The Inertia Tensor of a Magic Cube

Adam Rogers and Peter Loly
Department of Physics & Astronomy
The University of Manitoba, Winnipeg, Manitoba
Canada R3T 2N2

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Abstract

Magic cubes, interpreted as rigid body mass distributions, are shown to have maximally symmetric inertia tensors. This is due to their semi-magic property wherein each row, column and pillar has the same mass sum. The moment of inertia depends only on that property and the order of the cube (the number of point masses in each row, column and pillar), but not on a detailed numerology. Moreover the results for large order should be a good guide to the inertia of a random mass distribution, which would tend to have the same row, column and pillar mass sums within statistical fluctuations. [published in American Journal of Physics, 72(6), 786-9, June 2004]

1 Introduction

Magic squares are arrays of numbers that have the property that all rows, columns and both main diagonals add to the same value, called the magic constant. Magic cubes are the corresponding objects in three dimensions. If the numbers are interpreted as multiples of a unit mass on a cubic unit lattice, there is cartesian symmetry of row, column and pillar (RCP symmetry) masses. Qualitatively it follows that the inertia tensor of a magic cube is proportional to the unit tensor, because of a balancing of opposite pairs of rows, columns and pillars, and of opposite cartesian planes with respect to the center of mass which is located at the centre of the cube. So if one were to toss a magic cube, it would freely rotate just as a uniform sphere does, without any unstable rotations.

Normal magic squares are composed of a set of values ranging from $1..N^2$, where N is the order of the square ($N > 2$). The smallest normal magic square of order $N = 3$ is shown below.

$$\begin{array}{|ccc|} \hline 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \\ \hline \end{array} \tag{1}$$

A series of numbers ranging from $1..p$ has a sum

$$\sum_{k=1}^{p} k = \frac{p}{2}(p+1) \tag{2}$$

For a normal magic square $p = N^2$. However, this is the sum of all the entries in the figure. Since there are N rows and N columns, to obtain the sum of a single row or column the previous result must be divided by N. This yields the simple formula for the sum of row or column elements [1]:

$$C_2 = \frac{N}{2}(N^2+1) \tag{3}$$

Though there is only a single magic square of order three (apart from seven variants of the square due to rotations and reflections), there are 880 [2] distinct order four squares. As the order of the square increases, the number of magic squares that exist skyrocket. There are $275,305,224$ order five squares [3], and the number for order six cannot be counted exactly.

Extending magic squares into three dimensions produces a magic cube. In analogy with their two dimensional counterparts, normal magic cubes are comprised of numbers $1..N^3$ and have the sum of the entries in each row, column, pillar and the four main body diagonals equal to the magic constant. The formula for the the magic constant of a cube is found by putting $p = N^3$ in (2) and is [1]:

$$C_3 = \frac{N}{2}\left(N^3+1\right) \tag{4}$$

The smallest of these objects are the order three magic cubes of the sequence 1..27 with a magic constant of 42. There are actually 4 normal magic cubes of order three that exist, though each can be shown in 48 different ways due to reflections and rotations, giving a total of 192 order three magic cubes [4]. Just as for magic squares, the number of cubes per order rises rapidly. A recent calculation by Walter Trump estimates the number of order four normal magic cubes at approximately 7 trillion [3]. One of the 4 fundamental $N = 3$ cubes is shown below:

2	13	27		16	21	5		24	8	10
22	9	11		3	14	25		17	19	6
18	20	4		23	7	12		1	15	26
1^{st} Layer				2^{nd} Layer				3^{rd} Layer		

(5)

Recently Loly [5] discovered another property of magic squares which depends only on the order of the square and corresponds to the moment of inertia of the figure. This was done by treating the numbers in the magic square as multiples of a unit mass, separated by a unit distance. The scalar moment of inertia I is given by summing mr^2 for each entry in the square where m is the number in a given cell and r is the distance from the center of the square to the center of that cell. Here we detail a similar characteristic of magic cubes which depends only on the order of the cube and physically corresponds to the full inertia tensor. To do this, we treat the numbers in magic cubes as multiples of a unit mass, placed on a cubic unit lattice, in much the same way the moment of inertia was calculated for magic squares. The derivation of the formulae describing the elements of the inertia tensor depends only on the semi-magic nature of the arrangement (the property that all rows, columns and pillars add to the same constant value). Therefore, the inertia tensor formulae also apply to cubes which have more constraints than simply the semi-magic property, such as the normal magic cube above (5) whose body diagonals also share the common line sum.

2 The inertia tensor of an N$^{\text{th}}$ order magic cube

In classical mechanics the angular momentum vector $\vec{L}$ of an object is the product of the inertia tensor $\mathbf{I}$ and the angular velocity $\vec{\omega}$:

$$\vec{L} = \mathbf{I}\vec{\omega} \tag{6}$$

If $\vec{L}$ and $\vec{\omega}$ are parallel, the inertia tensor is replaced by a scalar. That is, if the axis of rotation is fixed the scalar moment of inertia is used as the proportionality between the angular velocity and angular momentum vectors. In this situation, the axis of rotation is a principal axis. However in general this is not the case and the angular momentum and angular velocity vectors need not be parallel [6]. Because of this, $\mathbf{I}$ is a second rank tensor which relates the angular momentum and angular velocity vectors. These tensors are comprised of nine entries and essentially the inertia tensor is similar to a 3×3 matrix. The tensor elements are given by:

$$I_{ij} = \sum_{\alpha} m_{\alpha} \left(\delta_{ij} \sum_{k} x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j} \right) \tag{7}$$

where m_α is the number centred in the α^{th} cell and the $x_{\alpha,n}$ terms are the components of the displacement from the origin to the α^{th} cell, measured in units of the nearest neighbour distance. To make use of this description of the inertia tensor, an origin must first be specified. When viewing the cube along a central axis perpendicular to a face of the cube, the cube can be visualized as an arrangement of pillars running parallel to the viewing axis. Since each of these pillars have identical line (and mass) sums and are in a symmetric arrangement about the axis, it is clear that the center of mass is located somewhere along the axis, since the contributions of opposite pairs of pillars exactly cancel out. However, the axis can be oriented in a similar manner on any face of the cube, and this result will hold true because of the fact that the pillars, columns and rows of the cube have the same line sum. This means that the centre of mass must lie at the exact centre of the cube.

The off-diagonal elements of the inertia tensor are called the products of inertia and are defined by (7) when $i \neq j$. An additional property of this tensor is that it is a 'symmetric tensor' having the property that $\mathbf{I}^T = \mathbf{I}$, or $I_{ij} = I_{ji}$. As such, the inertia tensor in general contains only six independent elements [7]. Because of the line sum symmetry of the magic cube it is obvious that the products of inertia will all be equal since the sum of the masses in each layer of the cube are identical due to the semi-magic property of the cube. Thus one need only calculate a single product of inertia to find the off-diagonal elements of the tensor. These tensor elements are also inherently obvious from inspection of the cube. Since each opposite layer of the cube contains the same amount of mass at the same distance from the origin there is a principal axis which lies perpendicular to each face. That is, when the magic cube is rotated about a central axis perpendicular to each face the angular momentum and angular velocity vectors will be parallel. This implies that the inertia tensor is diagonal when calculated from the center of mass of a magic cube. Therefore,

$$I_{ij} = I_{ji} = 0 \tag{8}$$

Since rotations about any axis central and perpendicular to a face of the cube should produce identical results (again due to the semi-magic property of line sum symmetry) one can also conclude that the moments of inertia (the diagonal entries of the tensor) are equal. One need only calculate one of these diagonal components to know them all. Then the inertia tensor of a magic cube is proportional to the identity tensor much as in the case of a uniform sphere. In fact, because of this property, the principal axes can be in any orientation as long as the origin is maintained at the center of mass of the cube. These properties also mirror those of a solid cube [7].

It is worthwhile to obtain a precise result for magic cubes. Firstly, to highlight the remarkable symmetry of the row, column and pillar mass sums (the RCP symmetry) for some surprising mass distributions in which the inertia tensor is simple in the absence of a truly symmetric rigid mass distribution. Secondly, to provide an additional example to the usual solid cubes studied in sophomore or junior courses. The expression for one of the moments of inertia, found using $i = j = 1$ in (7), is:

$$I_{xx} = \sum_{\alpha} m_{\alpha}(y_{\alpha}^2 + z_{\alpha}^2) \tag{9}$$

The origin is placed at the center of mass of the cube, with co-ordinate axes oriented perpendicular to the faces of the cube. The axis about which the cube is rotated is labelled as the x-axis. As was the case in determining the centre of mass of the magic cube, the cube can be regarded as being composed of an arrangement of pillars running parallel to the rotation (x) axis, in pairs of equal and opposite displacement from this axis. There are then N pillars of mass C_3 with the same y value (and similarly with respect to the z-axis because of the line sum symmetry of the cube). This means that since one knows the magic constant (the sum of the masses in a line), and as the order of the cube determines the number of pillars which have the same y (or z) co-ordinate value, it is possible to remove this product from the sum over α in (9) since it is constant. That is, the cube is composed of N layers of the same mass, m_{layer}, which are parallel to each axis and at constant y (or z) co-ordinate value from it. Using (4), the sum of the masses in a layer of the cube is:

$$m_{layer} = NC_3 = \frac{N^2}{2}\left(N^3 + 1\right) \tag{10}$$

Which is used to re-write (9) as:

$$I_{xx} = m_{layer}\sum_{\alpha}(y_\alpha^2 + z_\alpha^2) = \frac{N^2}{2}(N^3+1)(\sum_{\alpha} y_\alpha^2 + \sum_{\alpha} z_\alpha^2) \tag{11}$$

Because the shortest distance from each sheet to an axis is constant, it is possible to re-write the co-ordinate values as series in terms of N. Each of the co-ordinates range from $-\left(\frac{N-1}{2}\right)$ to $\left(\frac{N-1}{2}\right)$, in increments of one, since these masses are arrayed on a unit cubic lattice the mass layers are separated from each other by a unit distance. This is a symmetric interval, and so the sum of the co-ordinates can be expressed as twice the value of the series from 0 to $(\frac{N-1}{2})$:

$$\sum_{\alpha} y_\alpha^2 = \sum_{\alpha} z_\alpha^2 = \sum_{k=-(N-1)/2}^{(N-1)/2} k^2 = 2\sum_{k=0}^{(N-1)/2} k^2 = \frac{1}{12}N(N^2-1) \tag{12}$$

Then, substituting this result into (11), the resulting formula for the moments of inertia (since all of the moments are equal) is:

$$I_{ii} = \frac{N^3}{12}(N^3+1)\left(N^2-1\right) \tag{13}$$

The inertia tensor for a magic cube of arbitrary order is expressed by (13) with (8). The moments of inertia of the $N = 3$ magic cube presented earlier (5) are then $I_{ii} = 504$, which can be verified by direct calculation. Therefore the seven trillion magic cubes of order four all have the same moments of inertia.

3 Explicit Verification of Steiner's Parallel Axis Theorem

It is further possible to adapt Steiner's parallel axis theorem [7] for use with the inertia tensor of a magic cube. Then one can shift the origin of the co-ordinate system so that the inertia tensor describes the magic cube with respect to a new set of axes. Steiner's parallel axis theorem may be written as:

$$I_{ij}^{'} = I_{ij} + M\left(a^2\delta_{ij} - a_i a_j\right) \tag{14}$$

In this equation I_{ij} represents the components of the inertia tensor with respect to a co-ordinate system with origin at the center of mass of the body, and

I'_{ij} represents the components of the inertia tensor with respect to some other co-ordinate system whose origin can be inside or outside the body under examination. The axes of the I'_{ij} system must also be oriented in the same manner as the I_{ij} system so in this case there will be an axis along each edge of the cube. The mass M is the total mass of the body under consideration. For a magic cube, this value is N^2C_3. In (14) $\vec{a}$ represents the displacement by which the origin is translated, and the components of this displacement are represented by a_n. In shifting the origin from the center of the cube to the corner, the components have the values $a_1 = a_2 = a_3 = -(\frac{N-1}{2})$. When the parallel axis theorem is applied to the inertia tensor of a magic cube, the elements of the tensor transform to:

$$I'_{i\neq j} = -\frac{N^3}{8}(N^3+1)(N-1)^2 \tag{15}$$

$$I'_{ii} = \frac{N^3}{6}(N^3+1)(N-1)(2N-1) \tag{16}$$

When $N = 3$ equation (15) gives $I'_{i\neq j} = -378$ and (16) gives $I'_{ii} = 1260$, which is verifiable using (7).

Interestingly, the inertia tensor of a magic cube itself is semi-magic. This means that the rows and columns of the tensor all add to the same value. In this case, the inertia tensor consists of identical off-diagonal elements and different but identical diagonal elements as well. This is of the form:

$$\boxed{\begin{matrix} \alpha & \beta & \beta \\ \beta & \alpha & \beta \\ \beta & \beta & \alpha \end{matrix}} \tag{17}$$

Clearly, adding the entries of the tensor along a row always gives a value $\alpha+2\beta$ as does adding along a column. Note that the main diagonal does not add to this sum, and so the inertia tensor is said to be semi-magic only. Because of this property of the tensor, it is possible to predict the existence of a $[1,1,1]$ eigenvector (associated with the eigenvalue equal to the row or column sum of the square, $\alpha + 2\beta$) which is significant because it is also found by studying magic squares of any order as eigensystems as shown by Loly, Hruska, Steeds and Williams [8]. Premultiplication of the tensor by this vector will produce a vector whose components are the sum of the columns of the tensor. Postmultiplication of the tensor by this vector will produce a vector whose components are the sum of the rows of the tensor. When the origin is placed at the corner of the cube, α is given by (16) and β is given by (15); when the origin is located at the center of mass of the cube, α is given by (13) and $\beta = 0$.

Since the tensor is symmetric, it can be diagonalized by a rotation of axes [6]. This is done by finding the eigenvalues of the tensor, which are known as the principal moments of inertia. A new matrix is written in which the diagonal elements consist of these eigenvalues and all off-diagonal elements vanish. However, since the components of the original tensor depend only on the order of the cube N, the eigenvalues are also found to be dependent upon N only. Once the tensor is built from it's components, the characteristic equation can be found by setting the determinant of $\lambda U - I^{'}$ to zero, where $I^{'}$ is the inertia tensor at the corner of the cube and U is the 3×3 identity matrix. The solution of the resulting characteristic equation results in the eigenvalues of the tensor. The eigenvectors associated with each eigenvalue λ_i are found by solving the system $(\lambda_i U - I^{'})\vec{x} = 0$ by standard procedure [6]. In the case of a magic cube, the inertia tensor has eigenvalues and eigenvectors:

$$\lambda_1 = \tfrac{N^3}{12}\left(N^3+1\right)\left(N^2-1\right), \quad v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{18}$$

$$\lambda_2 = \tfrac{N^3}{24}\left(N^3+1\right)\left(N-1\right)\left(11N-7\right), \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \tag{19}$$

$$\lambda_3 = \lambda_2, \quad v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \tag{20}$$

As in the case for a solid cube, the first principal axis lies along a triagonal of the cube. Since the other two eigenvalues are equal, the orientation of the principal axes associated with these two moments are arbitrary as long as the origin is held fixed [7].

4 Conclusion

The reader may have noticed that our derivations only make use of the row, column and pillar line sums (the RCP symmetry) of magic cubes, making no reference to any diagonal line sums. This means that the results apply to the larger class of semi-magic cubes. In fact, the RCP symmetry of a magic cube makes for simpler results than for magic squares, which have only RC symmetry.

The formula for I_N (13) with large N is consistent with the moment of inertia of a solid cube with mass $M = NC_N$ and side length $L = N$, recovering the familiar formula $I = \frac{1}{6}ML^2$ [7]. It should also be clear that cubic arrays of N^3 random masses also have line sums which tend to an average value so that the present results apply for random cubes with large N.

It may also interest the reader to know that advanced physics techniques, especially from statistical mechanics, have been the only way of estimating the populations of magic squares beyond order five. Pinn and Wieczerkowski [9] obtained an estimate of $1.775399\ (42) \times 10^{19}$ for the number of order six magic squares.

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