# Exact temperature profile for the *hillingar* mirage

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In a *hillingar* mirage, the Earth's surface appears flat, because nearly horizontal light rays have the same curvature as the Earth. A linear temperature profile is traditionally inferred; its gradient is calculated to give this curvature to the exact horizontal ray. To see an image, however, a bundle of rays is required. To ensure that each ray in the bundle have the same curvature, the temperature profile must contain a small positive quadratic term, the coefficient of which is derived. 
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## I. INTRODUCTION

There is one type of superior mirage[1] that often goes unrecognized. It arises when the bundle of light rays entering the eye has a curvature similar to that of the Earth. The visual effect is a flattening of the Earth, which lifts objects into view that are normally beyond the horizon. Observers not familiar with the local geography do not know that they are seeing too far. On the shores of Lake Winnipeg, which is too wide to see across if one stands at the water's edge, the common reaction is, "The air is very clear today, because I can see the opposite shore." Strictly speaking, given that the root of the word "mirage" means "to mirror," this effect should not be called a mirage at all, because no image reflection (inversion) or other distortion is involved. But I will continue the tradition of calling all atmospheric refraction phenomena by this name.

The medieval Norse are thought to have known this effect, and had a name for it: *hillingar*. Cleasby and Vigfusson[2] define it as

> *Upheaving*, esp. of a mirage, when rocks and islands look as if lifted above the level of the sea.

Well-known historical sightings fall into this category. They have caused difficulties in arctic exploration for centuries. An excellent example is the 1818 expedition of Sir John Ross.[3] He abandoned his search for the Northwest Passage when, sailing north of Baffin Island, he saw Lancaster Sound completely blocked by a mountain range, which in fact does not exist. Hobbs,[4] who has collected a number of such occurrences, identifies the apparent mountains as a mirage of the very distant Somerset Island.

In a previous paper (1979) I defined *hillingar* in rather general terms as a mirage that makes the Earth appear flat or saucer-shaped.[5] It is seen when light rays propagate with a curvature that nearly equals the curvature of the Earth. For present purposes I would like to refine the definition to imply an exactly flat apparent Earth, with magnification[6] exactly equal to unity. While this is a special case that will be quite rare, it is still of interest as a point of reference, hence the derivation of the necessary conditions in the following paragraphs.

## II. PREVIOUS RESULT

A specific well-known temperature gradient is required to make a level ray propagate with constant elevation above the Earth. Pernter and Exner[7] publish the value of 0.112 °C/m for an air temperature of 0 °C. The curvature of such a ray exactly matches that of the Earth.

To see an image, however, the eye must receive a bundle of rays. Rees addressed this issue when he derived conditions for which a mirage would have a linear image diagram.[8] He showed that for objects subtending a few arcminutes at the eye, the necessary refractive index profile is a quadratic function of elevation. But when the magnification is unity, as in the *hillingar* form, this function becomes linear, and the corresponding temperature profile is very nearly linear as well.[9]

The linear functions are not quite adequate. The curvature of a light ray, discussed in Sec. III, depends on both temperature and pressure. A ray whose initial slope is positive (nonzero) will traverse higher elevations, where lower pressure and higher temperature serve to reduce the curvature below the required value. Thus only the level ray will satisfy the *hillingar* definition. For all others, elevation above the Earth will increase more rapidly than the linear rate required for unit magnification.

## III. MODIFIED RESULT

For the conditions under discussion (small initial ray slopes), the *hillingar* occurs if each ray in the bundle has exactly the same curvature as the Earth. This assertion is intuitively reasonable; the proof has been omitted for the sake of brevity. The necessary temperature profile can then be derived simply by setting curvature equal to the reciprocal of the Earth's radius, $R_E$.

Fleagle and Businger[10] derive ray curvature $\kappa$ for the case of level rays:

$$\kappa=\frac{n(z)-1}{n(z)T(z)}\left[\frac{dT}{dz}+g\beta\right]\mathrm{m}^{-1}, \tag{1}$$

where $z$ = elevation (m) above the observer's eye, $T$ = temperature (K), $g$ = acceleration of gravity = 9.81 m s$^{-2}$, $\beta$ = reciprocal of the specific gas constant for dry air$=3.48\times10^{-3}$ K kg J$^{-1}$, $n$=refractive index of air$=1+\varepsilon\rho$, where $\varepsilon=226\times10^{-6}\,\mathrm{m^3\,kg^{-1}}$, and $\rho$=density of air $=\beta p/T\,\mathrm{kg\,m^{-3}}$.

For rays inclined with an angle $\phi$ to the horizontal, a factor cos $\phi$ must be inserted on the right-hand side of Eq. (1).[11] Then, when $n(z)-1$ is replaced in terms of pressure $p$ and temperature, the equation becomes

$$\kappa=\frac{\varepsilon\beta p(z)\cos\phi}{n(z)T^2(z)}\left[\frac{dT}{dz}+g\beta\right]. \quad (2)$$

Next, pressure can be eliminated in favor of temperature. In a static gravitational field, pressure satisfies the relation $dp/dz=-g\rho$; when $\rho$ is replaced by $\beta p/T$ the equation integrates to

$$p(z)=p_0\exp\left[-g\beta\int_0^z\frac{dz'}{T(z')}\right] \quad (3)$$

for substitution into Eq. (2).

Consider now some limits on the range of $z$ and $\phi$. Elevation $z$ will be limited to 100 m, to prevent inordinate demands on the temperature inversion (exceeding 10 °C if we use the previously quoted temperature gradient). Elevation angle $\phi$ will be limited to 15 arcmin. This value is based on long-term experience in observing mirages; mirage heights above 15 arcmin are extremely rare.

The significant variables in Eq. (2) are temperature and pressure. Over the range of interest, the variations in refractive index and cos $\phi$ are about four orders of magnitude smaller, and will be neglected. In the following, $n(z)$ will be replaced by $n_0$, the value at zero elevation, and cos $\phi$ by 1.

The temperature profile is now written as a quadratic function of elevation, i.e., a power series expansion one order more precise than the previously used linear approximation:

$$T(z)=T_0+az+bz^2. \quad (4)$$

When this form is substituted into Eq. (3), the integral can be evaluated in closed form: Its value is the difference between two arctangent functions. For the expected range of parameters ($a\approx0.1$, $b$ much smaller), a Taylor expansion of the arctangent to the quadratic term is sufficiently accurate. Then, as the exponent in (3) is quite small, a first-order expansion of the exponential yields accuracy to the same level. The result is

$$p(z)\approx p_0\left[1-\frac{g\beta z}{T_0}+\frac{g\beta az^2}{2T_0^2}\right]. \quad (5)$$

We can now substitute into Eq. (2). First take the $T^2$ term to the left-hand side, and set $\kappa$ to $1/R_E$. Then insert Eqs. (4) and (5), expand, and equate coefficients of the constant and linear terms to get equations for $a$ and $b$:

$$a=\frac{n_0T_0^2}{R_E\varepsilon\beta p_0}-g\beta, \quad (6)$$

$$b=\frac{n_0T_0}{2R_E\varepsilon\beta p_0}[2a+g\beta]. \quad (7)$$

It may be noted that the equation defining the linear coefficient $a$ is exactly the same as that derived by Pernter and Exner[7] for the level ray.

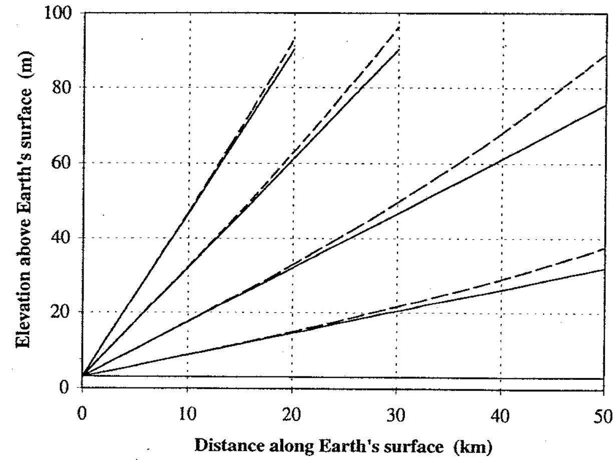


Fig. 1. Comparison of light rays for the case $T_0=273.15$ K, with and without the quadratic correction to the temperature profile. The solid lines represent rays in the quadratic case ($a=0.1127$ K/m and $b=6.98\times10^{-5}\,\mathrm{K\,m^{-2}}$), while the dashed lines are the rays in the linear case ($a=0.1127$ K/m and $b=0$). At the observer's eye, 3 m above the surface, the ray elevation angles are respectively 0, 2, 5, 10, and 15 arcmin.

Some typical numerical values follow. They use $R_E=6.37\times10^6$ m and $p_0=1.013\times10^5$ Pa. For $T_0=273.15$ K(0 °C), the results are $a=0.1127$ K/m and $b=6.98\times10^{-5}\,\mathrm{K\,m^{-2}}$. At 20 °C a stronger inversion is required: $a=0.1350$ K/m and $b=8.78\times10^{-5}\,\mathrm{K\,m^{-2}}$. The correction provided by the quadratic term is quite small, less than 1° at 100 m, but it is necessary.

Figure 1 shows the effect of each parameter on the shape of the rays. The dashed rays arise from the linear temperature profile (corresponding to $b=0$), and the solid rays are calculated from the corrected quadratic temperature profile. Only for the level trajectory do the rays match. In the case of the sloping rays, the dashed rays gain altitude more quickly than they should, because the curvature weakens with increasing elevation when the temperature profile is linear. The quadratic profile, on the other hand, keeps the curvature constant independent of elevation, so that the corresponding rays gain elevation linearly with distance.

## IV. CONCLUSION

To observe a *hillingar* mirage, in which the Earth appears exactly flat, and the magnification of all objects is exactly unity, it is necessary that the well-known linear increase of air temperature with elevation be augmented by a small quadratic correction.

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[1]Several good introductory books on atmospheric optical phenomena are the following: R. Greenler, *Rainbows, Halos, and Glories* (Cambridge U.P., Cambridge, 1980); M. G. J. Minnaert, *Light and Color in the Outdoors* (Springer, New York, 1993); and D. K. Lynch and W. Livingston, *Color and Light in Nature* (Cambridge U.P., Cambridge, 1995).

[2]R. Cleasby and G. Vigfusson, *An Icelandic/English Dictionary* (Clarendon, Oxford, 1957), 2nd ed. The antiquity of the word *hillingar* is not discussed, but it is used in the discussion of Norse exploration in G. Jones, *The Norse Atlantic Saga* (Oxford U.P., London, 1964), p. 6.

[3]John Ross, *A Voyage of Discovery* (John Murray, London, 1819). This book includes a beautiful engraving of the image seen by Ross. The image

contains none of the usual features such as distortion, anomalous magnification, or inversion that characterize most mirages.

[4]W. H. Hobbs, "Conditions of Exceptional Visibility within High Latitudes, Particularly as a Result of Superior Mirage," Ann. Assoc. Am. Geographers **27**, 229–240 (1937).

[5]W. H. Lehn, "The Novaya Zemlya effect: An arctic mirage," J. Opt. Soc. Am. **69**, 776–781 (1979).

[6]Magnification is defined as the size of an observed image with respect to a reference image that would be seen with exactly straight light rays.

[7]J. M. Pernter and F. Exner, *Meteorologische Optik* (Braumüller, Vienna, 1922), 2nd ed., p. 63.

[8]W. G. Rees, "Mirages with linear image diagrams," J. Opt. Soc. Am. A **7**, 1351–1354 (1990). The image diagram (transfer characteristic) is a plot of ray elevation angle at the eye versus the actual elevation at which the same ray intersects the object being viewed.

[9]If the refractive index is a linear function of elevation, then the corresponding temperature profile does contain a small quadratic term. But this term is an order of magnitude smaller than the one derived here, and moreover it has the opposite sign.

[10]R. G. Fleagle and J. A. Businger, *An Introduction to Atmospheric Physics* (Academic, New York, 1980), 2nd ed., Sec. 7.13.

[11]M. Born and E. Wolf, *Principles of Optics* (Pergamon, Oxford, 1986), 6th ed., Sec. 3.2.1.

## THE CHICAGO EXPOSITION

Historical exhibits were common, but they never went far enough; none were thoroughly worked out. One of the best was that of the Cunard steamers, but still a student hungry for results found himself obliged to waste a pencil and several sheets of paper trying to calculate exactly when, according to the given increase of power, tonnage, and speed, the growth of the ocean steamer would reach its limits. His figures brought him, he thought, to the year 1927; another generation to spare before force, space, and time should meet. The ocean steamer ran the surest line of triangulation into the future, because it was the nearest of man's products to a unity; railroads taught less because they seemed already finished except for mere increase in number; explosives taught most, but needed a tribe of chemists, physicists, and mathematicians to explain; the dynamo taught least because it had barely reached infancy, and, if its progress was to be constant at the rate of the last ten years, it would result in infinite costly energy within a generation. One lingered long among the dynamos, for they were new, and they gave to history a new phase. Men of science could never understand the ignorance and naïveté of the historian, who, when he came suddenly on a new power, asked naturally what it was; did it pull or did it push? Was it a screw or thrust? Did it flow or vibrate? Was it a wire or a mathematical line? And a score of such questions to which he expected answers and was astonished to get none.

Henry Adams, *The Education of Henry Adams* (The Modern Library, New York, 1931—Originally published by the Massachusetts Historical Society, 1918), pp. 341–2.