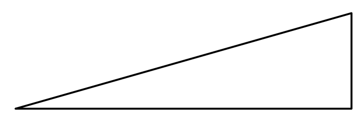
Chapter 6. Isoparametric Formulation

Same function that is used to define the element geometry is used to define the displacements within the element

2 Node Truss Element

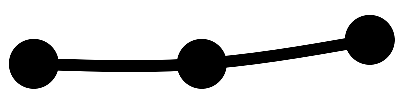
Linear geometry

Linear displacements

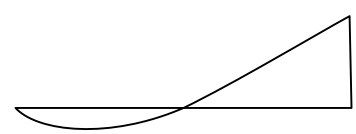


3 Node Beam Element

Quadratic geometry



Quadratic displacements



We assign the same local coordinate system to each element. This coordinate system is called the ***natural*** coordinate system.

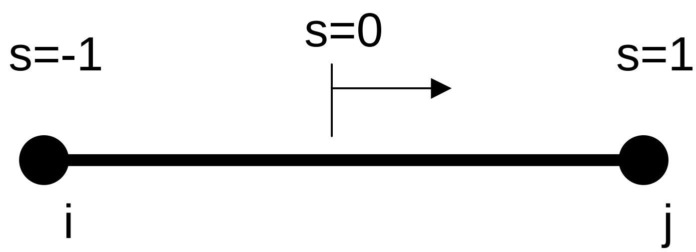
The advantage of choosing this coordinate system is 1) it is easier to define the shape functions and 2) the integration over the surface of the element is easier (we will use numerical integration which is much simpler in the natural coordinate systems and can be scaled to the actual area)

The steps in deriving the elemental stiffness matrices are the same:

- Step 1 Select element type
- Step 2 Select displacement function
- Step 3 Define strain/displacement, stress/strain relation
- Step 4 Derive element stiffness matrix and equations

1-D Truss Elements

For 1-D linear truss elements the natural coordinate system for an element is:



The natural coordinates are related to the global coordinates through

$$x = a_1 + a_2 s$$

which we can solve for the a's as:

$$x = \frac{1}{2}\left[(1-s)x_1 + (1+s)x_2\right]$$

or in matrix form as:

$$\mathbf{x} = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

where

$$N_1 = \frac{1-s}{2} \qquad N_2 = \frac{1+s}{2}$$

Now following the remainder of the steps becomes much simpler.

Step 2 Select a displacement function

$$\mathbf{u} = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix}$$

Step 3 Define $\mathbf{u}/\varepsilon$ and ε/σ relations

Recall that we had the following relation:

$$\varepsilon_x(x) = \frac{du}{dx}$$

Then by applying the chain rule of differentiation, we have

$$\varepsilon_x(x) = \frac{du}{ds} \bigg/ \frac{dx}{ds}$$

Thus,

$$\boldsymbol{\varepsilon}_{\mathbf{x}} = \begin{bmatrix} \frac{-1}{L} & \frac{1}{L} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \mathbf{Bu}$$

The stress/strain relation is expressed as:

$$\sigma_{\mathbf{x}} = \mathbf{D}\boldsymbol{\varepsilon}_{\mathbf{x}} \qquad \text{where} \qquad \mathbf{D} = E$$

Thus: $\boldsymbol{\sigma}_{\mathbf{x}} = E\mathbf{B}\,\mathbf{u}$

Step 4. Derive the element stiffness matrix and equations

The stiffness matrix is

$$\mathbf{K}^{(e)} = \int_L AE\, \mathbf{B}^T \mathbf{B}\ dx$$

which has an integral over x which we have to convert to an integral over s. This is done through the transformation:

$$\int_0^L f(x)dx = \int_{-1}^{1} f(s)\left|J\right|ds$$

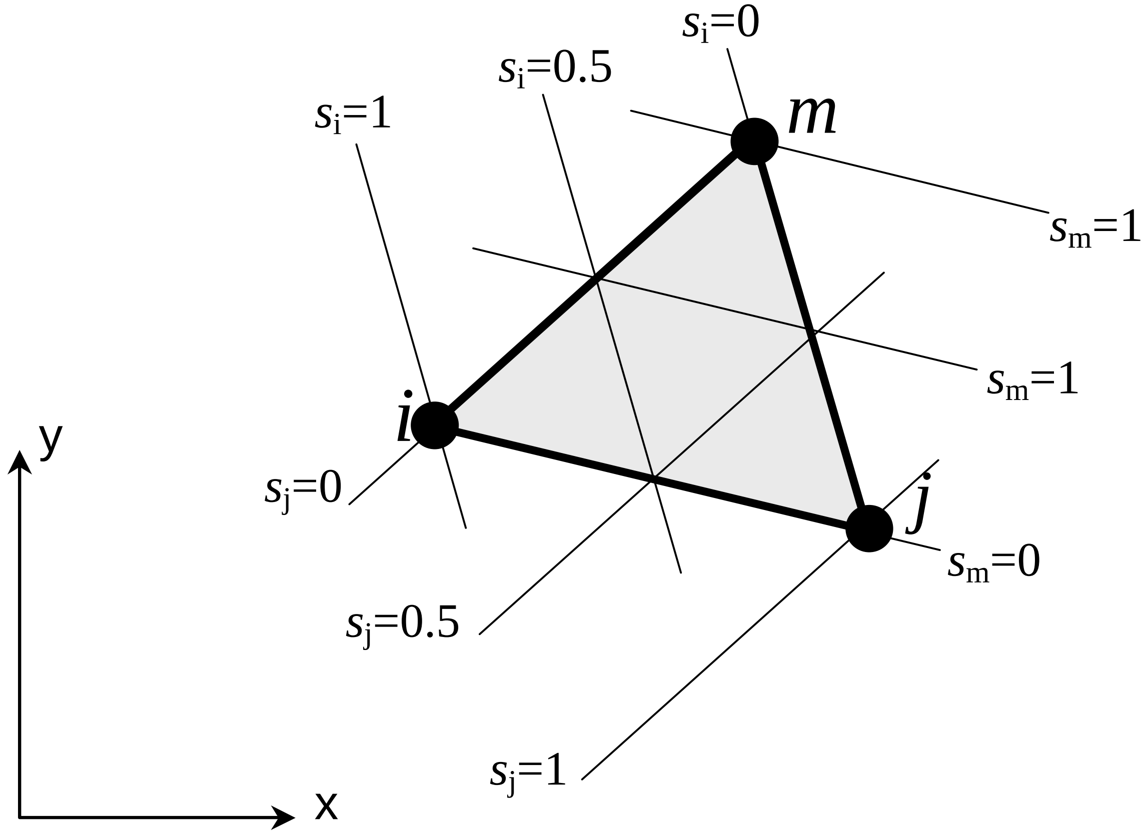
where $\left|J\right|$ is the Jacobian and for the simple truss element it is:

$$\left|J\right| = \frac{dx}{ds} = L/2$$

And Voila!!

$$\mathbf{K}^{(e)} = AE\int_{-1}^{1}\left(\begin{Bmatrix} \frac{-1}{L} \\ \frac{1}{L} \end{Bmatrix}\begin{bmatrix} \frac{-1}{L} & \frac{1}{L} \end{bmatrix}\frac{L}{2}\right)ds = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

CST Elements



we choose a natural coordinate system as shown and define the geometry in terms of the natural coordinate system as:

$$x = s_1 x_i + s_2 x_j + s_3 x_m$$

$$y = s_4 y_i + s_5 y_j + s_6 y_m$$

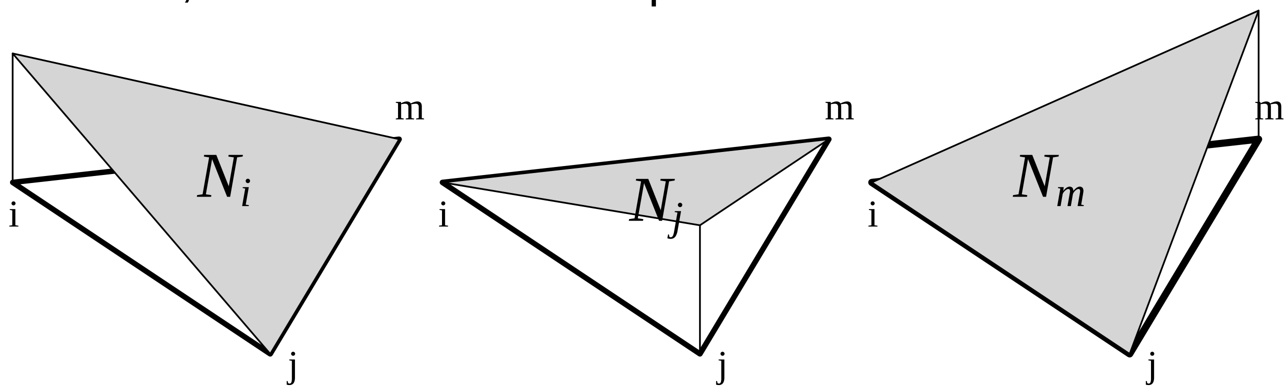
which we can write in matrix form as:

$$\begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_i & x_j & x_m \\ y_i & y_j & y_m \end{bmatrix} \begin{Bmatrix} s_1 \\ s_2 \\ s_3 \end{Bmatrix}$$

Which can be solved as:

$$\begin{Bmatrix} s_1 \\ s_2 \\ s_3 \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_i & x_j & x_m \\ y_i & y_j & y_m \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \alpha_i & \beta_i & \gamma_i \\ \alpha_j & \beta_j & \gamma_j \\ \alpha_m & \beta_m & \gamma_m \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$

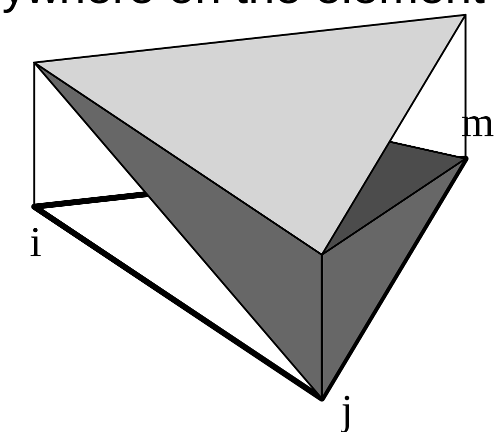
In this case, these s's are the shape functions



$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \begin{Bmatrix} x_i \\ y_i \\ x_j \\ y_j \\ x_m \\ y_m \end{Bmatrix}$$

The sum of the shape functions anywhere on the element add to 1

$$N_i + N_j + N_m = 1$$



Note that in this case, the N_i's are simply s_1 s_2 and s_3.

Step 2: Choose the displacement function

we can simply write the element displacement as a function of nodal dof in the same form as used to describe the geometry:

$$\begin{Bmatrix} u_x(x,y) \\ u_y(x,y) \end{Bmatrix} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \begin{Bmatrix} u_{xi} \\ u_{yi} \\ u_{xj} \\ u_{yj} \\ u_{xm} \\ u_{ym} \end{Bmatrix} \quad \text{or} \quad \mathbf{\Psi} = \mathbf{N}\mathbf{u}$$

Step 3: Strain/displacement and stress/strain relations

In 2-D the strain displacement relations are:

$$\varepsilon_x = \frac{\partial u_x}{\partial x}, \quad \varepsilon_y = \frac{\partial u_y}{\partial y}, \quad \text{and} \quad \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

or in matrix form as:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \begin{Bmatrix} u_{xi} \\ u_{yi} \\ u_{xj} \\ u_{yj} \\ u_{xm} \\ u_{ym} \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & \frac{\partial N_j}{\partial x} & 0 & \frac{\partial N_m}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial y} & 0 & \frac{\partial N_j}{\partial y} & 0 & \frac{\partial N_m}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & \frac{\partial N_j}{\partial y} & \frac{\partial N_j}{\partial x} & \frac{\partial N_m}{\partial y} & \frac{\partial N_m}{\partial x} \end{bmatrix} \begin{Bmatrix} u_{xi} \\ u_{yi} \\ u_{xj} \\ u_{yj} \\ u_{xm} \\ u_{ym} \end{Bmatrix}$$

$$\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{u}$$

In 2-D the stress/strain relations are:

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} = \mathbf{D}\mathbf{B}\mathbf{u}$$

Where $\mathbf{D}$ depends on whether plane stress or plane strain conditions prevails(see Chapter 5 for details)

Since the shape functions are functions of the natural coordinate s_i and not x and y, we apply the chain rule as:

$$\frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial s_1}\frac{\partial s_1}{\partial x} + \frac{\partial N_i}{\partial s_2}\frac{\partial s_2}{\partial x} + \frac{\partial N_i}{\partial s_3}\frac{\partial s_3}{\partial x} \qquad \frac{\partial N_i}{\partial y} = \frac{\partial N_i}{\partial s_1}\frac{\partial s_1}{\partial y} + \frac{\partial N_i}{\partial s_2}\frac{\partial s_2}{\partial y} + \frac{\partial N_i}{\partial s_3}\frac{\partial s_3}{\partial y}$$

Let us consider the following

$$\mathbf{B}_o = \begin{bmatrix} \frac{\partial N_1}{\partial s_1} & \frac{\partial N_2}{\partial s_1} & \frac{\partial N_3}{\partial s_1} \\ \frac{\partial N_1}{\partial s_2} & \frac{\partial N_2}{\partial s_2} & \frac{\partial N_3}{\partial s_2} \\ \frac{\partial N_1}{\partial s_3} & \frac{\partial N_2}{\partial s_3} & \frac{\partial N_3}{\partial s_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

then for the derivatives of the shape functions with respect to the global coordinate system we simply have:

$$\frac{\partial N_i}{\partial x} = \frac{\partial s_i}{\partial x} = \beta_i, \quad \frac{\partial N_j}{\partial x} = \frac{\partial s_j}{\partial x} = \beta_j, \quad \frac{\partial N_m}{\partial x} = \frac{\partial s_m}{\partial x} = \beta_m$$

$$\frac{\partial N_i}{\partial y} = \frac{\partial s_i}{\partial y} = \gamma_i, \quad \frac{\partial N_j}{\partial y} = \frac{\partial s_j}{\partial y} = \gamma_j, \quad \frac{\partial N_m}{\partial y} = \frac{\partial s_m}{\partial y} = \gamma_m$$

And the strains are written as:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix} \begin{Bmatrix} u_{xi} \\ u_{yi} \\ u_{xj} \\ u_{yj} \\ u_{xm} \\ u_{ym} \end{Bmatrix}$$

Step 4. Derive the element stiffness matrix and equations

Lastly, we use the PMPE to obtain the stiffness equations as:

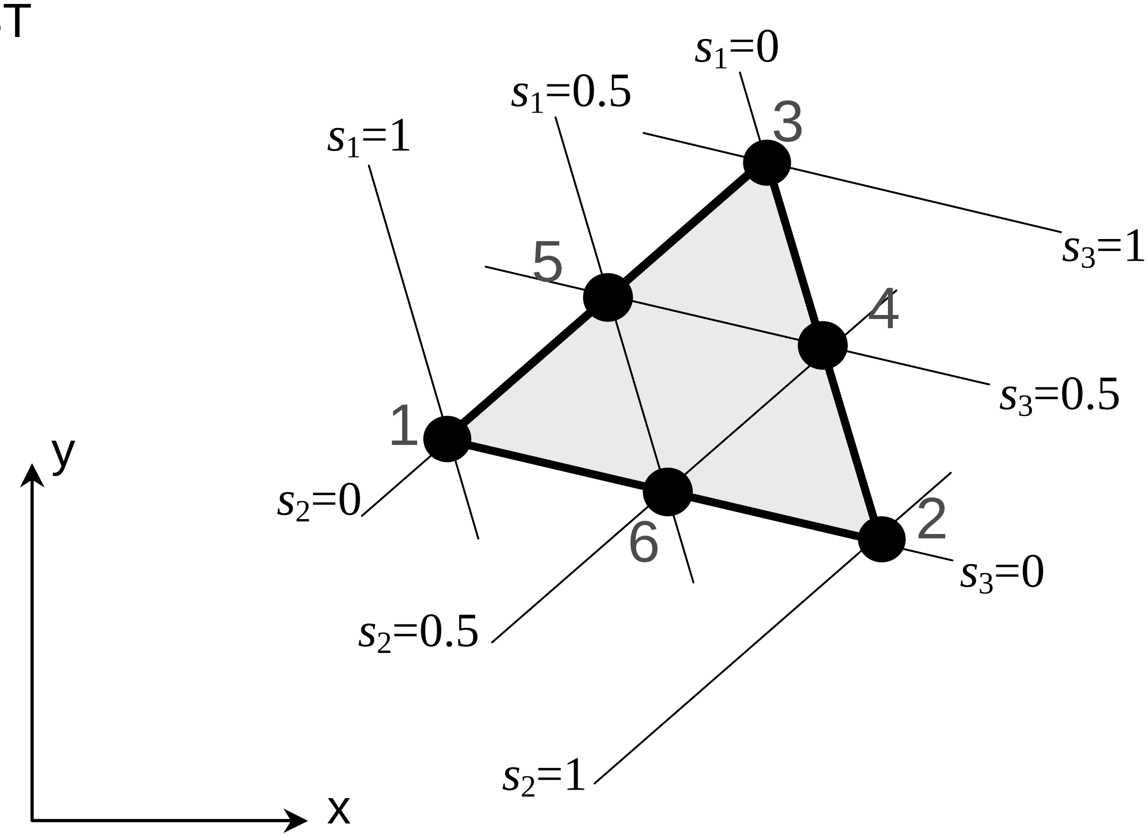
$$\iiint_V \mathbf{B}^T \mathbf{D} \mathbf{B} \mathbf{u}\, dv - \mathbf{P} - \iiint_V \mathbf{N}^T \mathbf{X}_{body}\, dV - \iint_S \mathbf{N}^T \mathbf{T}_{tract}\, dS = 0$$

Since all the terms in $\mathbf{B}$ are constant and assuming the thickness and material properties are constant over the element we have:

$$\mathbf{K}\,\mathbf{u} = \mathbf{f} \qquad \text{where} \qquad \mathbf{K} = tA\mathbf{B}^T\mathbf{D}\mathbf{B}$$

Linear Strain Triangle (LST) Elements

Again we choose the same natural coordinate system as for the CST



we define the geometry in terms of the natural coordinate system through the shape functions as:

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4 + N_5 x_5 + N_6 x_6$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4 + N_5 y_5 + N_6 y_6$$

Which we can solve for the shape functions in terms of the natural coordinates as follows:

Let N_i be a quadratic function of s_1 and s_2 (Nt we can express s_3 as a function of s_1 and s_2 as: $s_3=1-s_1-s_2$)

$$N_i = a_{oi} + a_{1i} s_1 + a_{2i} s_2 + a_{3i} s_1^2 + a_{4i} s_2^2 + a_{5i} s_1 s_2$$

which means that there are 6 unknown coefficients to be determined for each shape function

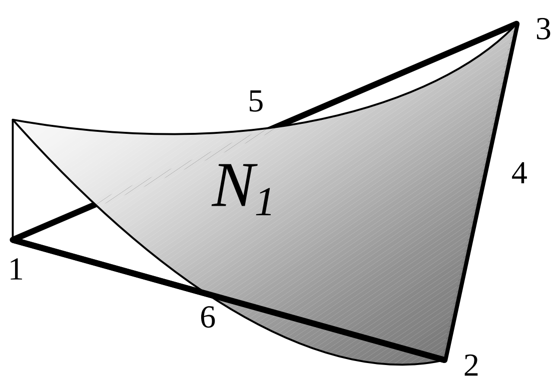
Using the following information that at node i we want N_i=1 and all other $N_{j\neq i}$= 0, then we get 6 equations for each shape function and we can solve for the coefficients and we have:

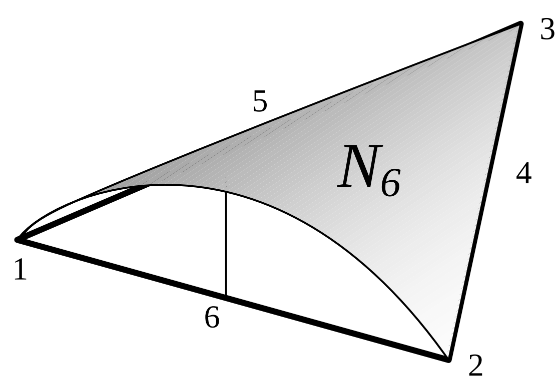
$$N_1 = s_1(2s_1 - 1)$$
$$N_2 = s_2(2s_2 - 1)$$
$$N_3 = s_3(2s_3 - 1)$$
$$N_4 = 4s_2 s_3$$
$$N_5 = 4s_3 s_1$$
$$N_6 = 4s_1 s_2$$

or recognizing that $s_3 = 1 - s_1 - s_2$, then we have

$$N_1 = s_1(2s_1 - 1)$$
$$N_2 = s_2(2s_2 - 1)$$
$$N_3 = (1 - s_1 - s_2)(2(1 - s_1 - s_2) - 1)$$
$$N_4 = 4s_2(1 - s_1 - s_2)$$
$$N_5 = 4(1 - s_1 - s_2)s_1$$
$$N_6 = 4s_1 s_2$$

In this case, these look like





The sum of the shape functions anywhere on the element add to 1

$$N_1 + N_2 + N_3 + N_4 + N_5 + N_6 = 1$$

Incidentally, the shape functions in the global coordinate system for a nice element with sides aligned with the x and y axes would look something like this:

$$N_1 = 1 - 3x/b - 3y/h + 2x^2/b^2 + 4xy/(bh) + 2y^2/h^2$$
$$N_2 = -x/b + 2x^2/b^2$$
$$N_3 = -y/h + 2h^2/h^2$$
$$N_4 = 4xy/(bh)$$
$$N_5 = 4y/h - 4xy/(bh) - 4y^2/h^2$$
$$N_6 = 4x/b - 4xy/(bh) - 4x^2/b^2$$

This is why we use the Isoparametric formulation!!!

Step 2: Choose the displacement function

We can simply write the element displacement as a function of nodal dof in the same form as used to describe the geometry:

$$\begin{Bmatrix} u_x(x,y) \\ u_y(x,y) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{y4} \\ u_{y4} \\ u_{y5} \\ u_{y5} \\ u_{x6} \\ u_{y6} \end{Bmatrix}$$

or $\boldsymbol{\Psi} = \mathbf{Nu}$

Step 3: Strain/displacement and stress/strain relations
In 2-D the strain/displacement relations are:

$$\varepsilon_x = \frac{\partial u_x}{\partial x}, \quad \varepsilon_y = \frac{\partial u_y}{\partial y}, \quad \text{and} \quad \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

or in matrix form as:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u_x \\ u_y \end{Bmatrix}$$

In 2-D the stress/strain relations are:

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} = \mathbf{DBu}$$

Where $\mathbf{D}$ depends on whether plane stress or plane strain conditions prevails(see Chapter 5 for details)

So how do we construct the B matrix?
Let us define the following matrix

$$\mathbf{B}_o = \begin{bmatrix} \frac{\partial N_1}{\partial s_1} & \frac{\partial N_2}{\partial s_1} & \frac{\partial N_3}{\partial s_1} & \frac{\partial N_4}{\partial s_1} & \frac{\partial N_5}{\partial s_1} & \frac{\partial N_6}{\partial s_1} \\ \frac{\partial N_1}{\partial s_2} & \frac{\partial N_2}{\partial s_2} & \frac{\partial N_3}{\partial s_2} & \frac{\partial N_4}{\partial s_2} & \frac{\partial N_5}{\partial s_2} & \frac{\partial N_6}{\partial s_2} \end{bmatrix}$$

$$= \begin{bmatrix} 4s_1 - 1 & 0 & 4s_1 + 4s_2 - 3 & -4s_2 & 4 - 8s_1 - 4s_2 & 4s_2 \\ 0 & 4s_1 - 1 & 4s_1 + 4s_2 - 3 & 4 - 4s_1 - 8s_2 & -4s_1 & 4s_1 \end{bmatrix}$$

and let the Jacobian matrix be (note that it is a 2by2)

$$\mathbf{J} = \begin{bmatrix} \frac{\partial N_1}{\partial s_1} & \frac{\partial N_2}{\partial s_1} & \frac{\partial N_3}{\partial s_1} & \frac{\partial N_4}{\partial s_1} & \frac{\partial N_5}{\partial s_1} & \frac{\partial N_6}{\partial s_1} \\ \frac{\partial N_1}{\partial s_2} & \frac{\partial N_2}{\partial s_2} & \frac{\partial N_3}{\partial s_2} & \frac{\partial N_4}{\partial s_2} & \frac{\partial N_5}{\partial s_2} & \frac{\partial N_6}{\partial s_2} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \\ x_5 & y_5 \\ x_6 & y_6 \end{bmatrix}$$

Then the terms in the B matrix are simply extracted from the product

$$\mathbf{J}^{-1}\mathbf{B}_o = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} & \frac{\partial N_5}{\partial x} & \frac{\partial N_6}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} & \frac{\partial N_5}{\partial y} & \frac{\partial N_6}{\partial y} \end{bmatrix}$$

Step 4. Derive the element stiffness matrix and equations

Lastly, we use the PMPE to obtain the stiffness equations as:

$$\iiint_V \mathbf{B}^T\mathbf{D}\mathbf{B}\mathbf{u}\,dv - \mathbf{P} - \iiint_V \mathbf{N}^T\mathbf{X}_{body}\,dV - \iint_S \mathbf{N}^T\mathbf{T}_{tract}\,dS = 0$$

We use Gaussian quadrature to perform the integration over the element

(Note that B and N in the above are functions of the natural coordinates s_1 and s_2)

Gaussian Quadrature (Numerical Integration)

As we saw, the derivation of the stiffness requires that we perform an integration over the element (this comes from the definition of the internal strain energy and when we assemble the force vector). Often this is difficult to do explicitly, unless you are using Mathematica so we turn to numerical integration techniques.

Note that in the element formulation, we are choosing the function form of the displacement (hence indirectly the form of the strains and stress which appear in the internal strain energy) – The principle behind Gaussian Quadrature is that if we know the functional form of what we are trying to integrate, then there is certain number of points where we need to evaluate the function which will give us an exact representation of the integral.

Gauss Formula:

$$I = \int_{-1}^{1} f(x)\,dx I = \sum_{i=1}^{n_{\text{int}}} W_i \;\; f(x_i)$$

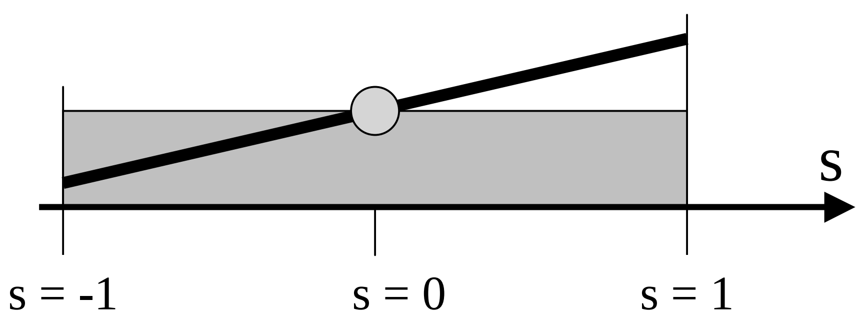
We evaluate an integral by evaluating the function we want to integrate at discrete point n_{int} and multiply this by an appropriate weight

Rule:

n_{int} integration point rule $\quad\Rightarrow\quad$ $2 \bullet n_{int} - 1$ order accuracy

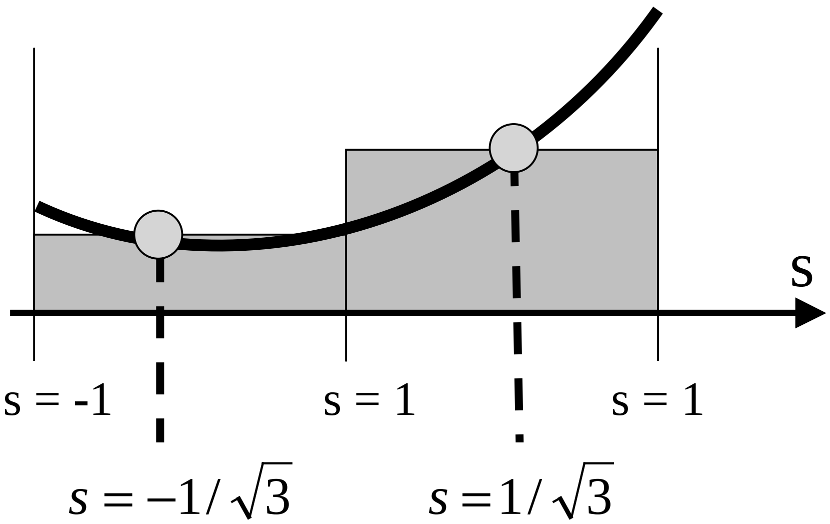
Examples:

1 integration point will integrate a 1st order polynomial exactly



$n_{int} = 1$
$s_1 = 0$
$W_1 = 2$

2 integration points will integrate a 3rd order polynomial exactly



$n_{int} = 2$

$s_{1,2} = \pm 1/\sqrt{3}$

$W_1 = W_2 = 1$

Gauss Formula in 2-Dimensions:

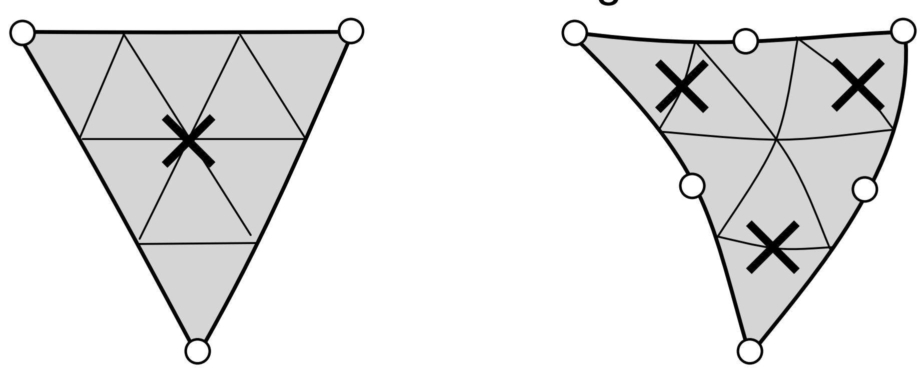
$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\,ds\,dt = \int_{-1}^{1}\left[\sum_{i=1} W_i \; f(s_i,t)\right]dt$$

$$= \left(\sum_j W_j \left[\sum_i W_i \; f(s_i,t)\right]\right)\Bigg|_j = \sum_i \sum_j W_i W_j \; f(s_i,t_j)$$

Example: CST (constant strain triangle)

Here we assumed a linear displacement function – which means that the strain field (and the stress field) is constant over the element. The find the integral of a constant, i.e. the area under the curve, we need only evaluate that function at one point. For a CST, this point is located in the center of the triangle, in the natural coordinate system, this point is located at s_1=s_2=s_3= 0.333 and the corresponding weight is 1.

3-node versus 6-node triangular elements



Returning to the six node LST element, we had **B** and **N** which were expressed in terms of the natural coordinates. For these elements we have 3 Gauss points with location and weights as:

$$
\begin{array}{lll}
gp1 & s_2 = s_3 = 0.1666,\;\; s_1 = 0.666 & W_1 = 0.333 \\
gp2 & s_1 = s_3 = 0.1666,\;\; s_2 = 0.666 & W_2 = 0.333 \\
gp3 & s_1 = s_2 = 0.1666,\;\; s_3 = 0.666 & W_3 = 0.333
\end{array}
$$

This gives us a degree of precision of 2 (integrates a 2nd order polynomial exactly) so we now have for the stiffness matrix

$$
\mathbf{k}^e = t \iint_{x-y} \mathbf{B}^T \mathbf{D} \mathbf{B}\; dA_{x-y\ plane} = t \iint_{s_1-s_2} \mathbf{B}^T \mathbf{D} \mathbf{B} \left|\mathbf{J}\right|\; dA_{s_i\ plane} = \frac{t}{2} \sum_{i=1}^{n} W_i \left(\mathbf{B}^T \mathbf{D} \mathbf{B}\right)_i \left|\mathbf{J}\right|_i
$$

$$
= t\,\frac{0.33}{2}\left(\mathbf{B}^T \mathbf{D} \mathbf{B}\left|\mathbf{J}\right|\right)_{gp1} + t\,\frac{0.33}{2}\left(\mathbf{B}^T \mathbf{D} \mathbf{B}\left|\mathbf{J}\right|\right)_{gp2} + t\,\frac{0.33}{2}\left(\mathbf{B}^T \mathbf{D} \mathbf{B}\left|\mathbf{J}\right|\right)_{gp3}
$$

Note: the factor of ½ comes from the area of the triangle in $s_1 - s_2$ space

where the Jacobian has already been constructed (when we formed the **B** matrix) as:

$$
\left|\mathbf{J}\right| = \det\left|
\begin{bmatrix}
\frac{\partial N_1}{\partial s_1} & \frac{\partial N_2}{\partial s_1} & \frac{\partial N_3}{\partial s_1} & \frac{\partial N_4}{\partial s_1} & \frac{\partial N_5}{\partial s_1} & \frac{\partial N_6}{\partial s_1} \\
\frac{\partial N_1}{\partial s_2} & \frac{\partial N_2}{\partial s_2} & \frac{\partial N_3}{\partial s_2} & \frac{\partial N_4}{\partial s_2} & \frac{\partial N_5}{\partial s_2} & \frac{\partial N_6}{\partial s_2}
\end{bmatrix}
\begin{bmatrix}
x_1 & y_1 \\
x_2 & y_2 \\
x_3 & y_3 \\
x_4 & y_4 \\
x_5 & y_5 \\
x_6 & y_6
\end{bmatrix}
\right|
$$

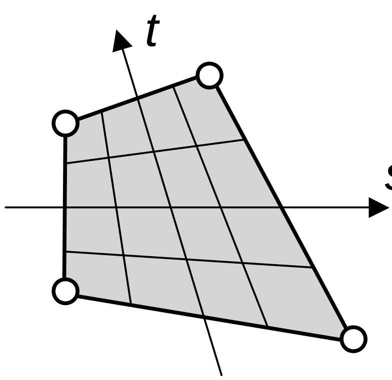
Similarly, we can perform the integrals appearing in the force vector

2-D Brick Elements

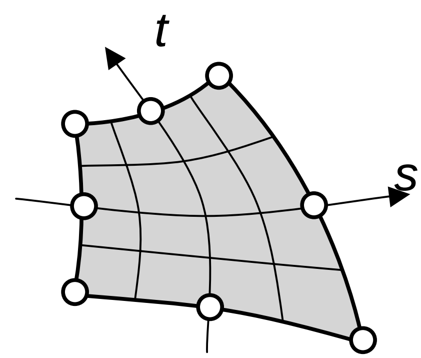
Natural coordinate system

For each element we assign a local coordinate system represented by *s* and *t* which both span the range from –1 to 1 over the area of the element

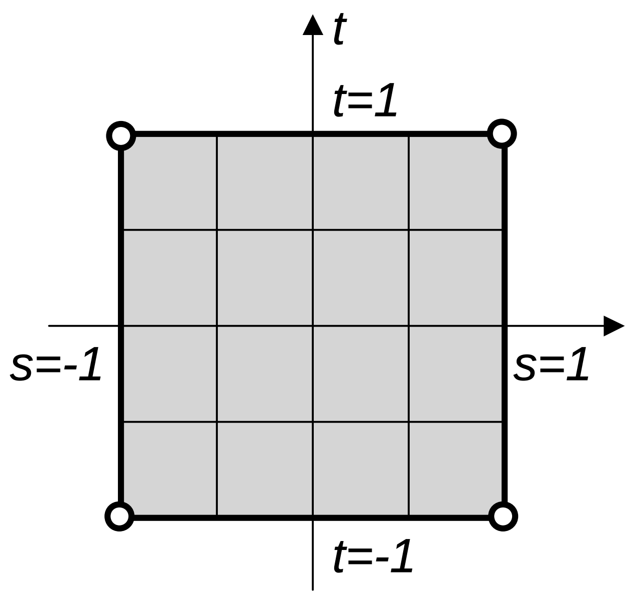
Linear displacement function (4 noded)



Quadratic displacement function (8 noded)



4 Node Bricks – BiLinear Quads



we choose a natural coordinate system as shown and define the geometry in terms of the natural coordinate system as:

$$x = a_1 + a_2 s + a_3 t + a_4 st$$

$$y = b_1 + b_2 s + b_3 t + b_4 st$$

or rather in terms of the shape functions and the nodal coordinates as:

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$

which we can write in matrix form as:

$$\begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{bmatrix} \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{Bmatrix}$$

Here the shape functions are

$$N_1 = \frac{1}{4}(1-s)(1-t)$$

$$N_2 = \frac{1}{4}(1+s)(1-t)$$

$$N_3 = \frac{1}{4}(1+s)(1+t)$$

$$N_4 = \frac{1}{4}(1-s)(1+t)$$

Thus we have

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{Bmatrix}$$

The sum of the shape functions anywhere on the element add to 1

Step 2: Choose the displacement function

we can simply write the element displacement as a function of nodal dof in the same form as used to describe the geometry:

$$\begin{Bmatrix} u_x(x,y) \\ u_y(x,y) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{Bmatrix} \quad \text{or} \quad \mathbf{\Psi} = \mathbf{N}\mathbf{u}$$

Step 3: Strain/displacement and stress/strain relations

Again the 2-D strain displacement relations are:

$$\varepsilon_x = \frac{\partial u_x}{\partial x}, \quad \varepsilon_y = \frac{\partial u_y}{\partial y}, \quad \text{and} \quad \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

or in matrix form as:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{Bmatrix}$$

$$\boldsymbol{\varepsilon} = \mathbf{Bu}$$

In 2-D the stress/strain relations are:

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} = \mathbf{DBu}$$

Where $\mathbf{D}$ depends on whether plane stress or plane strain conditions prevails (again see Chapter 5 for details)

But N_i's are functions of s and t and not x and y so we have to apply the chain rule of differentiation again. This time we have:

$$\frac{\partial N_i}{\partial x} = \frac{\partial N_i}{ds}\frac{\partial s}{\partial x} + \frac{\partial N_i}{\partial t}\frac{\partial t}{\partial x}$$

$$\frac{\partial N_i}{\partial y} = \frac{\partial N_i}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial N_i}{\partial t}\frac{\partial t}{\partial y}$$

Or in matrix form as:

$$\begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} = \begin{bmatrix} \frac{\partial s}{\partial x} & \frac{\partial t}{\partial x} \\ \frac{\partial s}{\partial y} & \frac{\partial t}{\partial y} \end{bmatrix} \begin{Bmatrix} \frac{\partial N_i}{\partial s} \\ \frac{\partial N_i}{\partial t} \end{Bmatrix}$$

but $\frac{\partial s}{\partial x}, \frac{\partial t}{\partial x}, \frac{\partial s}{\partial y}, \text{and } \frac{\partial t}{\partial y}$ are difficult to evaluate, but $\frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}, \frac{\partial x}{\partial t}, \text{and } \frac{\partial y}{\partial t}$ are not so we can write:

$$\frac{\partial N_i}{\partial x} = \frac{1}{|J|}\left(\frac{\partial N_i}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial N_i}{\partial t}\frac{\partial y}{\partial s}\right) \text{ and } \frac{\partial N_i}{\partial y} = \frac{1}{|J|}\left(-\frac{\partial N_i}{\partial s}\frac{\partial x}{\partial t} + \frac{\partial N_i}{\partial t}\frac{\partial x}{\partial s}\right)$$

where the determinant of the Jacobian, $|J|$ is:

$$|J| = \det\begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix} = \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s}$$

So we get a new $\mathbf{B}$ which equals $\mathbf{B}$ but is now a function of s and t.

Step 4. Derive the element stiffness matrix and equations

Lastly, we use the PMPE to obtain the stiffness equations as:

$$t\iint_A \mathbf{B}^T\mathbf{D}\mathbf{B}\mathbf{u}\,dA - \mathbf{P} - \iiint_V \mathbf{N}^T\mathbf{X}_{body}\,dV - \iint_S \mathbf{N}^T\mathbf{T}_{tract}\,dS = 0$$

in which we transform the integrals in the x-y plane to integrals over the s-t plane from –1 to 1 through the transformation and use Gaussian Quadrature to perform the integration

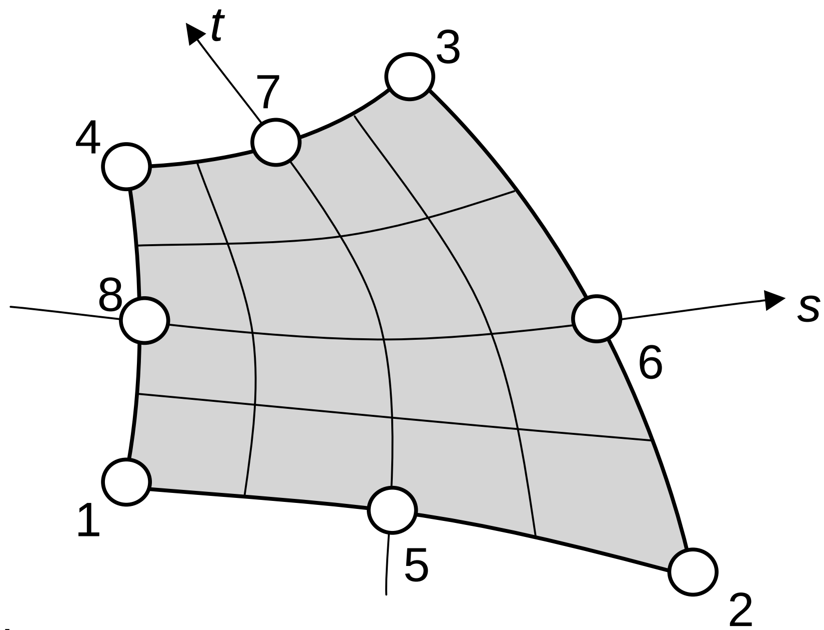
$$\int_A f(x)\,dxdy = \int_{-1}^{1}\int_{-1}^{1} f(s)|J|\,dsdt = 4\sum_{i=1}^{n} W_i\left(\mathbf{B}^T\mathbf{D}\mathbf{B}\right)_i |\mathbf{J}|_i$$

8-Node Brick Elements

Natural coordinate system

For each element we assign a local coordinate system represented by *s* and *t* which both span the range from –1 to 1 over the area of the element

Quadratic displacement function



the shape functions are:

$$N_1 = \frac{1}{4}(1-s)(1-t)(-s-t-1) \qquad N_2 = \frac{1}{4}(1+s)(1-t)(s-t-1)$$

$$N_3 = \frac{1}{4}(1+s)(1+t)(s+t-1) \qquad N_4 = \frac{1}{4}(1-s)(1+t)(-s+t-1)$$

$$N_5 = \frac{1}{2}(1+t)(1-s)(1+s) \qquad N_6 = \frac{1}{2}(1+s)(1-t)(1+t)$$

$$N_7 = \frac{1}{2}(1-t)(1-s)(1+s) \qquad N_8 = \frac{1}{2}(1-s)(1-t)(1+t)$$

alternatively we choose a natural coordinate system as shown and define the geometry in terms of the natural coordinate system as:

$$x = a_1 + a_2 s + a_3 t + a_4 st + a_5 s^2 + a_6 t^2 + a_7 s^2 t + a_8 st^2$$

$$y = b_1 + b_2 s + b_3 t + b_4 st + b_5 s^2 + b_6 t^2 + b_7 s^2 t + b_8 st^2$$

Note that this is an incomplete quadratic polynomial.

Gauss points at 4 node and 8 node brick elements

Recall that we had assumed the form of the displacements as:

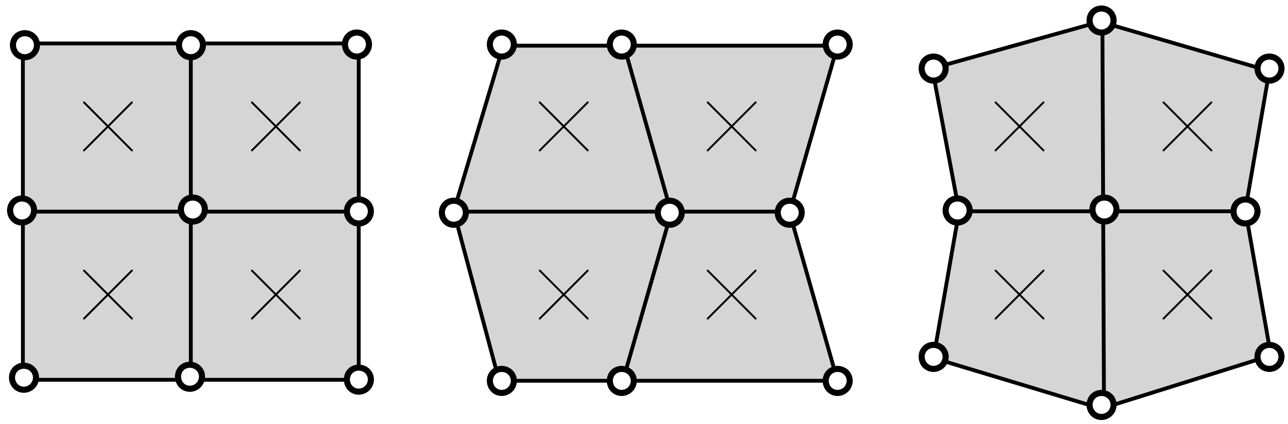
$$u_x = a_1 + a_2 s + a_3 t + a_4 st = \sum N_i u_{xi}$$

$$u_y = b_1 + b_2 s + b_3 t + b_4 st = \sum N_i u_{yi}$$

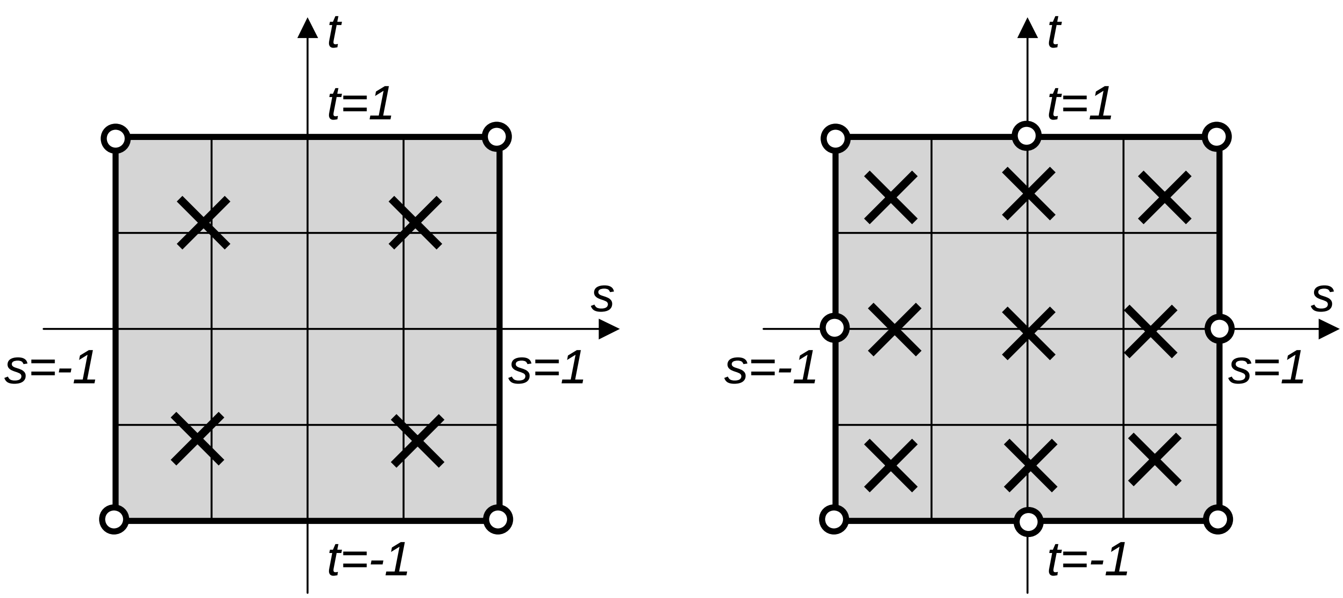
This is a bilinear approximation, it is not linear in every direction, and so perhaps 1 point is not sufficient

Quads tend to exhibit instabilities.

The following modes have no strains



Therefore, we use more points.



Validity of Isoparametric Elements

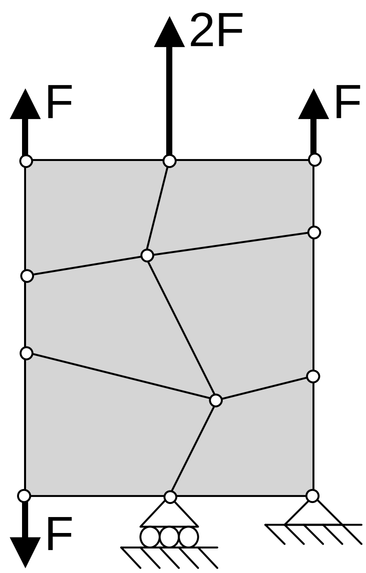
PATCH TEST

Critical test for validity is the patch test
Serves as a necessary and sufficient condition for the correct convergence of a finite element formulation

Basic idea is to assemble a small number of elements so that at least one node within the patch is shared by more than two elements. The boundary nodes of the model are loaded by a set of consistently derived nodal loads corresponding to a state of constant stress.

Example

A patch test for σ_y for 4-node elements



A patch test must be performed for all constant stress states demanded of the element

A successful patch test reveals that the element

- will display a state of constant strain
- will not strain when subjected to a rigid body motion
- is compatible with adjacent elements when subjected to a state of constant strain

Numerical example of LST

Specify the nodal coordinates

```
xccord = 881.5, 1<, 84, 2<, 82, 3<, 83, 2.5<, 81.75, 2<, 82.75, 1.5<<;
```

Material properties and plain stress D matrix

```
plainstress;
E1 = 100.;
ν1 = 0.3;
```

$$\text{Dmat} = \frac{\text{E1}}{1-\nu 1^2}\ \text{:81, ν1, 0<, 8ν1, 1, 0<, :0, 0, } \frac{1-\nu 1}{2} \text{>>;}$$

Specify the shape functions

```
s3 = 1 - s1 - s2;
n1 = s1 H2 s1 - 1L;
n2 = s2 H2 s2 - 1L;
n3 = s3 H2 s3 - 1L;
n4 = 4 s2 s3;
n5 = 4 s1 s3;
n6 = 4 s1 s2;
```

Form the Bnot matrix and determine the Jacobian

```
MatrixForm@
 Bnot = 88D@n1, s1D, D@n2, s1D, D@n3, s1D, D@n4, s1D, D@n5, s1D, D@n6, s1D<,
   8D@n1, s2D, D@n2, s2D, D@n3, s2D, D@n4, s2D, D@n5, s2D, D@n6, s2D<<D
```

$$\begin{pmatrix} -1+4\,s1 & 0 & 1-4\,(1-s1-s2) & -4\,s2 & -4\,s1+4\,(1-s1-s2) & 4\,s2 \\ 0 & -1+4\,s2 & 1-4\,(1-s1-s2) & 4\,(1-s1-s2)-4\,s2 & -4\,s1 & 4\,s1 \end{pmatrix}$$

```
MatrixForm@Jmat = Simplify@Bnot.xccordD êê ChopD
```

$$\begin{pmatrix} -0.5 & -2 \\ 2 & -1. \end{pmatrix}$$

```
MatrixForm@Bno = Simplify@Inverse@JmatD.BnotDD
```

$$\begin{pmatrix} 0.222222-0.888889\,s1 & -0.444444+1.77778\,s2 & -0.666667+0.888889\,s1+0.888889\,s2 & 1.77778-1.77778\,s1-2.66667\,s2 & -0.888889+0.\,s1+0.888889\,s2 & 1.77778\,s1-0.888889\,s2 \\ 0.444444-1.77778\,s1 & 0.111111-0.444444\,s2 & 1.66667-2.22222\,s1-2.22222\,s2 & -0.444444+0.444444\,s1+2.66667\,s2 & -1.77778+4.\,s1+1.77778\,s2 & -0.444444\,s1-1.77778\,s2 \end{pmatrix}$$

Form the B matrix

```
MatrixForm@
 Bmat = 88Bno@@1, 1DD, 0, Bno@@1, 2DD, 0, Bno@@1, 3DD, 0, Bno@@1, 4DD, 0,
     Bno@@1, 5DD, 0, Bno@@1, 6DD, 0<,
   80, Bno@@2, 1DD, 0, Bno@@2, 2DD, 0, Bno@@2, 3DD, 0, Bno@@2, 4DD, 0,
     Bno@@2, 5DD, 0, Bno@@2, 6DD<,
   8Bno@@2, 1DD, Bno@@1, 1DD, Bno@@2, 2DD, Bno@@1, 2DD, Bno@@2, 3DD,
     Bno@@1, 3DD, Bno@@2, 4DD, Bno@@1, 4DD, Bno@@2, 5DD, Bno@@1, 5DD,
     Bno@@2, 6DD, Bno@@1, 6DD<<D
```

$$\begin{pmatrix} 0.222222-0.888889s1 & 0 & -0.444444+1.77778s2 & 0 & -0.666667+0.888889s1+0.888889s2 & 0 & 1.77778-1.77778s1-2.66667s2 & 0 & -0.888889+0.s1+0.888889s2 & 0 & 1.77778s1-0.888889s2 & 0 \\ 0 & 0.444444-1.77778s1 & 0 & 0.111111-0.444444s2 & 0 & 1.66667-2.22222s1-2.22222s2 & 0 & -0.444444+0.444444s1+2.66667s2 & 0 & -1.77778+4.s1+1.77778s2 & 0 & -0.444444s1-1.77778s2 \\ 0.444444-1.77778s1 & 0.222222-0.888889s1 & 0.111111-0.444444s2 & -0.444444+1.77778s2 & 1.66667-2.22222s1-2.22222s2 & -0.666667+0.888889s1+0.888889s2 & -0.444444+0.444444s1+2.66667s2 & 1.77778-1.77778s1-2.66667s2 & -1.77778+4.s1+1.77778s2 & -0.888889+0.s1+0.888889s2 & -0.444444s1-1.77778s2 & 1.77778s1-0.888889s2 \end{pmatrix}$$

Start forming the stiffness matrix

```
MatrixForm@Kj@s1_, s2_D = Det@JmatD Simplify@Transpose@BmatD.Dmat.BmatDD
```

Perform numerical integration(evaluate at the gauss points)

```
MatrixForm@
 Kloc = 1ê3 .2ê2 HKj@.66666, .16666D + Kj@.166666, .6666D + Kj@.166666, .16666DLD
```

5.86064	3.17451	1.34302	1.01744	0.610504	0.0407003	0.0000338645	0.0000369036	−2.442	−0.162835	−5.3722	−4.06986
3.17451	10.6224	0.834302	−0.244186	0.223851	3.78512	0.0000251813	−0.000110059	−0.895414	−15.1401	−3.33728	0.976867
1.34302	0.834302	9.97885	−1.58685	1.98319	−1.3628	−7.93367	5.45212	0.000310882	−0.000485298	−5.3717	−3.33629
1.01744	−0.244186	−1.58685	4.02816	−1.54587	1.58655	6.18451	−6.34714	−0.000561474	0.000452564	−4.06867	0.976163
0.610504	0.223851	1.98319	−1.54587	7.78358	−3.9681	−7.93483	6.18568	−2.44125	−0.895912	−0.00119246	0.000351119
0.0407003	3.78512	−1.3628	1.58655	−3.9681	16.1166	5.45303	−6.34514	−0.163268	−15.1394	0.000439018	−0.003679
0.0000338645	0.0000251813	−7.93367	6.18451	−7.93483	5.45303	31.4979	−3.17146	−10.7473	−7.40719	−4.88214	−1.05891
0.0000369036	−0.000110059	5.45212	−6.34714	6.18568	−6.34514	−3.17146	41.0168	−7.40715	1.94975	−1.05922	−30.2742
−2.442	−0.895414	0.000310882	−0.000561474	−2.44125	−0.163268	−10.7473	−7.40715	31.5022	−3.17387	−15.872	11.6403
−0.162835	−15.1401	−0.000485298	0.000452564	−0.895912	−15.1394	−7.40719	1.94975	−3.17387	41.0245	11.6403	−12.6952
−5.3722	−3.33728	−5.3717	−4.06867	−0.00119246	0.000439018	−4.88214	−1.05922	−15.872	11.6403	31.4992	−3.17556
−4.06986	0.976867	−3.33629	0.976163	0.000351119	−0.003679	−1.05891	−30.2742	11.6403	−12.6952	−3.17556	41.02

Apply the boundary conditions

```
Rx2 = 1;
Ry2 = 0;
Rx2 = 1;
Ry3 = 0;
Rx4 = 0.5;
Ry4 = 0;
Ry5 = 0;
Rx6 = .5;
Ry6 = 0;
ux1 = 0;
uy1 = 0;
ux3 = 0;
ux5 = 0;
MatrixForm@
 fvect = 88Rx1<, 8Ry1<, 8Rx2<, 8Ry2<, 8Rx3<, 8Ry3<, 8Rx4<, 8Ry4<, 8Rx5<,
   8Ry5<, 8Rx6<, 8Ry6<<D
MatrixForm@
 uvect = 88ux1<, 8uy1<, 8ux2<, 8uy2<, 8ux3<, 8uy3<, 8ux4<, 8uy4<, 8ux5<,
   8uy5<, 8ux6<, 8uy6<<D
Solve@Kloc.uvect == fvectD
```

88ux2 → 0.210567, ux4 → 0.0729666, ux6 → 0.074885, uy2 → 0.0483552,
uy3 → −0.0211898, uy4 → −0.00110893, uy5 → −0.00970086, uy6 → 0.0198336,
Rx1 → −0.150302, Rx3 → −0.150302, Ry1 → -4.23449×10^{-17}, Rx5 → −1.6994<<

```
MatrixForm@
 fvect = 88Rx1<, 8Ry1<, 8Rx2<, 8Ry2<, 8Rx3<, 8Ry3<, 8Rx4<, 8Ry4<, 8Rx5<,
   8Ry5<, 8Rx6<, 8Ry6<<D
MatrixForm@
 uvect = 88ux1<, 8uy1<, 8ux2<, 8uy2<, 8ux3<, 8uy3<, 8ux4<, 8uy4<, 8ux5<,
   8uy5<, 8ux6<, 8uy6<<D
```

$$\begin{pmatrix} -0.150302 \\ 0 \\ 1 \\ 0 \\ -0.150302 \\ 0 \\ 0.5 \\ 0 \\ -1.6994 \\ 0 \\ 0.5 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0.210567 \\ 0.0483552 \\ 0 \\ -0.0211898 \\ 0.0729666 \\ -0.00110893 \\ 0 \\ -0.00970086 \\ 0.074885 \\ 0.0198336 \end{pmatrix}$$

```
MatrixForm@stressvect@s1_, s2_D = Simplify@Dmat.Bmat.uvectDD
```

$$\begin{pmatrix} 3.5683 + 0.341056\, s1 + 11.455\, s2 \\ -0.149976 + 0.0000323908\, s1 + 0.449932\, s2 \\ -0.374859 + 0.674735\, s1 + 0.449883\, s2 \end{pmatrix}$$

Determine the stresses at the Gauss points

```
MatrixForm@stressvect@.666, .166DD
MatrixForm@stressvect@.166, .666DD
MatrixForm@stressvect@.166, .166DD
```

$$\begin{pmatrix} 5.69697 \\ -0.0752658 \\ 0.149195 \end{pmatrix}, \begin{pmatrix} 11.2539 \\ 0.149684 \\ 0.0367692 \end{pmatrix}, \text{ and } \begin{pmatrix} 5.52644 \\ -0.075282 \\ -0.188172 \end{pmatrix}$$