10.3 (b)

For Figure (b)

(a)
$$s = \left[20 - \left(\frac{10+30}{2}\right)\right]\left(\frac{2}{30-10}\right)$$

$$s = [20 - 20]\,\frac{2}{20}$$

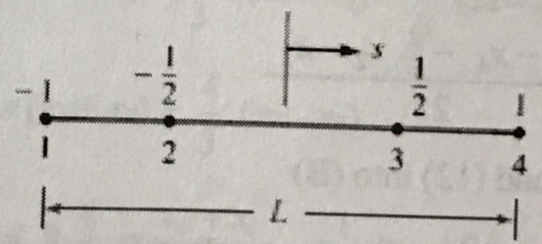
$$s = 0$$

(b)
$$N_1 = \frac{1-0}{2} = \frac{1}{2},\; N_2 = \frac{1+0}{2} = \frac{1}{2}$$

(c) $u_A = [1/2 \quad 1/2]\begin{Bmatrix}0.05\\0.10\end{Bmatrix} = 0.075$ mm

(d) $\varepsilon_x = \left[-\frac{1}{40} \quad \frac{1}{40}\right]\begin{Bmatrix}0.05\\0.10\end{Bmatrix} = 0.00125$

10.4



(1)
$$u = a_1 + a_2 s + a_3 s^2 + a_4 s^3 \tag{A}$$

$$x = a_1 + a_2 s + a_3 s^2 + a_4 s^3 \tag{B}$$

$$x_1 = a_1 + a_2(-1) + a_3(-1)^2 + a_4(-1)^3 \tag{1}$$

$$x_2 = a_1 + a_2\left(\frac{-1}{2}\right) + a_3\left(\frac{-1}{2}\right)^2 + a_4\left(\frac{-1}{2}\right)^3 \tag{2}$$

$$x_3 = a_1 + a_2\left(\frac{1}{2}\right) + a_3\left(\frac{1}{2}\right)^2 + a_4\left(\frac{1}{2}\right)^3 \tag{3}$$

$$x_4 = a_1 + a_2(1) + a_3(1)^2 + a_4(1)^3 \tag{4}$$

$$(1) + (4) \Rightarrow x_1 + x_4 = 2a_1 + 2a_3 \tag{5}$$

$$(2) + (3) \Rightarrow x_2 + x_3 = 2a_1 + \frac{a_3}{2} \tag{6}$$

(5) – (6) gives

$$x_1 + x_4 - (x_2 + x_3) = 2a_1 + 2a_3 - \left(2a_1 + \frac{a_3}{2}\right)$$

or
$$a_3 = \frac{2}{3}\,(x_1 + x_4 - x_2 - x_3) \tag{7}$$

(7) into (5)

$$x_1 + x_4 = 2a_1 + 2\left(\frac{2}{3}\right)\,(x_1 + x_4 - x_2 - x_3)$$

or
$$a_1 = \frac{-\frac{1}{3}\ x_1 + x_4\ + \frac{4}{3}\ x_2 + x_3}{2} \tag{8}$$

$$(1) - (4) \Rightarrow x_1 - x_4 = -2a_2 - 2a_4 \tag{9}$$

$$(2) - (3) \Rightarrow x_2 - x_3 = -a_2 - \frac{a_4}{4} \tag{10}$$

(9) – 2 (10) gives

$$x_1 - x_4 - 2(x_2 - x_3) = \frac{-3a_4}{2}$$

$$\therefore \quad a_4 = \frac{2}{3}\,[2(x_2 - x_3) - (x_1 - x_4)] \tag{11}$$

(11) into (9) yields

$$a_2 = \frac{\frac{1}{3}\ x_1 - x_4\ - \frac{8}{3}\ x_2 - x_3}{2} \tag{12}$$

Substituting (7), (8), (11) and (12) into (B)

$$x = \frac{4(x_2 + x_3) -\ x_1 + x_4}{6} + \frac{x_1 - x_4\ - 8\ x_2 - x_3\ \ s}{6}$$

$$+ \frac{4\ x_1 + x_4 - x_2 - x_3}{6}\ s^2 + \frac{8\ x_2 - x_3\ - 4\ x_1 - x_4\ \ s^3}{6} \tag{13}$$

Combine like x_1, x_2, x_3 and x_4 coefficients

$$x = \left(-\frac{2}{3}s^3 + \frac{2}{3}s^2 + \frac{s}{6} - \frac{1}{6}\right)x_1 + \left(\frac{4}{3}s^3 - \frac{2}{3}s^2 - \frac{4}{3}s + \frac{2}{3}\right)x_2$$

$$+ \left(-\frac{4}{3}s^3 - \frac{2}{3}s^2 + \frac{4}{3}s + \frac{2}{3}\right)x_3 + \left(\frac{2}{3}s^3 + \frac{2}{3}s^2 - \frac{s}{6} - \frac{1}{6}\right)x_4 \tag{14}$$

By (14) then

$$\{x\} = [N_1 \quad N_2 \quad N_3 \quad N_4] \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}$$

$$\therefore \; N_1 = \frac{-2}{3}s^3 + \frac{2}{3}s^2 + \frac{s}{6} - \frac{1}{6}$$

$$N_2 = \frac{4}{3}s^3 - \frac{2}{3}s^2 - \frac{4}{3}s + \frac{2}{3}$$

$$N_3 = \frac{-4}{3}s^3 - \frac{2}{3}s^2 + \frac{4}{3}s + \frac{2}{3}$$

$$N_4 = \frac{2}{3}s^3 + \frac{2}{3}s^2 - \frac{s}{6} - \frac{1}{6}$$

(2)
$$\frac{du}{ds} = \left[-2s^2 + \frac{4}{3}s + \frac{1}{6} \quad 4s^2 - \frac{4}{3}s - \frac{4}{3} \quad -4s^2 - \frac{4}{3}s + \frac{4}{3} \quad 2s^2 + \frac{4}{3}s - \frac{1}{6}\right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Differentiating (13)

$$\frac{dx}{ds} = \left(-2s^2 + \frac{4}{3}s + \frac{1}{6}\right)x_1 + \left(4s^2 + \frac{4}{3}s - \frac{4}{3}\right)x_2$$

$$+ \left(-4s^2 - \frac{4}{3}s + \frac{4}{3}\right)x_3 + \left(2s^2 + \frac{4}{3}s - \frac{1}{6}\right)x_4$$

Simplifying

$$= 2s^2(x_4 - x_1) + \frac{4}{3}s(x_4 + x_1) - \frac{1}{6}(x_4 - x_1)$$

$$-4s^2(x_3 - x_2) - \frac{4}{3}s(x_3 + x_2) + \frac{4}{3}(x_3 - x_2)$$

$$= 2s^2 L + \frac{8}{3}s\frac{x_4 + x_1}{2} - \frac{1}{6}L - 4s^2\left(\frac{L}{2}\right) - \frac{8}{3}s\left(\frac{x_3 + x_2}{2}\right) + \frac{4}{3}\left(\frac{L}{2}\right)$$

$$= 2s^2 L + \frac{8}{3}s\,x_c - \frac{L}{6} - 2s^2 L - \frac{8}{3}s\,x_c + \frac{2}{3}L$$

$$\frac{dx}{ds} = \frac{L}{2}$$

Now

$$\frac{du}{dx} = \frac{\frac{du}{ds}}{\frac{dx}{ds}} \quad \text{and} \quad \frac{du}{dx} = \varepsilon_x = [B]\{d\}$$

$$\therefore \quad \varepsilon_x = \left[\frac{-12s^2+8s+1}{3L} \quad \frac{12s^2-4s-4}{\frac{3L}{2}} \quad \frac{-12s^2-4s+4}{\frac{3L}{2}} \quad \frac{12s^2+8s-1}{3L} \right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$\therefore \quad [B] = \left[\frac{-12s^2+8s+1}{3L} \quad \frac{12s^2-4s-4}{\frac{3L}{2}} \quad \frac{-12s^2-4s+4}{\frac{3L}{2}} \quad \frac{12s^2+8s-1}{3L} \right]$$

r Figure (a)

Using Equation (10.5.6)

$$x = x_A = 13 = 15 + \left(\frac{20-10}{2}\right) s + \frac{10+20-2\ 15}{2} s^2$$

$. \ 0s^2 + 5s + 2 = 0$

$$s = \frac{-2}{5} = -0.4$$

$$N_1 = \frac{s(s-1)}{2} = \frac{-0.4(-0.4-1)}{2}$$

$$= 0.28$$

$$N_2 = \frac{s(s+1)}{2} = \frac{-0.4(-0.4+1)}{2}$$

$$= -0.12$$

$$N_3 = 1 - s^2 = 1 - (-0.4)^2$$

$$= 0.84$$

$$\Sigma N\text{'s} = 0.28 - 0.12 + 0.84 = 1$$

$$u = a_1 + a_2 s + a_3 s^2$$

$$u_1 = 0.006 = a_1 + a_2(-1) + a_3(-1)^2$$

$$u_3 = 0 = a_1 + a_2(0) + a_3(0)$$

$$u_2 = -0.006 = a_1 + a_2(1) + a_3(1)^2$$

$$\therefore \quad a_3 = 0,\ a_2 = -0.006,\ a_1 = 0$$

$$\therefore \quad u = -0.006s \quad \text{and} \quad s = -0.4 \quad \text{at} \quad x_A = 13$$

$$\therefore \quad u = -0.006(-0.4) = 0.0024 \text{ in. at A}$$

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix}$$

x and y from Problem 10.9

$$\frac{\partial x}{\partial s} = \frac{1}{4}(-1)(1-t)x_1 + \frac{1}{4}(1-t)x_2 + \frac{1}{4}(1+t)x_3 + \frac{1}{4}(-$$

$$= N_{1,s}\, x_1 + N_{2,s}\, x_2 + N_{3,s}\, x_3 + N_{4,s}\, x_4$$

$$\frac{\partial x}{\partial t} = \frac{1}{4}(-1)(1-s)x_1 + \frac{1}{4}(-1)(1+s)x_2 + \frac{1}{4}(1+s)x_3 +$$

$$= N_{1,t}\, x_1 + N_{2,t}\, x_2 + N_{3,t}\, x_3 + N_{4,t}\, x_4$$

$$\frac{\partial y}{\partial s} = \frac{1}{4}(-1)(1-t)y_1 + \frac{1}{4}(1-t)y_2 + \frac{1}{4}(1+t)y_3 + \frac{1}{4}(-1$$

$$= N_{1,s}\, y_1 + N_{2,s}\, y_2 + N_{3,s}\, y_3 + N_{4,s}\, y_4$$

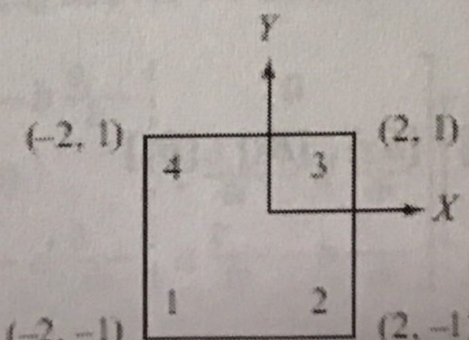
$$\frac{\partial y}{\partial t} = \frac{1}{4}(-1)(1-s)y_1 + \frac{1}{4}(-1)(1+s)y_2 + \frac{1}{4}(1+s)y_3 +$$

$$= N_{1,t}\, y_1 + N_{2,t}\, y_2 + N_{3,t}\, y_3 + N_{4,t}\, y_4$$

$$[J] = \begin{bmatrix} N_{1,s} & N_{2,s} & N_{3,s} & N_{4,s} \\ N_{1,t} & N_{2,t} & N_{3,t} & N_{4,t} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$

10.11 (a)

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix} \begin{pmatrix} x_1 = -2,\ x_2 = 2 \\ x_3 = 2,\ x_4 = -2 \end{pmatrix}$$



$$\frac{\partial x}{\partial s} = \frac{1}{4}(-1)(1-t)(-2) + \frac{1}{4}(1-t)(2) + \frac{1}{4}(1+t)(2) + \frac{1}{4}(-1)(1+t)(-2) = 2$$

$$\frac{\partial x}{\partial t} = \frac{1}{4}(-1)(1-s)(-2) + \frac{1}{4}(-1)(1+s)(2) + \frac{1}{4}(1+s)(2)$$

$$+ \frac{1}{4}(1-s)(-2) = 0$$

$$y_1 = 1, y_2 = -1$$

$$y_3 = 1, y_4 = 1$$

$$\frac{\partial y}{\partial s} = \frac{1}{4}(-1)(1-t)(-1) + \frac{1}{4}(1-t)(-1) + \frac{1}{4}(1+t)(1)$$

$$+ \frac{1}{4}(-1)(1+t)(1) = 0$$

$$\frac{\partial y}{\partial t} = \frac{1}{4}(-1)(1-s)(-1) + \frac{1}{4}(-1)(1+s)(-1) + \frac{1}{4}(1+s)(1)$$

$$+ \frac{1}{4}(1-s)(1) = 1$$

$$|[J]| = \left|\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right| = 2 - 0 = 2$$

By Equation (10.2.22)

$$|[J]| = \frac{1}{8}[-2\ 2\ 2\ -2] \times \begin{bmatrix} 0 & 1-t & t-s & s-1 \\ t-1 & 0 & s+1 & -s-t \\ s-t & -s-1 & 0 & t+1 \\ 1-s & s+t & -t-1 & 0 \end{bmatrix} \begin{Bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{Bmatrix} \qquad \text{(A)}$$

Simplifying by multiplying the matrices in Equation (A) yields

$$|[J]| = 2 \text{ also}$$

and $\quad |[J]| = \dfrac{A}{4}$ as

$$A = 4 \times 2 = 8 \text{ (area of element)}$$

$$\therefore \quad = |[J]| = \frac{8}{4} = 2$$

10.15 (f) (g)

(f) $\int_{-1}^{1} s \cos s \, ds = \left(\frac{5}{9}\right)(0.7746) \cos(0.7746) + \left(\frac{8}{9}\right)(0) \cos(0) + \frac{5}{9}(-0.7746) \cos(-0.7746)$

$= 0.30756 + 0 - 0.30756$

$= 0$

(g)

Two Gauss points

$\int_{-1}^{1} (4^s - 2s) ds$

$I = 4^{(-0.57735)} - 2(-0.57735)$

$+4^{(0.57735)} - 2(0.57735)$

$= 2.6755$

Exact is 2.7051

Two Newton-Cotes Sampling Points

$I = 2(\frac{1}{2} y_0 + \frac{1}{2} y_1)$

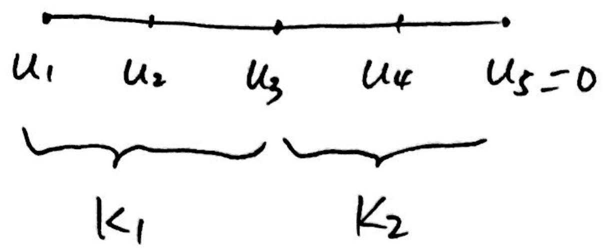
$y_0 = 4^{(-1)} - 2(-1) = 2.25$

$y_1 = 4^{(1)} - 2(1) = 2$

$I = 2(\frac{1}{2}(2.25) + \frac{1}{2}(2))$

$= 4.25$

10.7



$$[K] = \frac{AE}{2L} \begin{bmatrix} & 5\times5 & \end{bmatrix}$$ ——— 0.5

Calculate distribution $f_1 \sim f_5$ ——— 0.5

Calculate $u_1 \sim u_5$ ——— 0.5

Calculate $\sigma_1 \sim \sigma_5$ ——— 0.5

$$\{f\} = [K]\{u\}$$

$$\{\sigma\} = E\{B\}\{u\}.$$

If only considering 3 nodes instead of 5, you can still get 1.5 out of 2.