MECH 3502 MARCH 30, 2020

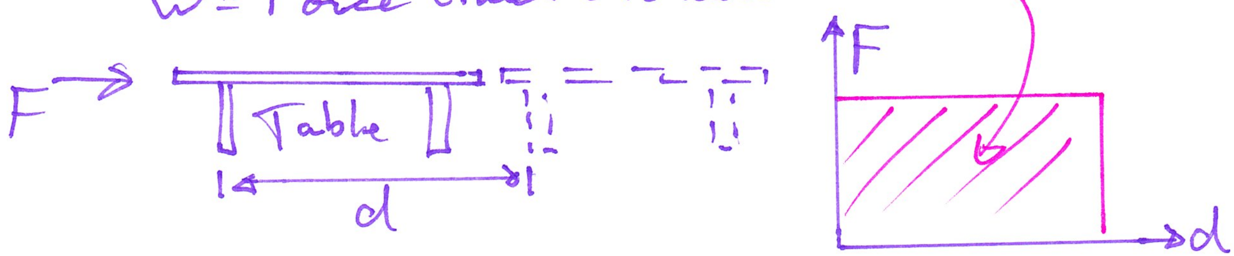
Development & Application of Castigliano's 2^{nd} Theorem

"If the strain energy of a linear elastic structure can be expressed as a fcn of the generalized force Q then the partial derivative of the strain energy with respect to the generalized force gives the generalized displacement V in the direction of Q".

First we need to understand the concept of Internal Strain Energy

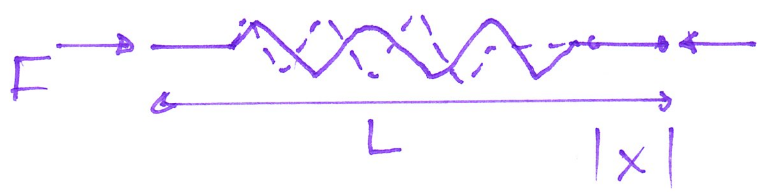
A form of potential energy (work) stored in the deformation of a body due to an applied loading

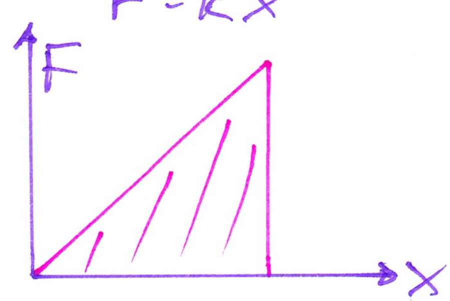
Your first concept of work is likely
W = Force times distance $W = F \cdot d$



Now consider the work done in compressing a simple spring

Spring Equation
$F = kx$



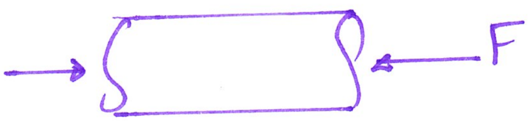


$$\text{Work done} = \frac{Fx}{2} = \frac{F^2}{2k}$$

Since the spring is assumed to be linear, the work done has to equal the energy stored in the relative deformation of the spring

$$\text{Strain Energy, } U = \frac{Fx}{2}$$

Now consider the compression of a bar of material



$$\text{Stress} = F/A$$

$$\text{Strain} = \frac{\Delta L}{L} \leftarrow x \text{ from spring eqn}$$

$$\text{So } U = \frac{Fx}{2} = \frac{\sigma \cdot A \cdot \varepsilon \cdot L}{2} = \frac{\sigma \cdot \varepsilon}{2} A \cdot L$$

This is volume

$$= \frac{\sigma \cdot \varepsilon}{2} \text{Vol}$$

Now in general we require 6 components of stress, $\sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{xz}$, and τ_{xy} to uniquely characterize the state of stress at every material point, and the same for strain

So the strain energy density at each point in a material is

$$u = \frac{1}{2} \sigma^T \varepsilon$$

Lowercase

Tensorial quantities with all 6 components of σ and ε

And the total Strain Energy becomes

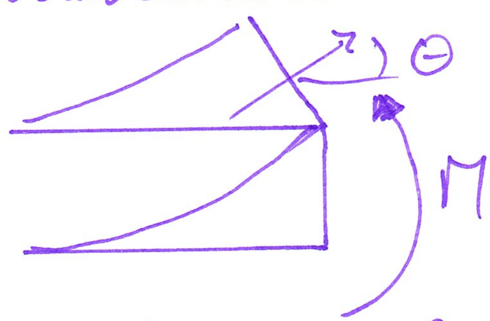
$$U = \int_V \frac{1}{2}\sigma^T \varepsilon \, dV$$

So for a bar in tension (or compression)

$$U = \frac{F^2 L}{2EA} = \int_0^L \frac{F^2}{2EA}\,dx$$ if E, A, and F depend on position

Notice that the strain energy will be positive regardless of whether the bar is stretched or compressed.

For a beam in bending



$$\sigma = -\frac{My}{I}$$

$$\varepsilon = \sigma/E = -\frac{My}{EI}$$

$$U = \int_{Vol} \frac{\sigma\varepsilon}{2}\,dV = \int_{Vol} \frac{M^2}{2EI^2}\,y^2\,dV = \int_0^L \frac{M^2}{2EI^2}\,dx \underbrace{\int_A y^2\,dA}_{=I}$$

$$\text{So } U = \int_0^L \frac{M^2}{2EI}\,dx$$

and for torsion $U = \int T\,d\theta$ and thus

$$U = \int_0^L \frac{T^2}{2GJ}\,dx$$

These expressions are all really similar. I'll write them one more time so you can try to recognize the differences:

$$\text{Bar} \quad U = \int_0^L \frac{F^2}{2EA}\,dx \qquad \text{Beam} \quad U = \int_0^L \frac{M^2}{2EI}\,dx \qquad \text{Shaft} \quad U = \int_0^L \frac{T^2}{2GJ}\,dx$$

All expressions have the square of the loading on top so will be always positive and the "stiffness" of the Bar, Beam, or Shaft on the bottom, specifically the material stiffness, E or G and the geometric stiffness A, I, or J.

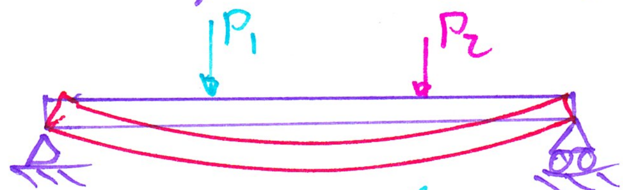
There is one more concept that we need to understand and that is:

the Reciprocal Work Theorem

If an elastic body is subjected to two systems of forces, then the work that would be done by the first system of forces in acting through the second displacements due to the second system of forces is equal to the work that would be done by the second system of forces acting through the displacements due to the first system of forces

Enrico Betti & Lord Rayleigh
~1872

Consider the following simply supported Beam



Case A, applying P_1 first: $U_A = \frac{P_1 v_{1,1}}{2}$

and now apply P_2 $\quad U_A = \frac{P_1 v_{1,1}}{2} + \frac{P_2 v_{2,2}}{2} + P_1 v_{1,2}$

$v_{1,1}$ is the displacement of point 1 due to load 1
$v_{2,2}$ is the displacement of point 2 due to load 2
and $v_{1,2}$ is the displacement of point 1
which moves <u>more</u> due to load 2.

Case B applying P_2 first: $\quad U_B = \frac{P_2 v_{2,2}}{2}$

and now apply P_1 $\quad U_B = \frac{P_2 v_{2,2}}{2} + \frac{P_1 v_{1,1}}{2} + P_2 v_{2,1}$

These expressions must be the same since the system is assumed to be linear so

$$P_1 v_{1,2} = P_2 v_{2,1}$$

Now if we consider the normalized displacements such that

$$f_{11} = \frac{v_{11}}{P_1}, \quad f_{22} = \frac{v_{22}}{P_2},$$

$$f_{12} = \frac{v_{12}}{P_2} \quad \text{and} \quad f_{21} = \frac{v_{21}}{P_1}$$

Then for case A we get

$$U_A = \frac{f_{11}P_1^2}{2} + \frac{f_{22}P_2^2}{2} + P_1 P_2 f_{12}$$

and for case B we get

$$U_B = \frac{f_{22}P_2^2}{2} + \frac{f_{11}P_1^2}{2} + P_1 P_2 f_{21}$$

and again since $U_A = U_B$ we get $f_{12} = f_{21}$

Now if we differentiate each expression as

$$\frac{\partial U_A}{\partial P_1} = P_1 f_{11} + P_2 f_{12} = v_{1,1} + v_{1,2} = v_1$$

or

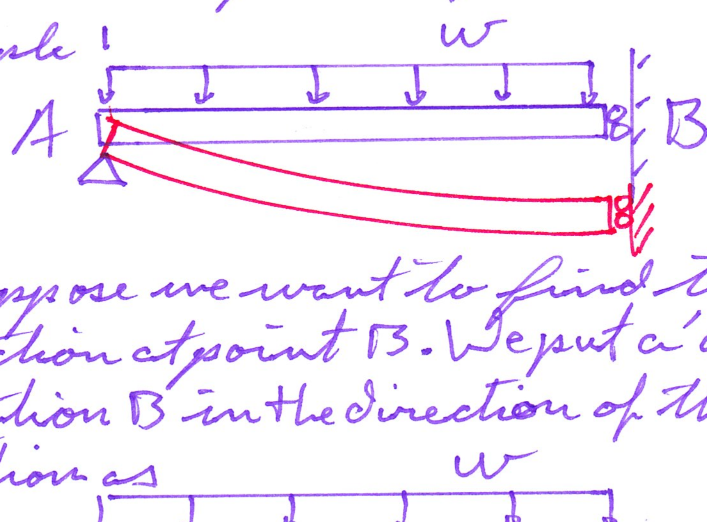
$$\frac{\partial U_B}{\partial P_2} = P_2 f_{22} + P_1 f_{21} = v_{2,2} + v_{2,1} = v_2$$

or simply $\dfrac{\partial U}{\partial P_i} = v_i$

This is Castigliano's 2nd Theorem

Let's look at a few examples

Example 1



Suppose we want to find the deflection at point B. We put a 'dummy force' at location B in the direction of the anticipated deflection as

Q ← dummy force

The moment as a function of position along the length of the beam is:

$$M(x) = -wLx + \int_0^x wx\,dx + Q(L-x)$$

$$= -wLx + \frac{wx^2}{2} + Q(L-x)$$

$U = \int_0^L \frac{M^2}{2EI}dx$ and we are going to need to diferentiate this w.r.t. Q as

$\frac{\partial U}{\partial Q} = \frac{\partial}{\partial Q}\int_0^L \frac{M^2}{2EI}dx$ where M is a fcn of Q

so $\frac{\partial U}{\partial Q} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial Q}dx$

Notice the 2 is gone since $\frac{\partial M^2}{\partial Q} = 2M\frac{\partial M}{\partial Q}$

$$\frac{\partial U}{\partial Q} = \int_0^L \left[\frac{-wLx + \frac{wL^2}{2} - Q(L-x)}{EI}\right]\left[-(L-x)\right]dx$$

Incidentaly, this dummy force Q isn't actually there, so it turns out we can set $Q=0$ and then do the integration and we get

$\frac{dU}{\partial Q} = \frac{5}{24}\frac{wL^4}{EI} = v$ displacement at location B

Formal Procedure for using Castigliano's 2^{nd} Theorem

Step 1 Put a dummy force (or moment) at the point and in the direction that the displacement (or rotation) component is to be found. (or in the case of a statically indeterminate structure, replace a reaction force with a dummy force)

Step 2 Use equilibrium to determine the bending moments, torques, etc. in the structure

Step 3 Diferentiate the Force, Moment, Torque with respect to the dummy force (or momet)

Step 4 Assemble the $\frac{\partial U}{\partial Q}$ (or $\frac{\partial U}{\partial M}$)

Steps 5a to find displacements: set Q, the dummy force, to zero and evaluate the integral to get the displacements in the direction of Q

Steps 5b to find reaction forces: evaluate the integral and set the resulting expression for the displacement to zero. Solve for the dummy load which represents the reaction force.

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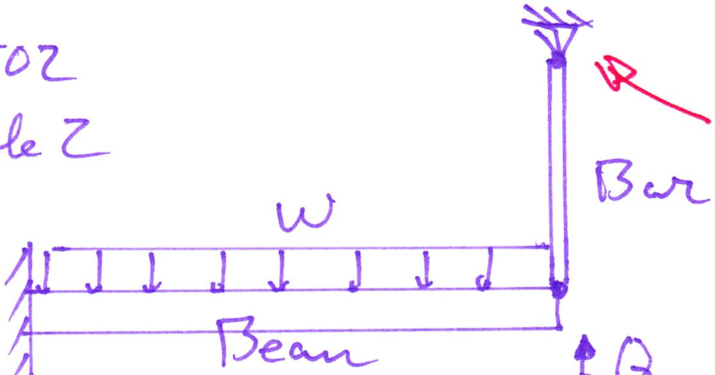
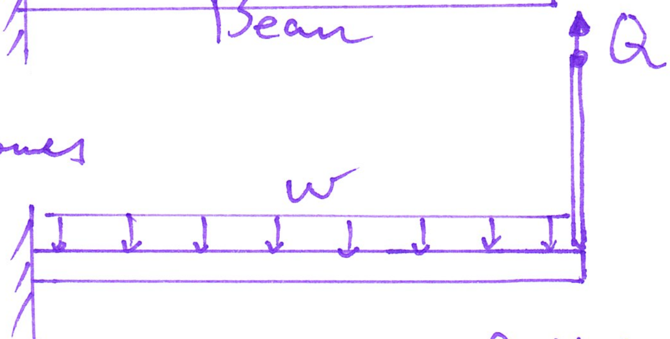
Example 2

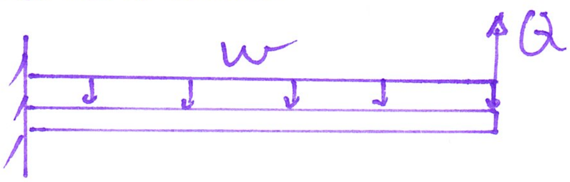


replace this support with a dummy force

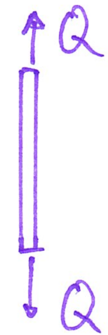
So it becomes



for the beam



for the bar



$$M_{beam} = Qx - \frac{wx^2}{2} \qquad F_{bar} = Q$$

$$\frac{\partial M}{\partial Q} = x \qquad \frac{\partial F}{\partial Q} = 1$$

$$\frac{\partial U}{\partial Q} = \frac{1}{EI}\int_{L_{beam}} \left(Qx - \frac{wx^2}{2}\right) x\,dx + \frac{1}{EA}\int_{L_{bar}} Q\,dx$$

Note: the strain energy is a scalar quantity and hence we can simply add pieces together

$$= \frac{\frac{QL_{beam}^3}{3} - \frac{wL_{beam}^4}{8}}{EI} + \frac{QL_{bar}}{EA}$$

And thus we get the reaction $\Rightarrow Q = \dfrac{\frac{wL_{beam}^4}{8EI}}{\frac{L_{bar}}{EA} + \frac{L_{beam}^3}{3EI}}$

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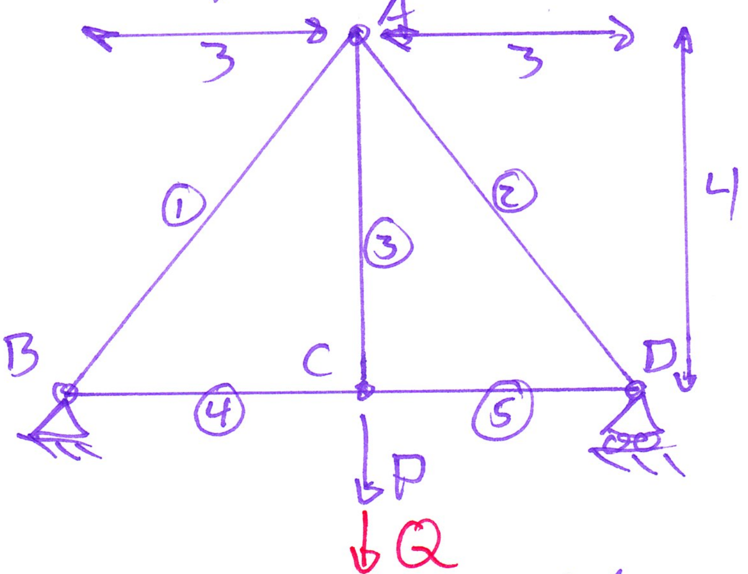
Example 3 truss



$L_1 = L_2 = 5$

$L_3 = 4$

$L_4 = L_5 = 3$

EA = same for all

applying equilibrium to determine the forces in each truss member

$$F_3 = P + Q \qquad F_1 = F_2 = -\frac{5}{4}\left(\frac{P}{2} + \frac{Q}{2}\right) = -\frac{5}{8}(P+Q)$$

and $F_4 = F_5 = \frac{3}{8}(P+Q)$

Now Castigliano's theorem $\displaystyle v = \frac{\partial U}{\partial Q} = \int_L \frac{F}{EA}\frac{\partial F}{\partial Q}\,dx$

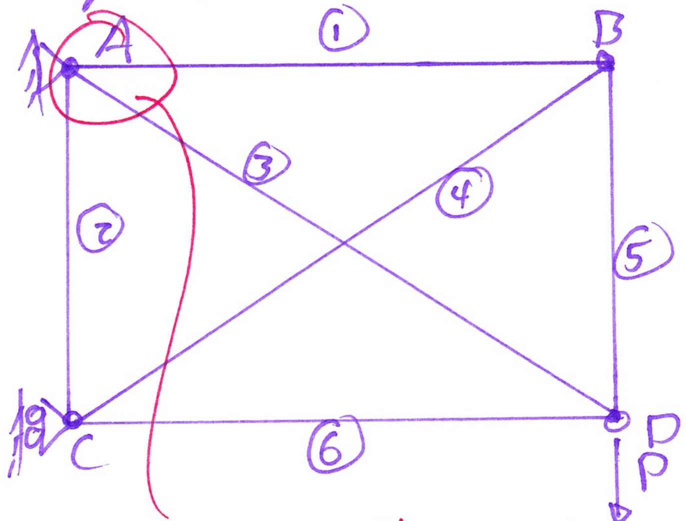
applying this for each member and add them up

$$v = \frac{1}{EA}\Big[\underbrace{\Big(\underbrace{-\frac{5}{8}(P+\cancel{Q}^{\to 0})}_{F}\Big)\underbrace{\Big(-\frac{5}{8}\Big)}_{\partial F/\partial Q}\underbrace{5}_{L}}_{\text{for member 1}} + -\frac{5}{8}(P+\cancel{Q}^{\to 0})\Big(-\frac{5}{8}\Big)5 + (P+\cancel{Q}^{\to 0})(1)\,4 + \underbrace{2\cdot\frac{3}{8}(P+\cancel{Q}^{\to 0})\Big(\frac{3}{8}\Big)3}_{\text{members 4 and 5}}\Big]$$

displacement @ C $= \dfrac{35}{4EA}P$ in the direction of Q

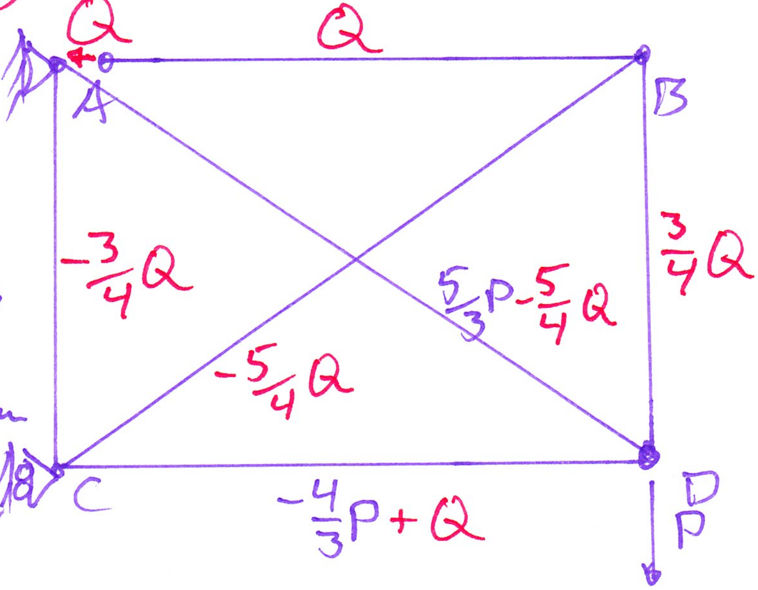
This example takes advantage of a table to help keep track of all the Members

Example 4



This is a statically indeterminate structure, too many unknowns to use equilibrium only to determine the internal forces in each member.

Break up this joint and put in a dummy force to make it statically determinate



We can now determine the internal forces in each member of the truss using equilibrium ΣF and ΣM

And construct the following table

Multiply this way

Member	F	$\frac{\partial F}{\partial Q}$	L	$\frac{1}{EA}$	$F\frac{\partial F}{\partial Q}L\frac{1}{EA}$
1	Q	1	L	$1/EA$	QL/EA
2	$-\frac{3}{4}Q$	$-\frac{3}{4}$	$\frac{3}{4}L$	"	$\frac{27}{64}QL/EA$
3	$\frac{5}{3}P-\frac{5}{4}Q$	$-\frac{5}{4}$	$\frac{5}{4}L$		|
4	$-\frac{5}{4}Q$	$-\frac{5}{4}$	$\frac{5}{4}L$	"	|
5	$\frac{3}{4}Q$	$\frac{3}{4}$	$\frac{3}{4}L$		|
6	$-\frac{4}{3}P+Q$	1	L	"	

add these up

$$\sum F\frac{\partial F}{\partial Q}L\frac{1}{EA} =$$

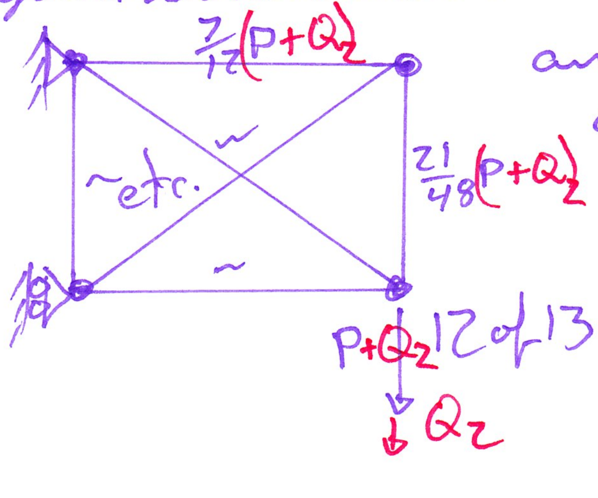
So displacement at A

$$= \frac{1}{EA}\left[-\frac{63}{16}PL + \frac{27}{4}QL\right] = 0$$

Since joint A was originally connected

and thus $Q = \frac{7}{12}P$

Now we can put this back into the original structure as



and apply a New dummy force @ D to solve for the deflection @ D

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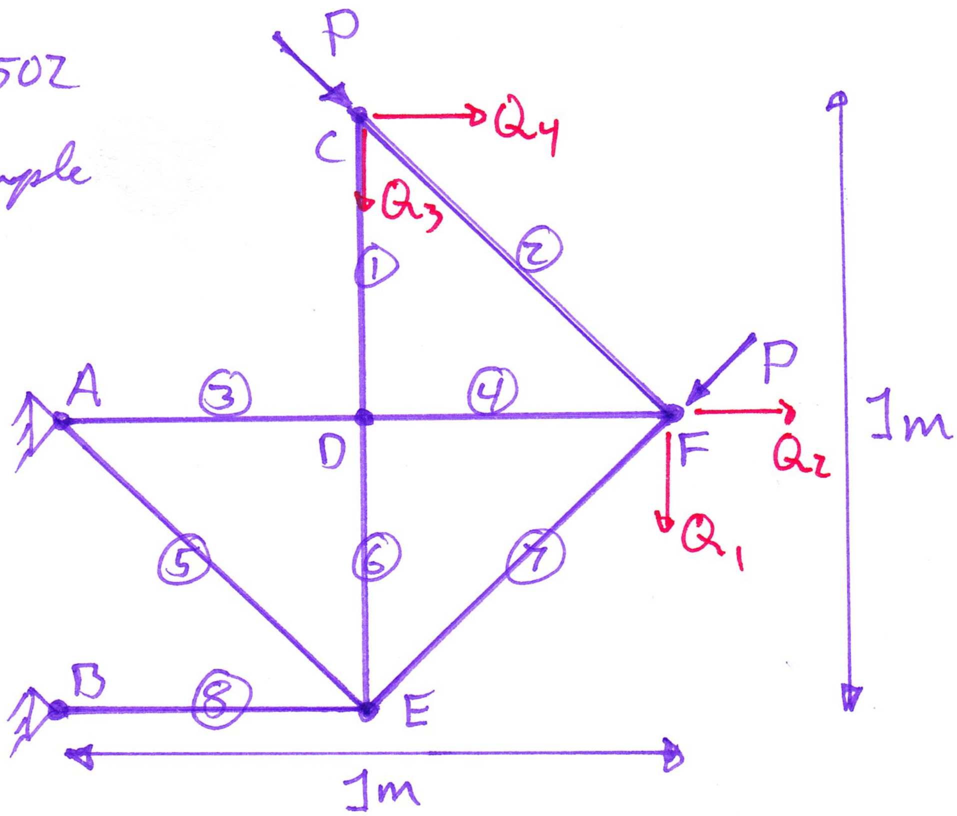
Example



We'll do these dummy forces one at a time to determine the magnitude of the deflection at the locations of the applied loads.

Note: This example is being left as an excercise for the reader. Did you really think I would be solving this for you? Also my last two lecture traditionaly involve doing review examples in preparation for the final, which we will not be doing, so get to work on those projects, practice social distancing, take care of yourselves, friends, and family and we'll see you in MECH 3652!