

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we denote the degrees of the best unconstrained and coconvex approximations of a function f by polynomials from $\mathbb{P}_n$ (the space of algebraic polynomials of degree $< n$). In particular,

$$E_n^{(2)}(f) := E_n^{(2)}(f, Y_0) = \inf\left\{\|f - P_n\|\colon\ P_n \in \mathbb{P}_n \cap \Delta^2\right\}$$

is the degree of the best convex approximation of f.

Although it is obvious that $E_n(f) \le E_n^{(2)}(f)$, Lorentz and Zeller [1] showed that the inverse inequality $E_n^{(2)}(f) \le cE_n(f)$ is invalid even if a constant c is allowed to depend on the function $f \in \Delta^2$. There are many examples showing that the same is true for piecewise convex functions from $\Delta^2(Y_s)$. Despite the existence of counterexamples, we have recently proved the following result:

Theorem A [2]. *For each $\alpha > 0$ and integer $s \ge 0$, there is a constant $c(\alpha, s)$, such that, for every collection $Y_s \in \mathbb{Y}_s$ and a function $f \in \Delta^2(Y_s)$,*

$$\sup\left\{n^\alpha E_n^{(2)}(f, Y_s)\colon n \ge \mathcal{N}^*\right\} \le c(\alpha, s) \sup\left\{n^\alpha E_n(f)\colon n \ge 1\right\}, \tag{1.1}$$

where $\mathcal{N}^* = 1$ if either $s = 0$ or $s = 1$ and $\alpha \ne 4$ and $\mathcal{N}^* = \mathcal{N}^*(\alpha, Y_s)$ is a constant depending only on α and Y_s if either $s \ge 2$ or $s = 1$ and $\alpha = 4$.

It has also been shown that Theorem A cannot be improved, i.e., if either $s \ge 2$ or $s = 1$ and $\alpha = 4$, then the constant $\mathcal{N}^*$ cannot be made independent of Y_s.

Theorem B [2]. *Let $s \ge 2$. Then, for any $\alpha > 0$ and $m \in \mathbb{N}$, there exist a collection $Y_s \in \mathbb{Y}_s$ and a function $f \in \Delta^2(Y_s)$ such that*

$$m^\alpha E_m^{(2)}(f, Y_s) \ge c(\alpha, s) m^{\alpha+1-\lceil\alpha\rceil} \sup\left\{n^\alpha E_n(f)\colon n \ge 1\right\}, \tag{1.2}$$

where $c(\alpha, s)$ is a positive constant and $\lceil\alpha\rceil$ is the ceiling function (i.e., the smallest integer not smaller than α).

Theorem C [2]. *For every $Y_1 \in \mathbb{Y}_1$, there exists a function $f \in \Delta^2(Y_1)$ satisfying the equality*

$$\sup\left\{n^4 E_n(f)\colon n \in \mathbb{N}\right\} = 1,$$

such that, for any $m \in \mathbb{N}$, we have

$$m^4 E_m^{(2)}(f, Y_1) \ge \left(c \ln \frac{m}{1 + m^2\varphi(y_1)} - 1\right) \tag{1.3}$$

and

$$\sup\left\{n^4 E_n^{(2)}(f, Y_1)\colon n \in \mathbb{N}\right\} \ge c\,|\ln \varphi(y_1)|, \tag{1.4}$$

where $\varphi(y) := \sqrt{1 - y^2}$ and c is an absolute positive constant.

Everywhere in what follows, by $c(\dots)$ we denote positive real constants that depend only on the parameters, sets, and functions in the parentheses. Generally speaking, these constants are different in different cases even if they appear in the same line. In particular, absolute positive constants are also denoted by c. Similarly, $\mathcal{N}(\dots)$

denote natural numbers that depend only on the quantities in the parentheses. Thus, $\mathcal{N}(\alpha, Y_s)$ is a natural number that depends only on α and Y_s but is independent of any other parameters.

The main goal in the present paper is to answer the following questions:

> What happens if we replace $n \geq 1$ in (1.1) by $n \geq \mathcal{N}$, where $\mathcal{N} \in \mathbb{N}$? Is Theorem A still valid? What can we say about the dependence of $\mathcal{N}^*$ on α, $\mathcal{N}$, Y_s, and f?

Our first result is the following generalization of Theorem A:

Theorem 1.1. *For each $\alpha > 0$, $\mathcal{N} \in \mathbb{N}$, $s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $Y_s \in \mathbb{Y}_s$, and $f \in \Delta^2(Y_s)$, there exists $\mathcal{N}^* \in \mathbb{N}$ such that*

$$\sup\left\{n^\alpha E_n^{(2)}(f, Y_s) : n \geq \mathcal{N}^*\right\} \leq c(\alpha, \mathcal{N}, s) \sup\left\{n^\alpha E_n(f) : n \geq \mathcal{N}\right\}. \tag{1.5}$$

Note that $\mathcal{N}^* \in \mathbb{N}$ in the statement of Theorem 1.1 may depend on α, $\mathcal{N}$, Y_s, and f or may be independent of these parameters. Theorem 1.2 proved in what follows gives a complete answer to the question when and how this dependence occurs.

It is easy to see that the assertion of Theorem 1.1 in the case $\mathcal{N} = 2$ immediately follows from Theorem A. Namely,

> if $\mathcal{N} = 2$, then Theorem 1.1 is true with $\mathcal{N}^* = 2$ if either $s = 0$ or $s = 1$ and $\alpha \neq 4$ and $\mathcal{N}^* = \mathcal{N}^*(\alpha, Y_s)$ if either $s \geq 2$ or $s = 1$ and $\alpha = 4$.

Indeed, in view of the fact that the function $g := f - p_2$, where $p_2 := \arg\inf_{p \in \mathbb{P}_2} \|f - p\|$, is such that $E_n(g) = E_n(f)$, $E_n^{(2)}(g, Y_s) = E_n^{(2)}(f, Y_s)$ for all $n \geq 2$, and $E_1(g) \leq \|g\| = E_2(f)$, we conclude that

$$\begin{aligned} \sup\left\{n^\alpha E_n^{(2)}(f, Y_s) : n \geq \mathcal{N}^*\right\} &= \sup\left\{n^\alpha E_n^{(2)}(g, Y_s) : n \geq \mathcal{N}^*\right\} \\ &\leq c(\alpha, s) \sup\left\{n^\alpha E_n(g) : n \geq 1\right\} = c(\alpha, s) \sup\left\{n^\alpha E_n(f) : n \geq 2\right\}. \end{aligned}$$

Moreover, Theorems B and C imply that,

> for $\mathcal{N}^* = 2$, the number $\mathcal{N}^*$ cannot be made independent of Y_s if either $s \geq 2$ or $s = 1$ and $\alpha = 4$.

We now emphasize that, except the case $3 \leq \mathcal{N} \leq s + 2$, the number $\mathcal{N}^*$ cannot be smaller than $\mathcal{N}$. Indeed, to show this, it suffices to consider any function $f_s \in \Delta^2(Y_s)$ in the form of a polynomial of degree exactly $\mathcal{N} - 1$, e.g., such that $f_s''(x) := (x + 2)^{\mathcal{N} - s - 3} \Pi(x; Y_s)$ for $\mathcal{N} \geq s + 3$ and $f_s(x) := x$ for $\mathcal{N} = 2$. Then $E_n(f_s) = 0$ for all $n \geq \mathcal{N}$ and we immediately arrive at a contradiction by assuming that $\mathcal{N}^*$ in (1.5) is strictly smaller than $\mathcal{N}$. If $3 \leq \mathcal{N} \leq s + 2$, then $\mathbb{P}_{\mathcal{N}} \cap \Delta^2(Y_s) = \mathbb{P}_2 \cap \Delta^2(Y_s)$ (any polynomial of degree $\leq s + 1$ with s changes of convexity must be linear) and, hence,

$$E_{\mathcal{N}}^{(2)}(f, Y_s) = E_2^{(2)}(f, Y_s) = E_2(f),$$

i.e., if (1.5) is true with $\mathcal{N}^* = \mathcal{N}$, then it is also true with $\mathcal{N}^* = 2$.

In addition, by Theorem B, we cannot expect that $\mathcal{N}^*$ is independent of Y_s for $s \geq 2$, Given a triple $(\alpha, \mathcal{N}, s)$, we want to determine the exact dependences of $\mathcal{N}^*$ on all quantities appearing in the statement of Theorem 1.1 such that inequality (1.5) is satisfied.

We now show that there are three different types of behavior of $\mathcal{N}^*$. In order to describe these types of behavior, we introduce the following notation:

Definition. *Let* $(\alpha, \mathcal{N}, s) \in \mathbb{R}_+ \times \mathbb{N} \times \mathbb{N}_0$.

1. *We write* $(\alpha, \mathcal{N}, s) \in$ *"+", if Theorem* 1.1 *holds with* $\mathcal{N}^* = \mathcal{N}$.
2. *We write* $(\alpha, \mathcal{N}, s) \in$ *"*$\oplus$*", if*
   (a) *Theorem 1.1 holds with* $\mathcal{N}^* = \mathcal{N}^*(\alpha, \mathcal{N}, Y_s)$ *and*
   (b) *Theorem 1.1 is not valid with* $\mathcal{N}^*$ *independent of* Y_s*, i.e., for each* $A > 0$ *and* $M \in \mathbb{N}$*, one can find a number* $m > M$*, a collection* $Y_s \in \mathbb{Y}_s$*, and a function* $f \in \Delta^2(Y_s)$*, such that*

$$m^\alpha E_m^{(2)}(f, Y_s) \geq A \sup\left\{n^\alpha E_n(f) \colon n \geq \mathcal{N}\right\}. \tag{1.6}$$

3. *We write* $(\alpha, \mathcal{N}, s) \in$ *"*$\ominus$*", if*
   (a) *Theorem 1.1 holds with* $\mathcal{N}^* = \mathcal{N}^*(\alpha, \mathcal{N}, Y_s, f)$ *and*
   (b) *Theorem 1.1 is not valid with* $\mathcal{N}^*$ *independent of* f*, i.e., for each* $A > 0$*,* $M \in \mathbb{N}$*, and* $Y_s \in \mathbb{Y}_s$*, one can find* $m > M$ *and* $f \in \Delta^2(Y_s)$ *such that inequality (1.6) holds.*

It turns out that $\mathcal{N}^*$ depends on

$$\overline{\alpha} := \lceil \alpha/2 \rceil \tag{1.7}$$

but not on α itself with the only exception of the case $\overline{\alpha} = 2$, $\mathcal{N} \leq 2$, and $s = 1$, which has already been discussed above.

Theorem 1.2. *Let* $(\alpha, \mathcal{N}, s) \in \mathbb{R}_+ \times \mathbb{N} \times \mathbb{N}_0$*. Then*

(i) $(\alpha, \mathcal{N}, s) \in$ "+"] *if*

$s = 0$, $\overline{\alpha} \leq 2$, *and* $\mathcal{N} \leq 3$;
$s = 0$, $\overline{\alpha} \geq 3$, *and* $\mathcal{N} \in \mathbb{N}$;
$s = 1$, $\overline{\alpha} = 1$, *and* $\mathcal{N} \leq 2$;
$s = 1$, $\overline{\alpha} = 2$, $\alpha \neq 4$, *and* $\mathcal{N} \leq 2$;
$s = 1$, $\overline{\alpha} = 3$, *and* $\mathcal{N} \leq 4$;
$s = 1$, $\overline{\alpha} \geq 4$, *and* $\mathcal{N} \in \mathbb{N}$.

(ii) $(\alpha, \mathcal{N}, s) \in$ "$\ominus$"] *if*

$s \geq 0$, $\overline{\alpha} \leq 2$, *and* $\mathcal{N} \geq s + 4$;
$s \geq 1$, $\overline{\alpha} = 1$, *and* $\mathcal{N} = s + 3$.

(iii) $(\alpha, \mathcal{N}, s) \in$ "$\oplus$" *in all other cases, except possibly the case* $s \geq 3$, $\overline{\alpha} = 2$, *and* $\mathcal{N} = s + 3$.

Recall that the cases $\mathcal{N} = 1$ and $\mathcal{N} = 2$ in this theorem follow from Theorems A – C and the discussion following the assertion of Theorem 1.1.

In order to make it easier to understand and remember the assertions of Theorem 1.2 and recognize the patterns of behavior of the triples $(\alpha, \mathcal{N}, s)$, we summarize the results in the following tables relating $\mathcal{N}$ and $\overline{\alpha}$ to various values of s.

$\overline{\alpha}$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
4	$+$	$+$	$+$	$+$	$+$	$\cdots$
3	$+$	$+$	$+$	$+$	$+$	$\cdots$
2	$+$	$+$	$+$	$\ominus$	$\ominus$	$\cdots$
1	$+$	$+$	$+$	$\ominus$	$\ominus$	$\cdots$
	1	2	3	4	5	$\mathcal{N}$

$s = 0$

$\overline{\alpha}$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
5	$+$	$+$	$+$	$+$	$+$	$+$	$\cdots$
4	$+$	$+$	$+$	$+$	$+$	$+$	$\cdots$
3	$+$	$+$	$+$	$+$	$\oplus$	$\oplus$	$\cdots$
2	$\overset{\circ}{+}$	$\overset{\circ}{+}$	$\oplus$	$\oplus$	$\ominus$	$\ominus$	$\cdots$
1	$+$	$+$	$\oplus$	$\ominus$	$\ominus$	$\ominus$	$\cdots$
	1	2	3	4	5	6	$\mathcal{N}$

$s = 1$

$\overline{\alpha}$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
4	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
3	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
2	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\ominus$	$\ominus$	$\cdots$
1	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\ominus$	$\ominus$	$\ominus$	$\cdots$
	1	2	3	4	5	6	7	$\mathcal{N}$

$s = 2$

$\overline{\alpha}$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
4	$\oplus$	$\oplus$	$\cdots$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
3	$\oplus$	$\oplus$	$\cdots$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
2	$\oplus$	$\oplus$	$\cdots$	$\oplus$	$\oplus$	?	$\ominus$	$\ominus$	$\cdots$
1	$\oplus$	$\oplus$	$\cdots$	$\oplus$	$\oplus$	$\ominus$	$\ominus$	$\ominus$	$\cdots$
	1	2	$\cdots$	$s+1$	$s+2$	$s+3$	$s+4$	$s+5$	$\mathcal{N}$

$s \geq 3$

The symbol "$\overset{\circ}{+}$" in the positions $(\overline{\alpha}, \mathcal{N}) = (2,1)$ and $(2,2)$ for $s = 1$ (exceptional case) means that $(\alpha, \mathcal{N}, s) \in$ "$+$" for $\alpha \neq 4$ (i.e., $2 < \alpha < 4$) and $(\alpha, \mathcal{N}, s) \in$ "$\oplus$" for $\alpha = 4$.

We also write "?" in the position $(\overline{\alpha}, \mathcal{N}) = (2, s+3)$ for $s \geq 3$ because we do not know exactly what happens in this case. We know, however, that $(\alpha, \mathcal{N}, s) \in$ "$\ominus$" or "$\oplus$" for $s \geq 3$, $2 < \alpha \leq 4$ and $\mathcal{N} = s+3$ (see Theorem B and case 11 in Section 4.2).

2. Proofs of the Negative Results

We first formulate the following well-known result (see, e.g., [3, p. 418], Theorem 7.5.2):

Lemma 2.1. *Let $r \in \mathbb{N}$ and $G_r(x) = (x+1)^r \ln(x+1)$, $G_r(-1) := 0$. Then*

$$E_n(G_r) \leq c(r) n^{-2r}, \quad n \in \mathbb{N}. \tag{2.1}$$

Further, we prove the following lemma:

Lemma 2.2. *For every $A > 0$ and $m \in \mathbb{N}$, there are points $y_1 \in (-1,1)$ and $\tilde{y}_1 \in (-1,1)$ and functions $f \in \Delta^2(Y_1)$ and $\tilde{f} \in \Delta^2(\tilde{Y}_1)$, where $Y_1 := \{y_1\}$ and $\tilde{Y}_1 := \{\tilde{y}_1\}$, such that*

$$n^4 E_n(f) \leq 1, \quad n \geq 3, \quad \text{and} \quad n^6 E_n(\tilde{f}) \leq 1, \quad n \geq 5. \tag{2.2}$$

At the same time,

$$E_m^{(2)}(f, Y_1) \geq A \quad \text{and} \quad E_m^{(2)}(\tilde{f}, \tilde{Y}_1) \geq A.$$

Proof. Given $A > 0$ and $m \in \mathbb{N}$ in the proof of [4] (Theorem 2.4), we construct functions $g_4 \in \Delta^2(Y_1)$ and $g_6 \in \Delta^2(\tilde{Y}_1)$ for some $-1 < y_1 < 1$ and $-1 < \tilde{y}_1 < 1$ such that

$$E_m^{(2)}(g_4, Y_1) \geq A \quad \text{and} \quad E_m^{(2)}(g_6, \tilde{Y}_1) \geq A. \tag{2.3}$$

These functions have the representation $g_{2r} = P_{2r-1} + c_r G_r$, $r = 2, 3$, where $P_{2r-1} \in \mathbb{P}_{2r-1}$ and c_r is an absolute constant. Therefore, by virtue of (2.1), we conclude that

$$n^{2r} E_n(g_{2r}) \leq c, \quad n \geq 2r - 1,$$

and the proof is complete.

Remark 2.1. Note that Lemma 2.2 readily implies that if $s = 1$, then, for $\overline{\alpha} = 1, 2$ and all $\mathcal{N} \geq 3$, as well as for $\overline{\alpha} = 3$ and all $\mathcal{N} \geq 5$, the symbol "$+$" cannot occupy the position $(\overline{\alpha}, \mathcal{N})$.

Our next result is true for any $s \in \mathbb{N}_0$.

Lemma 2.3. *Let $s \in \mathbb{N}_0$ and let $Y_s \in \mathbb{Y}_s$. For any $A > 0$ and $m \in \mathbb{N}$, there exists a function $f \in \Delta^2(Y_s)$, such that*

$$n^4 E_n(f) \leq 1, \quad n \geq s + 4.$$

At the same time,

$$E_m^{(2)}(f, Y_s) \geq A.$$

Proof. Following [4], for each $b \in (-1,0)$, we denote

$$f_b(x) := \int_0^x (x-t)\Pi(t;Y_s)\left(\int_b^t \frac{t-u}{(u+1)^2}du\right)dt.$$

Clearly, $f_b''(x)\Pi(x;Y_s) \ge 0$, $x \in (-1,1)$, and, hence, $f_b \in \Delta^2(Y_s)$. Straightforward computations performed by using the Taylor expansion of $\Pi(x;Y_s)$ about $t=-1$ imply that

$$f_b = P_{s+4} - \sum_{r=0}^{s} \frac{\Pi^{(r)}(-1;Y_s)}{(r+2)!} G_{r+2},$$

where $P_{s+4} \in \mathbb{P}_{s+4}$. Hence, by virtue of Lemma 2.1, we obtain,

$$n^4 E_n(f_b) \le c(s), \quad n \ge s+4, \tag{2.4}$$

because $\left\|\Pi^{(r)}(\cdot;Y_s)\right\| \le c(s)$, $0 \le r \le s$.

The polynomial

$$p_{s+4}(x) := \int_0^x (x-t)\Pi(t;Y_s)\left(\int_b^1 \frac{t-u}{(u+1)^2}du\right)dt,$$

belongs to $\mathbb{P}_{s+4}$ and satisfies the equality

$$\Pi(-1;Y_s)p_{s+4}''(-1) = \Pi^2(-1;Y_s)\ln\frac{b+1}{2}.$$

Hence, for any polynomial $P_m \in \mathbb{P}_m \cap \Delta^2(Y_s)$, $m \ge s+4$, we get

$$\begin{aligned} -\Pi^2(-1;Y_s)\ln\frac{b+1}{2} &= -\Pi(-1;Y_s)p_{s+4}''(-1) \\ &\le \Pi(-1;Y_s)\left(P_m''(-1) - p_{s+4}''(-1)\right) \le m^4|\Pi(-1;Y_s)|\|P_m - p_{s+4}\|, \end{aligned} \tag{2.5}$$

where we have used Markov's inequality. In addition,

$$p_{s+4}(x) - f_b(x) = \int_0^x (x-t)\Pi(t;Y_s)\left(\int_t^1 \frac{t-u}{(u+1)^2}du\right)dt,$$

which is independent of b. Hence, in view of (2.5),

$$m^{-4}\left|\Pi(-1;Y_s)\right|\ln\frac{2}{b+1} \le \|P_m - f_b\| + \|f_b - p_{s+4}\| \le \|P_m - f_b\| + c(s)$$

and, thus,

$$E_m^{(2)}(f_b, Y_s) \geq m^{-4}\big|\Pi(-1; Y_s)\big| \ln \frac{2}{b+1} - c(s).$$

Taking $f := cf_b$ with suitable $c = c(s)$ and b, we complete the proof of the lemma.

Remark 2.2. Lemma 2.3 implies that if $\overline{\alpha} = 1$ or 2, then, for all $s \geq 0$ and $\mathcal{N} \geq s+4$, the symbol "+" or "⊕" cannot occupy the position $(\overline{\alpha}, \mathcal{N})$ (thus, the best possible situation is that these positions are occupied by the symbol "⊖;" as shown in what follows, this is indeed the case).

Finally, for $s \geq 1$, we have the following lemma:

Lemma 2.4. *Let $s \in \mathbb{N}$ and let $Y_s \in \mathbb{Y}_s$. For each $A > 0$ and $m \in \mathbb{N}$, there exists a function $f \in \Delta^2(Y_s)$, such that*

$$n^2 E_n(f) \leq 1, \quad n \geq s+3,$$

and

$$E_m^{(2)}(f, Y_s) \geq A.$$

Proof. Denote $D_j(x) := x^j \ln|x|$ $(D_j(0) := 0)$. It is well known (and easy to check) that, for $j \geq 1$, the function $D_j^{(j-1)}$ belongs to the Zygmund class, i.e., $\omega_2(D_j^{(j-1)}, t) \leq c(j)t$. Thus, for $j \geq 2$, we have $E_n(D_j) \leq c(j)n^{-j} \leq c(j)n^{-2}$, $n \geq 1$. Hence, for $D_{j,\gamma}(x) := D_j(x+\gamma)$, $-1 < \gamma < 1$, $j \geq 2$, we get

$$E_n(D_{j,\gamma}) \leq \frac{c(j)}{n^2} \quad n \geq 1. \tag{2.6}$$

We take $0 < b < \frac{1}{2}\min\{y_1 - y_2, 1 - y_1\}$. Further, let

$$\tilde{l}_b(x) := \frac{x}{b} - 1 + \ln b. \tag{2.7}$$

(Note that $y = l_b(x)$ is the tangent to the function $\ln|x|$ at the point $x = b$.) Also let b^* be the other (clearly, negative) root of the equation $\tilde{l}_b(x) = \ln|x|$. It is obvious that

$$|b^*| = -b^* < b, \tag{2.8}$$

and $(x - b^*)\big(\tilde{l}_b(x) - \ln|x|\big) \geq 0$, $x \neq 0$. This means that, for

$$l_b(x) := \tilde{l}_b(x + b^*), \tag{2.9}$$

we have

$$x(l_b(x) - \ln|x + b^*|) \geq 0, \quad x \neq |b^*|. \tag{2.10}$$

Denote

$$\Pi_1(x) := \Pi_{i=2}^s (x - y_i) \qquad (\Pi_1 :\equiv 1 \quad \text{for} \quad s = 1),$$

and let

$$L_b(x) := \int_0^x (x-u)\Pi_1(u) l_b(u-y_1)\,du$$

and

$$g_b(x) := \int_0^x (x-u)\Pi_1(u)\ln|u+b^*-y_1|\,du.$$

Finally, we write

$$f_b := L_b - g_b.$$

As a result of integration by parts, we obtain

$$\int_0^x (x-u)\ln|u+b^*-y_1|\,du = \frac{1}{2}D_2(x+b^*-y_1) + p_3(x),$$

where $p_3 \in \mathbb{P}_3$. Similarly,

$$g_b(x) = \sum_{r=0}^{s-1} \frac{\Pi_1^{(r)}(y_1-b^*)}{(r+2)!} D_{r+2}(x+b^*-y_1) + p_{s+2}(x), \tag{2.11}$$

where $p_{s+2} \in \mathbb{P}_{s+2}$. Further, since $L_b \in \mathbb{P}_{s+3}$, inequality (2.6) implies that

$$E_n(f_b) \le \frac{c(s)}{n^2}, \quad n \ge s+3. \tag{2.12}$$

At the same time, it follows from (2.10) that $f_b \in \Delta^2(Y_s)$.

On the other hand, given $P_m \in \mathbb{P}_m \cap \Delta^2(Y_s)$, by virtue of (2.7) and (2.9), we conclude that

$$0 < \Pi_1(y_1)\ln\frac{1}{b} < \Pi_1(y_1)\left(\ln\frac{1}{b} + 1 - \frac{b^*}{b}\right)$$

$$= -L_b''(y_1) = P_m''(y_1) - L_b''(y_1) \le c(s, y_1)m^2\|P_m - L_b\|,$$

where we have used Bernstein's inequality. Since

$$\|g_b\| \le 2\|\Pi_1\| \int_0^1 |\ln x|\,dx = 2\|\Pi_1\| \le 2^s,$$

we have

$$0 < \Pi_1(y_1)\ln\frac{1}{b} \le c(s, y_1)m^2\big(\|P_m - f_b\| + \|g_b\|\big) \le c(Y_s)m^2(\|P_m - f_b\| + 1).$$

Hence,

$$E_m^{(2)}(f_b, Y_s) \geq \frac{c(Y_s)}{m^2}\ln\frac{1}{b} - 1.$$

In combination with (2.12), this inequality yields the statement of the lemma for $f := cf_b$ with suitable $c = c(s)$ and b.

Remark 2.3. Lemma 2.4 implies that if $\overline{\alpha} = 1$ and $s \geq 1$, then, for all $\mathcal{N} \geq s + 3$, the symbols "$+$" and "$\oplus$" cannot occupy the position $(\overline{\alpha}, \mathcal{N})$ (thus, the best possible situation is that the corresponding positions are occupied by "$\ominus$;" as shown in what follows, this is indeed the case).

3. Auxiliary Results

Recall that $\varphi(x) = \sqrt{1-x^2}$. Let $\mathbb{C}^r_\varphi$, $r \geq 1$, be the space of functions $f \in \mathbb{C}^r(-1,1) \cap \mathbb{C}[-1,1]$ such that

$$\lim_{x\to\pm 1} \varphi^r(x) f^{(r)}(x) = 0,$$

and $\mathbb{C}^0_\varphi := \mathbb{C}[-1,1]$.

By

$$\Delta^k_\delta(g, x) := \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} g\left(x - \frac{k\delta}{2} + i\delta\right),$$

we denote the kth symmetric difference of a function g with step δ. Thus, the Ditzian–Totik-type modulus of smoothness of the rth derivative of a function $f \in \mathbb{C}^r_\varphi$, is defined as follows:

$$\omega^\varphi_{k,r}(f^{(r)}, t) := \sup_{h\in[0,t]} \ \sup_{x:|x|+(kh)\varphi(x)/2<1} W^r\left(x, \frac{kh}{2}\right) \left|\Delta^k_{h\varphi(x)}(f^{(r)}, x)\right| \tag{3.1}$$

with the weight

$$W(x,\mu) := \varphi\big(|x| + \mu\varphi(x)\big), \quad |x| + \mu\varphi(x) < 1. \tag{3.2}$$

If $r = 0$, then

$$\omega^\varphi_k(f,t) := \omega^\varphi_{k,0}(f,t)$$

is the (ordinary) Ditzian–Totik modulus of smoothness. Finally, let $\|f\|_{C[a,b]}$ be the uniform norm of a function $f \in \mathbb{C}[a,b]$ (in particular, $\|f\|_{C[-1,1]} = \|f\|$). We recall that the ordinary kth modulus of smoothness of $f \in \mathbb{C}[a,b]$ is

$$\omega_k\big(f, t, [a,b]\big) := \sup_{h\in[0,t]} \left\|\Delta^k_h(f,\cdot)\right\|_{\mathbb{C}[a+kh/2,\, b-kh/2]},$$

and denote $\omega_k(f,t) := \omega_k(f,t,[-1,1])$.

The following results are the so-called inverse theorems. They characterize the smoothness (i.e., describe the class) of functions with prescribed order of polynomial approximation.

First, we formulate a corollary of the classical Dzyadyk–Timan–Lebed–Brudnyi inverse theorem (see, e.g., [3], Theorem 7.1.2).

Theorem 3.1. *Let* $2r < \alpha < 2k + 2r$ *and let* $f \in \mathbb{C}[-1, 1]$. *If*

$$n^{\alpha} E_n(f) \leq 1, \quad n \geq k + r,$$

then $f \in \mathbb{C}^r[-1, 1]$ *and*

$$\omega_k(f^{(r)}, t^2) \leq c(\alpha, k, r) t^{\alpha - 2r}. \tag{3.3}$$

For the Ditzian–Totik-type moduli of smoothness, we need the following result obtained as a generalization of [5] (Theorem 7.2.4) to the case $p = \infty$.

By Φ we denote the set of nondecreasing functions $\phi\colon [0, \infty) \to [0, \infty)$ such that $\phi(0+) = 0$.

Theorem 3.2. *Given* $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $N \in \mathbb{N}$, *and* $\phi \in \Phi$ *such that*

$$\int_0^1 \frac{r\phi(u)}{u^{r+1}} du < +\infty.$$

If

$$E_n(f) \leq \phi\left(\frac{1}{n}\right), \quad \textit{for all} \quad n \geq N,$$

then $f \in \mathbb{C}^r_{\varphi}$ *and*

$$\omega^{\varphi}_{k,r}(f^{(r)}, t) \leq c(k, r) \int_0^t \frac{r\phi(u)}{u^{r+1}} du + c(k, r) t^k \int_t^1 \frac{\phi(u)}{u^{k+r+1}} du + c(k, r, N) t^k E_{k+r}(f), \quad t \in [0, 1/2].$$

If, in addition, $N \leq k + r$, *then the following Bari–Stechkin-type estimate holds:*

$$\omega^{\varphi}_{k,r}(f^{(r)}, t) \leq c(k, r) \int_0^t \frac{r\phi(u)}{u^{r+1}} du + c(k, r) t^k \int_t^1 \frac{\phi(u)}{u^{k+r+1}} du, \quad t \in [0, 1/2].$$

We present the proof of this theorem in the appendix.

In fact, we only need the following theorem which is an immediate consequence of Theorem 3.2 ($\phi(u) := u^{\alpha}$) but is of especial interest in the context of the present paper.

Theorem 3.3. *Let* $r \in \mathbb{N}_0$, $k \in \mathbb{N}$, *and* $\alpha > 0$, *be such that* $r < \alpha < k + r$ *and let* $f \in \mathbb{C}[-1, 1]$. *If*

$$n^{\alpha} E_n(f) \leq 1 \quad \textit{for all} \quad n \geq N,$$

where $N \ge k + r$, *then* $f \in \mathbb{C}_\varphi^r$ *and*

$$\omega_{k,r}^\varphi(f^{(r)}, t) \le c(\alpha, k, r)t^{\alpha-r} + c(N, k, r)t^k E_{k+r}(f).$$

In particular, if $N = k + r$, *then*

$$\omega_{k,r}^\varphi(f^{(r)}, t) \le c(\alpha, k, r)t^{\alpha-r}.$$

Lemma 3.1. [See [6], [4] (Theorems 2.7, 2.8, and 2.11), [2] (Lemma 2.8), and [7] (Theorem 3.1).]

I. *Let* $f \in \Delta^2$. *If* $f \in \mathbb{C}[-1, 1]$, *then*

$$E_n^{(2)}(f) \le c\omega_4^\varphi\left(f, \frac{1}{n}\right) + cn^{-6}\|f\|, \quad n \ge 3. \tag{3.4}$$

Moreover, if $f \in \mathbb{C}_\varphi^2 \cap \mathbb{C}^1[-1, 1]$, *then*

$$E_n^{(2)}(f) \le c(k)n^{-2}\omega_{k,2}^\varphi\left(f'', \frac{1}{n}\right) + c(k)n^{-2}\omega_2\left(f', \frac{1}{n^2}\right), \quad n \ge 3. \tag{3.5}$$

Furthermore, if $f \in \mathbb{C}_\varphi^2 \cap \mathbb{C}^2[-1, 1]$ *and* $k, l \in \mathbb{N}$, *then, for* $n \ge l + 2$, *the following inequality is true:*

$$E_n^{(2)}(f) \le c(k, l)n^{-2}\omega_{k,2}^\varphi\left(f'', \frac{1}{n}\right) + c(k, l)n^{-4}\omega_l\left(f'', \frac{1}{n^2}\right). \tag{3.6}$$

II. *Let* $f \in \Delta^2(Y_1)$. *If* $f \in \mathbb{C}[-1, 1]$, *then*

$$E_n^{(2)}(f, Y_1) \le c\omega_3^\varphi\left(f, \frac{1}{n}\right) + c\omega_2\left(f, \frac{1}{n^2}\right), \quad n \ge 2. \tag{3.7}$$

If, in addition, $f \in \mathbb{C}_\varphi^2 \cap \mathbb{C}^1[-1, 1]$, *then*

$$E_n^{(2)}(f, Y_1) \le cn^{-2}\omega_{3,2}^\varphi\left(f'', \frac{1}{n}\right) + cn^{-2}\omega_1\left(f', \frac{1}{n^2}\right), \quad n \ge 2, \tag{3.8}$$

and

$$E_n^{(2)}(f, Y_1) \le cn^{-2}\omega_{3,2}^\varphi\left(f'', \frac{1}{n}\right) + cn^{-2}\omega_2\left(f', \frac{1}{n^2}\right), \quad n\varphi(y_1) > 1. \tag{3.9}$$

If $f \in \mathbb{C}_\varphi^2$, *then*

$$E_n^{(2)}(f, Y_1) \le cn^{-2}\omega_{3,2}^\varphi\left(f'', \frac{1}{n}\right) + cn^{-4}\omega_{2,2}^\varphi\left(f'', \frac{1}{n}\right), \quad n \ge N(Y_1), \tag{3.10}$$

and

$$E_n^{(2)}(f, Y_1) \le cn^{-2}\omega_{3,2}^{\varphi}\left(f'', \frac{1}{n}\right), \quad n \ge N(f). \tag{3.11}$$

Moreover, if we actually have $f \in \mathbb{C}_{\varphi}^3 \cap \mathbb{C}^2[-1,1]$, *then, for any* $k \in \mathbb{N}$,

$$E_n^{(2)}(f, Y_1) \le c(k)n^{-3}\omega_{k,3}^{\varphi}\left(f^{(3)}, \frac{1}{n}\right) + c(k)n^{-4}\omega_2\left(f'', \frac{1}{n^2}\right), \quad n \ge 4. \tag{3.12}$$

Furthermore, if $f \in \mathbb{C}^3[-1,1]$, *then*

$$E_n^{(2)}(f, Y_1) \le c(k)n^{-3}\omega_{k,3}^{\varphi}\left(f^{(3)}, \frac{1}{n}\right) + c(k)n^{-6}\omega_k\left(f^{(3)}, \frac{1}{n^2}\right), \quad n \ge k+3. \tag{3.13}$$

III. *Let* $f \in \Delta^2(Y_s)$, $s \in \mathbb{N}$. *If* $f \in \mathbb{C}[-1,1]$, *then*

$$E_n^{(2)}(f, Y_s) \le c(s)\omega_3^{\varphi}\left(f, \frac{1}{n}\right), \quad n \ge N(Y_s). \tag{3.14}$$

Moreover, if $f \in \mathbb{C}_{\varphi}^3 \cap \mathbb{C}^2[-1,1]$, $s \in \mathbb{N}$, *and* $k, l \in \mathbb{N}$, *then there exists* $N(Y_s, k, l)$ *such that, for all* $n \ge N(Y_s, k, l)$,

$$E_n^{(2)}(f, Y_s) \le c(k,l,s)n^{-3}\omega_{k,3}^{\varphi}\left(f^{(3)}, \frac{1}{n}\right) + c(k,l,s)n^{-4}\omega_l\left(f'', \frac{1}{n^2}\right). \tag{3.15}$$

In addition, if $s \ge 2$ *and* $f \in \mathbb{C}_{\varphi}^2$, *then*

$$E_n^{(2)}(f, Y_s) \le c(s)n^{-2}\omega_{3,2}^{\varphi}\left(f'', \frac{1}{n}\right), \quad n \ge N(Y_s). \tag{3.16}$$

Remark 3.1. Estimate (3.13) was not proved in [2]. However, its proof is very similar to the proof presented in [2] and based on the fact that if $f \in \mathbb{C}^3[a,b]$ is such that f is concave on $[a, y_1]$ and convex on $[y_1, b]$ (i.e., $f''(x)(x - y_1) \ge 0$, $a \le x \le b$) and p_k is such that $p_k \ge f^{(3)}$ on $[a,b]$ and

$$\|f^{(3)} - p_k\| \le c(k)\omega_k(f^{(3)}, b-a, [a,b])$$

(e.g.,

$$p_k := \arg \inf_{p \in \mathbb{P}_k} \|f^{(3)} - p\|_{C[a,b]} + \inf_{p \in \mathbb{P}_k} \|f^{(3)} - p\|_{C[a,b]}),$$

then

$$P(x) := \int_a^x \int_a^t \int_{y_1}^s p_k(v)\,dv\,ds\,dt + f(a) + f'(a)(x-a)$$

is a polynomial from $\mathbb{P}_{k+3}$ coconvex with f on $[a,b]$ and such that $P(a) = f(a)$ and

$$\|f - P\|_{C[a,b]} \le c(k)(b-a)^3\omega_k\big(f^{(3)}, b-a, [a,b]\big). \tag{3.17}$$

We omit the details.

4. Proofs of the Positive Results

Since the cases $\mathcal{N} = 1$ and $\mathcal{N} = 2$ have already been discussed, we assume that $\mathcal{N} \ge 3$. Given $\alpha > 0$, integers $\mathcal{N} \ge 3$, $s \ge 0$, a collection $Y_s \in \mathbb{Y}_s$, and a function $f \in \Delta^2(Y_s)$, we suppose, without loss of generality, that

$$n^\alpha E_n(f) \le 1, \quad \text{for all} \quad n \ge \mathcal{N}. \tag{4.1}$$

Thus, it is necessary to prove the inequality

$$n^\alpha E_n^{(2)}(f, Y_s) \le c(\alpha, \mathcal{N}, s), \quad n \ge \mathcal{N}^*, \tag{4.2}$$

with proper $\mathcal{N}^*$.

4.1. Convex Approximation: $s = 0$.

1. $\mathcal{N} = 3,\ 0 < \alpha < 3$ ("+").

   Theorem 3.3 (with $r = 0$ and $k = 3$), inequality (4.1), and the estimate $E_n^{(2)}(f) \le c\omega_3^\varphi(f, 1/n)$, $n \ge 3$, proved in [8] yield $E_n^{(2)}(f) \le c\omega_3^\varphi(f, 1/n) \le cn^{-\alpha}$ for $n \ge 3 =: \mathcal{N}^*$.

2. $\mathcal{N} = 3,\ 3 \le \alpha \le 4$ ("+").

   Theorem 3.3 (with $r = 2$ and $k = 3$), Theorem 3.1 (with $r = 1$ and $k = 2$), and inequality (4.1) imply that $f \in \mathbb{C}_\varphi^2 \cap \mathbb{C}^1[-1,1]$, $\omega_{3,2}^\varphi(f'', t) \le c(\alpha)t^{\alpha-2}$, and $\omega_2(f', t^2) \le c(\alpha)t^{\alpha-2}$. Inequality (3.5) now yields $E_n^{(2)}(f) \le c(\alpha)n^{-\alpha}$ for $n \ge 3 =: \mathcal{N}^*$.

3. $\alpha > 4,\ \mathcal{N} > \alpha$ ("+").

   Theorem 3.3 (with $r = 2$ and $k = \mathcal{N} - 2$), Theorem 3.1 (with $r = 2$ and $k = \mathcal{N} - 2$), and inequality (4.1) imply that $f \in \mathbb{C}_\varphi^2 \cap \mathbb{C}^2[-1,1]$,

   $$\omega_{\mathcal{N}-2,2}^\varphi(f'', t) \le c(\alpha, \mathcal{N})t^{\alpha-2}, \qquad \text{and} \qquad \omega_{\mathcal{N}-2}(f'', t^2) \le c(\alpha, \mathcal{N})t^{\alpha-4}.$$

   Therefore, inequality (3.6) (with $k = l = \mathcal{N} - 2$) yields inequality (4.2) with $\mathcal{N}^* = \mathcal{N}$.

4. $\alpha > 4,\ 4 < \mathcal{N} \le \alpha$ ("+").

   Let $\mathcal{N}_1 := \lfloor \alpha \rfloor + 1$. Note that $\mathcal{N}_1 > \alpha \ge \mathcal{N}$. Since (4.1) is satisfied with $\mathcal{N}_1$ instead of $\mathcal{N}$, it follows from case 3 that $n^\alpha E_n^{(2)}(f) \le c(\alpha)$, $n \ge \mathcal{N}_1$. Now let $\alpha_1 := \mathcal{N}/2 + 2$ and note that $4 < \alpha_1 < \mathcal{N}$. It follows from (4.1) that $n^{\alpha_1} E_n(f) \le 1$ for all $n \ge \mathcal{N}$. Thus, by using case 3 once again, we get $n^{\alpha_1} E_n^{(2)}(f) \le c(\mathcal{N})$, $n \ge \mathcal{N}$. Hence, for $\mathcal{N} \le n < \mathcal{N}_1$, we find

   $$n^\alpha E_n^{(2)}(f) \le c(\mathcal{N})n^{\alpha-\alpha_1} \le c(\mathcal{N})\mathcal{N}_1^{\alpha-\alpha_1} \le c(\alpha, \mathcal{N}),$$

which proves (4.2) with $\mathcal{N}^* = \mathcal{N}$.

5. $\mathcal{N} = 3,\ \alpha > 4$ ("+").

   It follows from cases 3 and 4 that inequality (4.2) is true for $n \ge 5$. Note that the polynomial of the best approximation of degree ≤ 2 to a convex function f must be convex (this follows, e.g., from the Chebyshev equioscillation theorem) and, therefore, $E_3^{(2)}(f) = E_3(f)$. Hence, for $n = 3$ and 4, we have

$$E_n^{(2)}(f) \le E_3^{(2)}(f) = E_3(f) \le 1 \le 4^\alpha n^{-\alpha},$$

   and, thus, we arrive at inequality (4.2) with $\mathcal{N}^* = 3$.

6. $\mathcal{N} = 4,\ \alpha > 4$ ("+").

   As in case 5, it follows from cases 3 and 4 that inequality (4.2) is valid for $n \ge 5$ and, hence, it remains to show that $E_4^{(2)}(f) \le c(\alpha)$. Since (4.1) implies that $n^{\alpha_1} E_n(f) \le 1,\ n \ge 4$, where $\alpha_1 := \min\{\alpha, 5\}$, it follows from Theorem 3.1 (with $r = 2$ and $k = 2$) that $f \in \mathbb{C}^2[-1,1]$ and $\omega_2(f'', t^2) \le c(\alpha) t^{\alpha_1 - 4}$, and in particular, $\omega_2(f'', 1) \le c(\alpha)$. Therefore, $E_2(f'') \le c\omega_2(f'', 1) \le c(\alpha)$. Further, since the inequality $E_4^{(2)}(f) \le 2E_2(f'')$ holds for each $f \in \mathbb{C}^2[-1,1] \cap \Delta^2$, we conclude that $E_4^{(2)}(f) \le c(\alpha)$, as required.

7. $\mathcal{N} \ge 4,\ 0 < \alpha < 4$ ("$\ominus$").

   Theorem 3.3 (with $k = 4$ and $N = \mathcal{N}$) and inequalities (4.1) and (3.4) yield

$$E_n^{(2)}(f) \le c(\alpha) n^{-\alpha} + c(\mathcal{N}) n^{-4} \|f\| \le c(\alpha) n^{-\alpha}$$

   for all $n \ge \max\left\{3, c(\alpha, \mathcal{N}) \|f\|^{1/(4-\alpha)}\right\} =: \mathcal{N}^*$.

8. $\mathcal{N} \ge 4,\ \alpha = 4$ ("$\ominus$").

   Theorem 3.3 (with $r = 2$ and $k = 3$), Theorem 3.3 (with $r = 1$ and $k = 3$), and inequality (4.1) imply that $f \in \mathbb{C}_\varphi^2 \cap \mathbb{C}^1[-1,1]$, $\omega_{3,2}^\varphi(f'', t) \le ct^2$, and $\omega_3(f', t^2) \le ct^2$. By the Marchaud classical inequality (see, e.g., [5], (4.3.1)), the last estimate implies that $\omega_2(f', t) \le ct + ct^2 \|f'\|$. Inequality (3.5) (with $k = 3$) now yields $E_n^{(2)}(f) \le cn^{-4} + cn^{-6}\|f'\|,\ n \ge 3$, and hence, we arrive at inequality (4.2) with $\mathcal{N}^* := \max\left\{3, c\|f'\|^{1/2}\right\}$.

4.2. Coconvex Approximation: Case $s \ge 1$. For some cases considered in what follows, we need the fact (see [9]) that, for any $f \in \Delta^2(Y_s),\ s \ge 1$,

$$E_2(f) \le c(Y_s) E_{s+2}(f). \tag{4.3}$$

Remark 4.1. For the sake of convenience, for each case analyzed in what follows, we present the full range of the values of α for which the corresponding proof can be used. Hence, the same triple $(\alpha, \mathcal{N}, s)$ may be covered by more than one case.

1. $s = 1,\ 4 < \alpha < 8,\ \mathcal{N} = 4$ ("+").

   Theorem 3.3 (with $r = 3$ and $k = 5$), Theorem 3.3 (with $r = k = 2$), and inequality (4.1), imply that $f \in \mathbb{C}_\varphi^3 \cap \mathbb{C}^2[-1,1]$,

$$\omega_{5,3}^\varphi(f^{(3)}, t) \le c(\alpha) t^{\alpha-3}, \qquad \text{and} \qquad \omega_2(f'', t^2) \le c(\alpha) t^{\alpha-4}.$$

Therefore, (3.12) (with $k = 5$), yields inequality (4.2) with $\mathcal{N}^* = 4$.

2. $s = 1,\ 4 < \alpha < 8,\ \mathcal{N} = 3$ ("+").

   It follows from case 1 that (4.2) is true for $n \geq 4$. Thus, in order to show that $\mathcal{N}^* = 3$, it remains to check the inequality $E_3^{(2)}(f, Y_1) \leq c(\alpha)$. Indeed, since inequality (4.1) is true for $\alpha_1 := \min\{\alpha, 5\}$, it follows from Theorem 3.2 (with $r = 2$ and $k = 1$), that $f \in \mathbb{C}^2[-1, 1]$ (so that $f''(y_1) = 0$), $\omega_1(f'', t^2) \leq c(\alpha)t^{\alpha_1 - 4}$, and in particular, $\omega_1(f'', 1) \leq c(\alpha)$. We now set $p_2(x) := f(y_1) + f'(y_1)(x - y_1)$. This yields

$$E_3^{(2)}(f, Y_1) = E_2^{(2)}(f, Y_1) = E_2(f) \leq \|f - p_2\|$$

$$= \left\| \int_{y_1}^{x} \int_{y_1}^{u} (f''(s) - f''(y_1))\, ds\, du \right\| \leq c\omega_1(f'', 1) \leq c(\alpha).$$

3. $s = 1,\ \alpha > 6,\ \mathcal{N} > \alpha$ ("+").

   Theorems 3.3 and 3.1 (with $r = 3$ and $k = \mathcal{N} - 3$) and inequality (4.1) imply that $f \in \mathbb{C}^3$, $\omega^{\varphi}_{\mathcal{N}-3,3}(f^{(3)}, t) \leq c(\alpha, \mathcal{N})t^{\alpha-3}$, and $\omega_{\mathcal{N}-3}(f^{(3)}, t^2) \leq c(\alpha, \mathcal{N})t^{\alpha-6}$. Estimate (3.13) (with $k = \mathcal{N} - 3$) now yields inequality (4.2) with $\mathcal{N}^* = \mathcal{N}$.

4. $s = 1,\ \alpha > 6,\ 6 < \mathcal{N} \leq \alpha$ ("+").

   Let $\mathcal{N}_1 := \lfloor \alpha \rfloor + 1$. Note that $\mathcal{N}_1 > \alpha \geq \mathcal{N}$. Since (4.1) is satisfied with $\mathcal{N}_1$ instead of $\mathcal{N}$, it follows from case 3 that $n^{\alpha} E_n^{(2)}(f, Y_1) \leq c(\alpha)$, $n \geq \mathcal{N}_1$. Now let $\alpha_1 := (\mathcal{N} + 6)/2$. Note that $6 < \alpha_1 < \mathcal{N}$. It follows from (4.1) that $n^{\alpha_1} E_n(f) \leq 1$ for all $n \geq \mathcal{N}$. Further, by using case 3 once again, we get $n^{\alpha_1} E_n^{(2)}(f, Y_1) \leq c(\mathcal{N})$, $n \geq \mathcal{N}$.

   Hence, for $\mathcal{N} \leq n < \mathcal{N}_1$, we get

$$n^{\alpha} E_n^{(2)}(f, Y_1) \leq c(\mathcal{N}) n^{\alpha - \alpha_1} \leq c(\mathcal{N}) \mathcal{N}_1^{\alpha - \alpha_1} \leq c(\alpha, \mathcal{N}),$$

   which proves (4.2) with $\mathcal{N}^* = \mathcal{N}$.

5. $s = 1,\ \alpha > 6,\ \mathcal{N} = 3$ ("+").

   It follows from cases 3 and 4, that (4.2) is valid with $n \geq 7$. Thus, since (4.1) is obviously true, say, with $\alpha = 5$, it follows from case 2 that $E_3^{(2)}(f, Y_1) \leq c$, and hence, for $3 \leq n \leq 6$,

$$n^{\alpha} E_n^{(2)}(f, Y_1) \leq 6^{\alpha} E_3^{(2)}(f, Y_1) \leq c(\alpha).$$

   This means that (4.2) is true with $\mathcal{N}^* = 3$.

6. $s = 1,\ \alpha > 6,\ \mathcal{N} = 4$ ("+").

   The proof is completely analogous to the proof in case 5 except the fact that the inequality $E_4^{(2)}(f, Y_1) \leq c$ follows from case 1.

7. $s = 1,\ \alpha > 6,\ \mathcal{N} = 6$ ("+").

It follows from cases 3 and 4, that inequality (4.2) is valid with $n \ge 7$. Hence, as in case 5, it suffices to show that $E_6^{(2)}(f, Y_1) \le c(\alpha)$. If $\alpha_1 := \min\{\alpha, 7\}$, it follows from (4.1) that $n^{\alpha_1} E_n(f) \le 1$ for all $n \ge 6$. Thus, by applying Theorem 3.2 (with $r = 3$ and $k = 3$), we conclude that $f \in \mathbb{C}^3[-1, 1]$, $\omega_3(f^{(3)}, t^2) \le c(\alpha) t^{\alpha_1 - 6}$, and in particular, $\omega_3(f^{(3)}, 1) \le c(\alpha)$. The inequality $E_6^{(2)}(f, Y_1) \le c(\alpha)$ now follows from (3.17) (with $k = 3$ and $[a, b] = [-1, 1]$).

8. $s = 1,\ \alpha > 6,\ \mathcal{N} = 5$ ("+").

   The argument is exactly the same as in the previous case with the only difference that we use $k = 2$ instead of $k = 3$.

9. $s \ge 1,\ 0 < \alpha < 3,\ 3 \le \mathcal{N} \le s + 2$ ("$\oplus$").

   Theorem 3.3 (with $k = 3$ and $f - p_3$ instead of f, where $p_3 := \arg \inf_{p \in \mathbb{P}_3} \|f - p\|$), implies that

$$\omega_3^{\varphi}(f, t) \le c(\alpha) t^{\alpha} + c(s) t^3 E_3(f).$$

   Further, by using (4.3) and (4.1), we get

$$E_3(f) \le E_2(f) \le c(Y_s) E_{s+2}(f) \le c(\alpha, Y_s).$$

   Therefore,

$$\omega_3^{\varphi}(f, 1/n) \le c(\alpha) n^{-\alpha} + c(\alpha, Y_s) n^{-3} \le c(\alpha) n^{-\alpha}$$

   for $n \ge N(\alpha, Y_s)$. Inequality (4.2) now follows from (3.14).

10. $s \ge 2,\ 2 < \alpha < 5,\ 3 \le \mathcal{N} \le s + 2$ ("$\oplus$").

    Theorem 3.3 (with $r = 2,\ k = 3$) implies that $f \in \mathbb{C}_{\varphi}^2$ and

$$\omega_{3,2}^{\varphi}(f'', t) \le c(\alpha) t^{\alpha - 2} + c(s) t^3 E_5(f).$$

    Further, by virtue of (4.3) and (4.1), we conclude that

$$E_5(f) \le E_2(f) \le c(Y_s) E_{s+2}(f) \le c(\alpha, Y_s).$$

    Hence,

$$\omega_{3,2}^{\varphi}(f'', 1/n) \le c(\alpha) n^{-\alpha+2} + c(\alpha, Y_s) n^{-3} \le c(\alpha) n^{-\alpha+2}$$

    for $n \ge N(\alpha, Y_s)$. Inequality (4.2) now follows from (3.16).

11. $s \ge 1,\ 2 < \alpha < 5,\ \mathcal{N} \ge s + 3$ ("$\ominus$") (except all "$\oplus$" cases in these regions).

    As in case 10, we can prove that

$$\omega_{3,2}^{\varphi}(f'', 1/n) \le c(\alpha) n^{-\alpha+2} + c(s) n^{-3} \|f\| 4$$

and, hence,

$$\omega^{\varphi}_{3,2}(f'', 1/n) \le c(\alpha) n^{-\alpha+2}$$

for $n \ge N(\alpha, f)$. Thus, inequality (4.2) with $\mathcal{N}^* = \mathcal{N}^*(\alpha, f)$ follows from (3.16) for $s \ge 2$, and from (3.11) for $s = 1$.

12. $s = 2,\ 2 < \alpha < 5,\ \mathcal{N} = 5$ ("$\oplus$").

    Theorem 3.3 (with $r = 2$ and $k = 3$) and inequality (4.1) imply that $f \in \mathbb{C}^2_{\varphi}$ and $\omega^{\varphi}_{3,2}(f'', t) \le c(\alpha) t^{\alpha-2}$. Now inequality (3.16) implies inequality (4.2) with $\mathcal{N}^* = \mathcal{N}^*(\alpha, Y_s)$.

13. $s = 1,\ 4 < \alpha \le 6,\ \mathcal{N} \ge 5$ and $s \ge 2,\ \alpha > 4,\ \mathcal{N} \ge 3$ ("$\oplus$").

    If $\mathcal{N}_1 := \max\{\lfloor \alpha \rfloor + 1, \mathcal{N}\}$, then Theorem 3.3 (with $r = 3$ and $k = \mathcal{N}_1 - 3$), Theorem 3.3 (with $r = 2$ and $k = \mathcal{N}_1 - 2$), and inequality (4.1) imply that $f \in \mathbb{C}^3_{\varphi} \cap \mathbb{C}^2[-1, 1]$,

$$\omega^{\varphi}_{\mathcal{N}_1-3,3}(f^{(3)}, t) \le c(\alpha, \mathcal{N}) t^{\alpha-3}, \qquad \text{and} \qquad \omega_{\mathcal{N}_1-2}(f'', t^2) \le c(\alpha, \mathcal{N}) t^{\alpha-4}.$$

    Therefore, inequality (3.15) (with $k = \mathcal{N}_1 - 3$ and $l = \mathcal{N}_1 - 2$) yields inequality (4.2) with $\mathcal{N}^* = \mathcal{N}^*(\alpha, \mathcal{N}, Y_s)$.

14. $s = 1,\ 2 < \alpha < 5,\ \mathcal{N} = 3$ or 4 ("$\oplus$").

    Theorem 3.3 (with $r = 2$ and $k = 3$) and inequality (4.1) imply that $f \in \mathbb{C}^2_{\varphi}$ and $\omega^{\varphi}_{3,2}(f'', t) \le c(\alpha) t^{\alpha-2}$. We set $\alpha_1 := \min\{\alpha, 3\}$. Then Theorem 3.3 (with $r = k = 2$) implies that $\omega^{\varphi}_{2,2}(f'', t) \le c(\alpha) t^{\alpha_1-2}$. Therefore, it follows from inequality (3.10) that

$$E^{(2)}_n(f, Y_1) \le cn^{-\alpha} + cn^{-\alpha_1-2} \le cn^{-\alpha}, \quad n \ge N(Y_1),$$

    as required.

15. $s \ge 1,\ 0 < \alpha < 3,\ \mathcal{N} \ge s + 3$ ("$\oplus$").

    Theorem 3.3 (with $k = 3$ and $N = \mathcal{N}$) and inequalities (4.1) and (3.14) yield

$$E^{(2)}_n(f) \le c(\alpha) n^{-\alpha} + c(\mathcal{N}) n^{-3} \|f\| \le c(\alpha) n^{-\alpha}$$

    for all sufficiently large n, $n \ge \mathcal{N}^*(\alpha, \mathcal{N}, Y_s, f)$.

5. Appendix: Proof of Theorem 3.2

First, we present the proof for the case $r \ge 1$. Without loss of generality, we assume that $N \ge k + r$ and set $m_j := N2^j$ and $\phi_j := \phi(m_j^{-1})$. Further, we expand f in the telescopic series

$$f = P_{k+r} + (P_N - P_{k+r}) + \sum_{j=0}^{\infty} (P_{m_{j+1}} - P_{m_j}) =: P_{k+r} + Q + \sum_{j=0}^{\infty} Q_j, \tag{5.1}$$

where $P_n \in \mathbb{P}_n$ are the polynomials of the best approximation of f, i.e., $\|f - P_n\| = E_n(f)$. Hence, the polynomials Q_j are of degree $< m_{j+1}$ and satisfy the inequality $\|Q_j\| \le \phi_{j+1} + \phi_j \le 2\phi_j$. For fixed

$x \in (-1,1)$ and $h \in [0,t]$, satisfying the inequality $kh\varphi(x)/2 < 1-|x|$, we set $x_* := |x| + kh\varphi(x)/2$ and note that if

$$u \in [-x_*, x_*] \supseteq [x - kh\varphi(x)/2, x + kh\varphi(x)/2] =: A,$$

then $\varphi(u) \ge \varphi(x_*)$. Hence, for $u \in A$ and $l \in \mathbb{N}$, the Markov–Bernstein inequality implies that

$$\left|Q_j^{(l)}(u)\right| \le c(l) m_{j+1}^l \left(\frac{1}{m_{j+1}} + \varphi(u)\right)^{-l} \phi_j \le c(l) m_j^l \left(\frac{1}{m_j} + \varphi(x_*)\right)^{-l} \phi_j, \tag{5.2}$$

which, in turn, yields the inequality

$$\left|\Delta^k_{h\varphi(x)}(Q_j^{(r)}, x)\right| \le 2^k \max_{u\in A} |Q_j^{(r)}(u)| \le c(r) 2^k \frac{m_j^r}{\varphi^r(x_*)} \phi_j$$

for $l = r$. Therefore, if we denote $J := \min\{j : 1/m_j \le h\}$, then we find

$$\begin{aligned}
\varphi^r(x_*) \sum_{j=J+1}^{\infty} \left|\Delta^k_{h\varphi(x)}\big(Q_j^{(r)}, x\big)\right| &\le c(r) 2^k \sum_{j=J+1}^{\infty} m_j^r \phi_j \\
&= c(k,r) \sum_{j=J+1}^{\infty} \int_{m_j^{-1}}^{m_{j-1}^{-1}} \frac{\phi_j}{u^{r+1}} du \le c(k,r) \sum_{j=J+1}^{\infty} \int_{m_j^{-1}}^{m_{j-1}^{-1}} \frac{\phi(u)}{u^{r+1}} du \\
&= c(k,r) \int_0^{m_J^{-1}} \frac{\phi(u)}{u^{r+1}} du \le c(k,r) \int_0^h \frac{\phi(u)}{u^{r+1}} du.
\end{aligned} \tag{5.3}$$

We also note that

$$\frac{\varphi(x) - \varphi(x_*)}{kh/2} = \frac{\varphi(x) - \varphi(x_*)}{x_* - |x|} \varphi(x) < \frac{\varphi(x) - \varphi(x_*)}{x_* - |x|} \big(\varphi(x) + \varphi(x_*)\big) = x_* + |x| < 2$$

and, therefore,

$$\varphi(x) < kh + \varphi(x_*).$$

Hence, for $0 \le j \le J$, in view of the fact that $1/m_j > h/2$, by virtue of inequality (5.2) with $l = r + k$, we obtain

$$\begin{aligned}
\left|\Delta^k_{h\varphi(x)}(Q_j^{(r)}, x)\right| &\le (h\varphi(x))^k \max_{u\in A} \left|Q_j^{(k+r)}(u)\right| \\
&\le c(k,r) \frac{h^k m_j^{k+r} \varphi^k(x)}{(kh + \varphi(x_*))^{k+r}} \phi_j \le c(k,r) \frac{h^k m_j^{k+r}}{\varphi^r(x_*)} \phi_j
\end{aligned}$$

$$\le c(k,r)\frac{h^k}{\varphi^r(x_*)}\int\limits_{m_j^{-1}}^{m_{j-1}^{-1}}\frac{\phi_j}{u^{k+r+1}}du$$

$$\le c(k,r)\frac{h^k}{\varphi^r(x_*)}\int\limits_{m_j^{-1}}^{m_{j-1}^{-1}}\frac{\phi(u)}{u^{k+r+1}}du,$$

where $m_{-1} := N/2$. Hence, we get

$$\varphi^r(x_*)\sum_{j=0}^{J}\left|\Delta^k_{h\varphi(x)}\left(Q_j^{(r)},x\right)\right| \le c(k,r)h^k\sum_{j=0}^{J}\int\limits_{m_j^{-1}}^{m_{j-1}^{-1}}\frac{\phi(u)}{u^{k+r+1}}du$$

$$= c(k,r)h^k\int\limits_{m_J^{-1}}^{2/N}\frac{\phi(u)}{u^{k+r+1}}du \le c(k,r)h^k\int\limits_{h/2}^{1}\frac{\phi(u)}{u^{k+r+1}}du$$

$$\le c(k,r)h^k\int\limits_{h}^{1}\frac{\phi(u)}{u^{k+r+1}}du. \tag{5.4}$$

Note that

$$\int\limits_0^h\frac{\phi(u)}{u^{r+1}}du + h^k\int\limits_h^1\frac{\phi(u)}{u^{k+r+1}}du \le \int\limits_0^t\frac{\phi(u)}{u^{r+1}}du + t^k\int\limits_t^1\frac{\phi(u)}{u^{k+r+1}}du, \quad h\le t.$$

Finally, we arrive at the estimate

$$\left|\Delta^k_{h\varphi(x)}(Q^{(r)},x)\right| \le h^k\left\|Q^{(k+r)}\right\| \le 2N^{2(k+r)}h^kE_{k+r}(f), \tag{5.5}$$

which follows from Markov's inequality. Note that if $N = k + r,$ then $Q \equiv 0$ and, therefore, the left-hand side of inequality (5.5) vanishes and no estimates are required.

Finally, the fact that $\Delta^k_{h\varphi(x)}(P^{(r)}_{k+r},x) = 0$ combined with inequalities (5.3), (5.4), and (5.5) completes the proof of the theorem for $r \ge 1$.

For $r = 0,$ we can write

$$f = P_k + Q + \sum_{j=0}^{J}Q_j + (f - P_{m_{J+1}}),$$

where $Q := P_N - P_k$ and $Q_j := P_{m_{j+1}} - P_{m_j}$ [see (5.1)]. As above, the proof is completed by using inequalities (5.4) and (5.5) and the following inequality:

$$h^k \int_h^1 \frac{\phi(u)}{u^{k+1}} du \le 3t^k \int_t^1 \frac{\phi(u)}{u^{k+1}} du, \quad h \le t \le \frac{1}{2}.$$

Theorem 3.2 is proved.

Remark 5.1. In the definition of the modulus $\omega_{k,r}^{\varphi}$ in the present paper, we use the weight $W(x,\mu)$ from (3.2), where $\mu = kh/2$. Note that it is also possible to use the weights (see [4, 10])

$$W_1(x,\mu) := \left((1-\mu\varphi(x))^2 - x^2\right)^{1/2},$$

or (see [2])

$$W_2(x,\mu) := \left(\varphi^2(x) - \mu\varphi(x)(1+|x|)\right)^{1/2},$$

which would yield equivalent definitions of the modulus $\omega_{k,r}^{\varphi}$ because, for $\mu \in (0,1)$ and $x\colon |x| + \mu\varphi(x) < 1$, we have

$$\sqrt{\frac{1-\mu}{1+\mu}}\, W \le W_1 \le W_2 \le W.$$

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