The Inertial Properties of Magic Squares and Cubes

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1 The Moment of Inertia of Magic Squares

Magic squares of order N are composed of the entries $1..N^2$ arranged on a square unit lattice such that the sum of all entries along the rows, columns and main diagonals are equal to the magic constant of the square. An example of a magic square is shown below:

$$\begin{array}{|ccc|}\hline 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \\ \hline\end{array} \tag{1}$$

The magic constant can be easily found by summing the values $1..N^2$ and dividing by N, the number of rows and columns to find

$$C_2 = \frac{N}{2}(N^2 + 1). \tag{2}$$

For $N = 3$, the magic constant is equal to 15 [1].

Though there is only one magic square of order 3 apart from trivial rotations and reflections, the number of squares per order quickly skyrockets. There are 880 distinct order 4 squares, and 275, 305, 224 distinct order 5 magic squares [2].

If we interpret magic squares as being comprised of masses proportional to the entries of the squares we can determine their moment of inertia about a given axis of rotation. The scalar moment of inertia I is found by summing mr^2 for each entry in the square, where m is the mass of the entry and r is the distance of that entry from the axis of rotation. If we consider an axis of rotation through the middle row (in the case of even order squares the rotation axis lies between the two middle rows) and it's counterpart through the middle column, it is obvious that the moments of inertia should be equal, since there are an equal number of rows/columns of the same line mass equally displaced from these axes. We can use the perpendicular axis theorem, which states [3]:

$$I_z = I_x + I_y \tag{3}$$

With $I_x = I_y$, we have $I_z = 2I_x$. If we place an axis parallel to one edge of the square it is easy to derive a general formula for the moment of inertia about that axis. Because we know the sum of values in a line and the spacing of the masses, we can find a formula in terms only of N, the order of the square, much like the line sum formula. From here we can use the parallel axis theorem to shift the axis so it passes through the center of the square. An application of the perpendicular axis theorem will then give us the moment of inertia perpendicular to the plane of the square. We find the simple formula:

$$I_z = \frac{1}{12} N^2 (N^4 - 1) \tag{4}$$

For $N = 3$, $I_z = 60$ which can be verified explicitly using (1).

This is the only other property of magic squares, aside from the line sum, which is solely dependent on the order of the square, N. It is also worth noting that since we have only made use of the row and column line sums, the formula is general for semi-magic squares as well. These types of squares have only row and column line sums, but no restriction on the main diagonal sums. Eq.(4) is consistent for large N with the moment of inertia of a continuous plate with mass $M = NC_2$ and $L = N$, reducing to $I = \frac{1}{6}ML^2$ [4].

Because of the simple application of inertia principles and mathematics, this derivation is suitable for first year physics students. Magic squares have a few practical applications, including uses in cryptography [5] and image processing [6]. When treated as matrices, magic squares also serve as exceptional examples of some advanced linear algebra theorems. When treated as mass distributions magic squares give clear and accessible examples of the properties of the moment of inertia.

2 The Inertia Tensor of Magic Cubes

We can extend the concept of a magic square into the third dimension. When we do this we get a magic cube. These cubes have constant Row, Column, Pillar (referred to as RCP) and main diagonal line sums. An example is presented below.

1st layer			2nd layer			3rd layer		
2	13	27	16	21	5	24	8	10
22	9	11	3	14	25	17	19	6
18	20	4	23	7	12	1	15	26

(5)

Magic cubes are composed of entries $1..N^3$, and in an analogous manner to the procedure for magic squares, we can find a formula for the line sum [1]:

$$C_3 = \frac{N}{2}(N^3 + 1), \tag{6}$$

Since each of these layers is a magic square, though not of consecutive integers, it is easy to find the moment of inertia of a single layer as an extension of (4), and thus that N stacked layers give the moment of inertia of a magic cube:

$$I_z = \frac{1}{12} N^3 (N^3 + 1)(N^2 - 1), \tag{7}$$

By RCP symmetry, $I_x = I_y = I_z$. More formally, the inertia tensor is also diagonal (the off-diagonal elements vanish) with the origin of co-ordinates at the center of the cube. This shows that magic cubes have the same inertial form as a spherical top [7].

The derivation of this property of magic cubes is appropriate for a sophomore course in classical mechanics, and provides a great example of the properties of the inertia tensor [8].

3 Conclusions

Throughout our discussion of magic cubes, we have considered the entries in the magic cube only as masses. If we consider the entries instead as charges, we can neutralize the cube by subtracting the average entry from the rest of the cube. This will have the effect of causing the first three multipole moments of the charge distribution to vanish, with only off-diagonal elements of the octupole tensor [9] remaining. To third order a discrete charge distribution in the form of a magic cube will produce no electrical potential.

Just as magic squares are easily extended into the third dimension to create a magic cube, magic cubes can be extended further into the fourth dimension, forming magic hypercubes. These can, in fact, be generalized into N-dimensions, though these objects have not been the focus of our studies, since the inertia tensor and multipole expansion function only in three dimensions.

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References

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