Signatura of magic and Latin integer squares: isentropic clans and indexing.

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Abstract

The 2010 study of the Shannon entropy of order nine Sudoku and Latin square matrices by Newton and DeSalvo [Proc. Roy. Soc. A 2010] is extended to natural magic and Latin squares up to order nine. We demonstrate that decimal and integer measures of the Singular Value sets, here named SV clans, are a powerful way of comparing different integer squares.

Several complete sets of magic and Latin squares are included, including the order eight Franklin subset which is of direct relevance to magic square line patterns on chess boards. While early examples suggested that lower rank specimens had lower entropy, sufficient data is presented to show that some full rank cases with low entropy possess a set of singular values separating into a dominant group with the remainder much weaker. An effective rank measure helps understand these issues.

We also introduce a new measure for integer squares based on the sum of the fourth powers of the singular values which appears to give a useful method of indexing both Latin and magic squares. This can be used to begin cataloging a "library" of magical squares.

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1 Introduction

The Shannon entropies of Sudoku matrices were studied by Newton and DeSalvo [39, NDS], but while they did not mention magic squares, much of what they examine for 9th order Latin and Sudoku squares is immediately applicable to magic squares, as well as Latin squares of arbitrary order. The essential input to the calculation of the Shannon entropy is the singular value decomposition

[24, SVD] of the matrices. For matrix A we need the eigenvalues of the matrices AA^T or A^TA, also known as Gramian matrices, which give the squares of the singular values (SVs) σ_i [33]. The number of non-zero SVs give the rank of a matrix. These SVs were recently studied for selected magic squares from order 3 to order 8 by Loly, Cameron, Trump and Schindel [33, LCTS] in order to help understand the complicated behaviour of their eigenvalues, which change under the standard eight magic square symmetries. For instance, rotations and reflections of the matrix elements can cause the eigenvalues to flip sign, change from real to complex values or even vanish. The main focus in LCTS concerned the eigenvalues λ_i, and only selected SVs were reported. Magical squares, i.e. magic, semi-magic and Latin squares are integer matrices, and as such often exhibit more elegant results than real number matrices. Most of our entropy results are lower than the averages given by NDS, often dramatically. SVs give an advantage over the eigenvalues since the SVs are invariant under rotation and/or reflection of the matrices [32, 33], as well as tiling to semi-magic squares. We found many degenerate (equal) SVs in studies of small Latin squares, as well as a few for magic squares. Thus Shannon entropy is a very useful metric for comparing different magical squares, as well as for other semi-magic squares [58].

Following NDS, we also calculate a percentage of compression factor for the reduction of the entropy from a reference maximum entropy which goes as the logarithm of the order of the squares, $\ln(n)$, in order to provide a comparison across different orders. NDS found average compressions in the range of 21 to 25%. While there is only one magic square of order three (the ancient Chinese Loshu), with a remarkably small compression of 14%, there are complete sets for orders four and five, with 880 (see [31] for an unsorted list) and $275,305,224$ distinct members respectively. Most of our focus is on individual squares deemed interesting for their extreme values of Shannon entropy. For higher orders the populations are so large that they are only known statistically [61, 33], except in special cases such as the 8th order Franklin squares counted in 2006 by Schindel, Rempel and Loly [50].

An explicit calculation of the Shannon entropy is given for the Loshu magic square in section 2. Since sufficient data is later presented to show that some full rank cases with low entropy possess a set of SVs separating into a dominant group with the remainder much weaker, an effective rank measure is introduced there to help understand these issues. A discussion of the characteristic polynomials for matrix eigenvalues and singular values is given in section 3. As a result we introduce some integer measures in section 3.1 that are useful in keeping track of individual examples and in section 4.1 offer a method of indexing integer squares. Section 4.2 enables extreme bounds to be found for the entropies, compressions and indices.

After examining general aspects of magic squares in section 5, the complete set of magic squares of order four is examined in new detail in section 5.3. The complete set of magic squares of order five is briefly reported in section 5.4, and section 6 gives a selection of examples of order 8 and 9 magic squares. Special attention is paid to the cases of 3 non-zero eigenvalues (rank 3) for which calculations simplify. In section 7 we examine small Latin squares up to the complete set at order 6 to give deeper insight than afforded by the ensemble averages employed in the NDS study. We also report a wider range of Sudokus than NDS.

As physicists the authors are keenly aware that using "entropy" in the sense here may trouble some readers. According to legend Von Neumann [18] suggested Claude Shannon call his formula "entropy" for two reasons: first, his uncertainty function had been used in statistical mechanics under that name so it was already known, and second no one really knows what entropy actually is, so in a debate he would always have the advantage. NDS interpret the SVD expansion $A = \sum_{i=1}^{n} \sigma_i \mathbf{u}_i \mathbf{v}_i^T$ as a distribution of "energy" among the set of "normal modes" $\mathbf{u}_i \mathbf{v}_i^T$. The Shannon entropy is a scalar measure of this energy distribution quantifying the disorder (randomness) associated with a given matrix A, and depends on how fast the SVs σ_i decay with increasing i.

2 Shannon entropy for the ancient Loshu magic square

We set the scene for several issues by first examining the sole natural magic square of order three:

$$Loshu = \begin{bmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{bmatrix}, \tag{1}$$

with the characteristic polynomial $x^3 - 15x^2 + 24x - 360$, where 15 is the linesum eigenvalue, 360 is the determinant, and 24 is the sum of the determinants of the three 2-by-2 principal minors. It exhibits full cover of the matrix elements 1...9 in a manner such that all antipodal pair sums about the centre yield 10, so that it belongs to the type of magic square called regular (or associative). The Loshu has eigenvalues $\lambda_1 = 15$, the Perron root for a positive square matrix, and a signed pair $\pm 2i\sqrt{6}$ [33]. Note that $\sum_{i=1}^{n} \lambda_i = \lambda_1$ for all magic squares [33].

Let us take here $A = Loshu$ from (1), then:

$$AA^T = \begin{bmatrix} 101 & 71 & 53 \\ 71 & 83 & 71 \\ 53 & 71 & 101 \end{bmatrix}, \; A^T A = \begin{bmatrix} 89 & 59 & 77 \\ 59 & 107 & 59 \\ 77 & 59 & 89 \end{bmatrix}, \tag{2}$$

which both have the same eigenvalues (the squared SVs σ_i^2) since in general these products always have identical characteristic polynomials:

$$X^3 - 285X^2 + 14\,076X - 129,600; \; \sigma_i^2 = 225, 48, 12. \tag{3}$$

Observe that the sum of these eigenvalues is an integer: $\sum_{i=1}^{n} \sigma_i^2 = 285$. In fact Appendix A shows that all natural squares, not just magic squares, have an integer sum of the squares of the SVs which is an invariant for each order. While generally AA^T and $A^T A$ are each symmetric, here they are bisymmetric (symmetric about both main diagonals).

The eigenvalues of AA^T are the squares of the SVs, with the σ_i listed in non-increasing order ($\sigma_i \geq \sigma_j$, $i < j$):

$$\sigma_1 = 15,\ \sigma_2 = 4\sqrt{3},\ \sigma_3 = 2\sqrt{3}, \tag{4}$$

where σ_1 is the same as the trace of the magic square, which equals λ_1 since the other eigenvalues ($\lambda_2 ... \lambda_n$) add to zero [33]. We now follow NDS [39] and normalize the σ_i by their sum:

$$\hat{\sigma}_i = \frac{\sigma_i}{\sum \sigma_i}, 0 \leq \hat{\sigma}_i \leq 1. \tag{5}$$

The decimal values will usually be rounded to single precision or less, unless otherwise stated. Then we obtain the Shannon entropy

$$H = -\sum_i \hat{\sigma}_i \ln\left(\hat{\sigma}_i\right), \tag{6}$$

finding $H = 0.937...$ as per NDS. The SVs contribute differently to the entropy, 0.31096, 0.35439, 0.27175, respectively, with the largest contribution from σ_2 in this case.

Finally we find the percentage compression, C, by generalizing NDS's for the $n = 9$ Sudoku case to reference $\ln(n)$ instead of their $\ln(9)$:

$$C = \left(1 - \frac{H}{\ln(n)}\right) \times 100\%, \tag{7}$$

finding $C = 14.7\%$. Note that compression varies oppositely to the entropy for a given order.

We then go further than NDS and consider an effective rank measure [48]:

$$erank = \exp(H). \tag{8}$$

For the rank three Loshu (1), $erank = 2.55256$, the reduction from full rank reflecting the decreasing magnitudes of the SVs. Note that it is sufficient to list just one of H, $erank$, or C, since the other quantities can be obtained from (7) and (8) when H and n are known. Later we find $erank$ to be very useful in comparing different magical squares.

3 Applying the fundamental theorem of algebra

Consider a square matrix A of order n, with eigenvalues λ_i. The general nth order characteristic polynomial $\alpha(x)$ may be factored to show its n roots [24, 7]:

$$\alpha(x) = \prod_{i=1}^{n} (x - \lambda_i) = x^n - a_1 x^{n-1} + a_2 x^{n-2} - ... \pm a_n = 0. \tag{9}$$

In 1629, more than two centuries before the matrix theories of Cayley and Sylvester, Girard [22, 35, 20] showed that the first few coefficients a_1, a_2, a_3,... in (9) gives the sum of the roots (i.e., the trace of the matrix eigenvalues), the sum of the squares of the roots, the sum of the cubes, etc:

$$G_n = \sum_{i=1}^{n} \lambda_i^n;\ G_1 = a_1;\ G_2 = a_1^2 - 2a_2;\ G_3 = a_1^3 - 3a_1a_2 + 3a_3;\ ... \tag{10}$$

Later these identities were rediscovered by Newton, and are often known as Newton's identities.

3.1 Gramian matrices and application to singular values

Since the SVs are the square roots of the Gramian eigenvalues σ_i^2 they must satisfy a variant of (9) obtained by interchanging $x \to X$, hence:

Theorem 1 *If the characteristic polynomial $\beta(X)$ of the nth order Gramian of the square matrix A is factored to show its roots in σ_i^2*

$$\beta(X) = \prod_{i=1}^{n} \left(X - \sigma_i^2\right) = X^n - b_1X^{n-1} + b_2X^{n-2} - ... \pm b_n = 0, \tag{11}$$

then from the Girard identities the sums of the even power of the SVs σ_i are given by:

$$P_n = \sum_{i=1}^{n} \sigma_i^{2n};\ P_1 = b_1;\ P_2 = b_1^2 - 2b_2;\ P_3 = b_1^3 - 3b_1b_2 + 3b_3;\ ... \tag{12}$$

It is worth noting a connection with Schatten $p-$norms and Ky Fan's $p - k$ norms [24], both of which include even and odd powers p of the SVs, rather than just the even powers which flow from our use of Girard's results [22]. Note that these norms involve the $p-$th roots of the sums of powers p of the SVs. Ky Fan's k-norm of A uses the k largest singular values of A, so that his 1-norm is the largest singular value of A, while the last of his norms, the sum of all singular values, is called the trace norm. Schatten's 2-norm is the square root of the squares of all the singular values of A.

3.2 A further step

We form a reduced polynomial by factoring out $(X - \sigma_1^2)$ from (11) and so:

Theorem 2 *For the Gramian of the nth order square matrix A the characteristic polynomial $\beta(X)$ with coefficients b_i and n roots σ_i^2 can be expressed as the reduced polynomial $\gamma(X)$ given by*

$$\gamma(X) = \prod_{i=2}^{n} \left(X - \sigma_i^2\right) = X^{n-1} - d_1X^{n-2} + d_2X^{n-3} - ... \mp d_{n-1} = 0, \tag{13}$$

with the coefficients d_i expressed in terms of the b_i and σ_1^2:

$$d_1 = b_1 - \sigma_1^2,\ d_2 = b_2 - \sigma_1^2 d_1,\ d_3 = b_3 - \sigma_1^2 d_2,\ ... \tag{14}$$

4 Integer matrices

We examine integer functions of the SVs which are useful for keeping track of general integer square matrices. The a_i coefficients are now integers in (9) and (10), as are the b_i and d_i coefficients in (11), and (12). Appendix A shows that P_1 has an n-dependent value for all natural squares of any order. From (12) that $P_2 = \sum_{i=1}^{n} \sigma_i^4$ is also an integer, even though we can show that individual σ_i^2 are not always integer. In the case of semi-magic squares of sequential integers the d_i are now independent of whether the first element is 1 or 0.

4.1 Long and Short indices

From Theorem 1 and 2 we have, since b_1 and b_2 are integers, the sum of the fourth powers of the SVs gives a Long integer index, L, for all integer squares:

$$L = \sum_{i=1}^{n} \sigma_i^4 = b_1^2 - 2b_2 = \sigma_1^4 + d_1^2 - 2d_2, \tag{15}$$

where the b_i are the coefficients of the characteristic polynomial of the Gramian and the d_i are the coefficients of the reduced characteristic polynomial. From this, we define a Reduced integer index, R, such that

$$R = L - \sigma_i^4. \tag{16}$$

So while index L appears to be useful for indexing all integer squares, index R offers a shorter reduced index for semi-magic squares. For the Loshu square in (1), $L = 53073$ and $R = 2448$. Note that R, as the sum $\sum_{i=2}^{n} \sigma_i^4$, is independent of the choice between using elements $1...n^2$ or $0...\left(n^2 - 1\right)$, although L will change. One might consider using b_2 as an index since b_1^2 and λ_1^4 depend only on n, however numerically $b_2 > R$, and the smaller index is preferable. In any case the b_i depend on whether the elements run from 0 or 1.

A third integer key, Q, which might be useful in resolving degeneracies in R (and L) is obtained from the sixth powers of the SVs:

$$Q = \sum_{i=2}^{n} \sigma_i^6 = b_1^3 - 3b_1b_2 + 3b_3 - \sigma_1^6 = d_1^3 - 3d_1d_2 + 3d_3. \tag{17}$$

Note that we recommend that the integer b_i or d_i be used to obtain R, L, Q to avoid rounding errors in computed values of the SVs which may enter the direct calculation of sums of powers of the SVs.

The integer measures, L and R, vary amongst (integer) Latin and magic squares of a given order, and those for different orders are separated by large gaps. Duplicate (degenerate) values occur when transformations, e.g. certain row-column permutations, produce a distinct magic square with the same H. These isentropic sets will be called "clans". Later we find some cases where the same key occurs for different clans with distinct values of H, e.g. we find one order four example of the same L, R for different H in Section 5.

4.2 Bounds for integer magic and Latin squares

Two extremes can be identified: first, a rank two with $\sigma_{3...n}$ all zero. From P_1 (12) it then follows that:

$$\sigma_2 = \sqrt{b_1 - \sigma_1^2}, \tag{18}$$

where Appendix A gives general expressions for b_1 for any order, separately for Latin and for magic squares. These lead to a lower bound on the entropies and an upper bound on the compressions.

Secondly, a full rank n with $\sigma_{2...n} = \sigma_n$, i.e. all but the first equal to σ_n:

$$\sigma_n = \sqrt{\frac{b_1 - \sigma_1^2}{n-1}}, \tag{19}$$

which gives the opposite bound.

5 Magic squares $1...n^2$ (full cover)

The number of distinct magic squares grows rapidly from the unique third order Loshu, through 880 at fourth order and $275,305,224$ at 5th order, with only statistical estimates available for $n > 5$ [61]. The magic linesum is:

$$\text{Magic line sum: } S_n = \lambda_1 = n(1+n^2)/2 \tag{20}$$

5.1 Principal types of magic squares

There are five particularly important types of magic squares:

- Associative (or regular) have elements antipodal about the centre with the same pair sum, with the Loshu (1) as an example:

$$a_{ij} + a_{n-i+1,n-j+1} = (1+n^2), \; i,j = 1,...,n \tag{21}$$

- In pandiagonal (also called Nasik) squares the broken diagonals (n consecutive elements parallel to the main diagonals under tiling, or periodic boundary condtions) have the same sum as the main diagonals. We note that there are integer squares which are pandiagonal, but not magic [29].

- Ultramagic squares have both the associative and pandiagonal properties. For singly even orders, e.g. $n = 6, 10, 14, ...$, there are no associatives or pandiagonals, and so no ultramagics.

- Franklin's famous squares [50] for $n = 8, 16$ have 'bent' diagonals with elements adding to the magic sum, but not necessarily the main diagonals. They also have a fixed sum for the half rows and columns, and with all 2-by-2 quartets having a fixed sum. At order 8 one third are pandiagonal and therefore magic. Franklin squares are interesting here because they exhibit the minimum rank of three expected for a magic square.

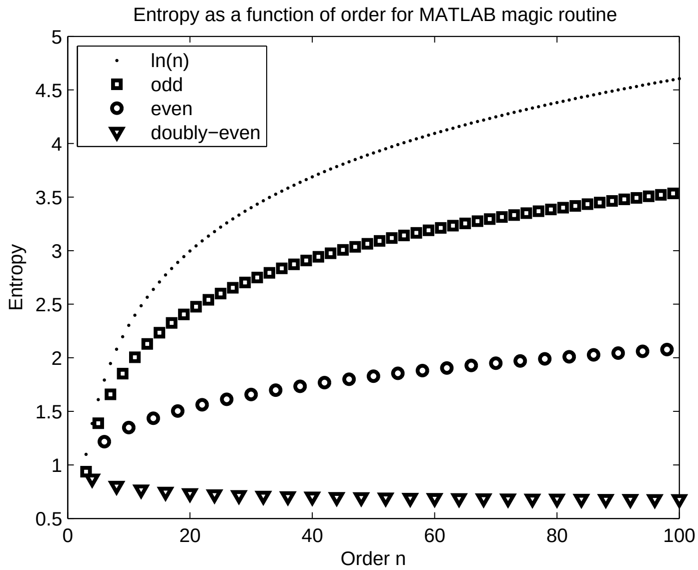

Figure 1: Entropy as a function of order n for MATLAB routine magic(n).

- Compound magic squares (CMSs) [47] of order $n = pq$ have order p or q tiled magic subsquares. These begin at $n = 9$ and are interesting because they have lower ranks than all but Franklin type magic squares when they share the same doubly even order ($n = 12, 16, 20, 24, ...$).

5.2 Overview of magic squares as order increases

Figure 1 shows the results of using the MATLAB MAGIC(n) function (we use *magicn* to identify these in later tables), together with the upper bound of $\ln(n)$ for $n = 3...100$. MATLAB [37] uses one algorithm to produce non-singular (full rank) associative magic squares for odd order, and a second algorithm for singular even order magic squares which are not associative but have rank $(n+4)/2$. MATLAB's third algorithm gives a family of rank 3 (singular) doubly even magic squares.

Figure 1, while based on the MATLAB algorithms, suggests that higher rank magic squares have the higher entropies, with rank three magic squares having the lower entropies. This figure is analogous to one presented in Rao et al [46] in the context of the Indus script.

5.2.1 Large n limit for doubly even cases

Kirkland and Neumann [25] showed that MATLAB's third algorithm leads to algebraic results for the three non-zero eigenvalues and SVs. As $n \to \infty$, the ratio $\sigma_1/\sigma_2 \to \sqrt{3}$, while $\sigma_3/\sigma_1 \to 0$, with the result that $H \to 0.6568...$, in

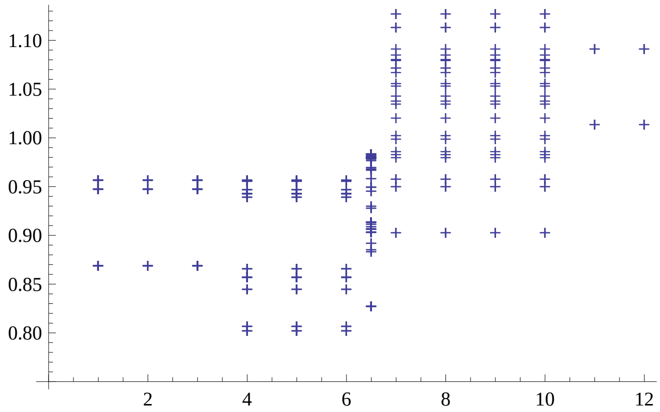

Figure 2: H-values for Dudeney's groups

accord with the asymptotic lower branch of Figure 1. The compression (7) increases gradually with n as $\left(1 - \frac{H}{\ln(n)}\right)$, becoming 95.25% for $n = 10^6$. Also, index R can be exactly calculated from their SV formula since their squares are integers.

5.3 Complete set of order 4 magic squares

Dudeney [14] gave the first complete classification of Frénicle's [3] 880 magic squares into a dozen Groups, I-XII, distinguishing arrangements in order four magic squares made by linking pairs of elements which sum to $17 = 1 + (n = 4)^2$. Group VI splits into two distinct sets of SV values, the semi-pandiagonal set VI-P sharing SVs with Groups IV and V, and VI-S with quite distinct SVs, justifying the split made by Dudeney.

For this set Loly, Cameron, Trump and Schindel [33] used values of σ_2^2 to find 63 distinct SVs, however the NDS work was not available to suggest any further exploration at that time. Now it is clear that there are five Supergroups; A, with Groups I (pandiagonal), II (semi-pandiagonal and semi-bent), and III (associative); B, with Groups IV, V, and V-P which are all semi-pandiagonal; C, with just Group VI-S which stands alone (and which Dudeney [14] had distinguished from Group VI-P); D, with Groups VII, VIII, IX and X and finally E, with Groups XI and XII. The 640 members of Groups I-VI are singular with rank 3, while the 240 members of Groups VII-XII are non-singular with rank 4. For numerical convenience in sorting and plotting results we list the Dudeney Groups by a numerical label $1 - 12$, with the subgroup VI-S as 6.5, since it fits logically between Groups VI-P and VII.

Figure 2 shows the Shannon entropy H versus these numerical group labels in order to emphasize the Supergroup sets, and so that the high and low values of the entropies of the Supergroups can be compared.

In Table 1 we examine entropies of this set for representative individual magic squares to provide a single decimal value for comparisons. We prefer *erank*, but also list H and C for comparison with NDS. The R-index is also recorded. These magic squares are labelled by Frénicle's 1693 index, from $F1$ to $F880$, for which Benson and Jacoby's Appendix [3] is a standard reference. We also provide a new order 4 clan index:

r^{a}	Group	$F^{\mathrm{b}}, clan^{\mathrm{c}}$	H	C (%)	R-index	*erank*
2	*bound*2		0.64845	53.22	115600	(2)
3	**G1-3**(α)	**F21, c9**	**0.87020**	**37.23**[d]	**102800**	**2.38739**
3	G1-3(β)	$F56, c35$	0.94874	31.56	78608	2.58246
3	G1-3(γ)	$F82, c41$	0.95819	30.88[e]	74000	2.60697
3	G4-6-P[f]	$F25, c1$	0.80375	42.02[d]	111376	2.23390
3	G4-6-P	$F73, c40$	0.95794	30.90[e]	74128	2.60632
3	G6-S	$F51, c4$	0.82847	40.24[d]	109000	2.28981
3	G6-S	$F46, c62$	0.98500	28.95[e]	58000	2.67781
3	**G6-S**[g]	$F1, c26$	**0.92925**	**32.97**	**86728**	**2.53260**
4	**G8**[g]	$F278, c25$	**1.0810**	**22.02**	**86728**	**2.94756**
4	G7-10	$F10, c3$	0.90403	34.79[d]	109264	2.46954
4	G7-10[f]	$F26, c63$	1.1286	18.59[e]	57232	3.09137
4	G11,12	$F3, c19$	1.0152	26.77[d]	93584	2.75996
4	G11,12	$F88, c42$	1.0925	21.19[e]	73424	2.98179
4	*boundn*		1.2248	11.65	$38533\frac{1}{3}$	(4)

Table 1: Order four magic squares [$\ln(4) = 1.3863...$]

[a] Rank; [b] Frenicle index; [c] clan number; [d] Highest compression of group; [e] Lowest compression of group; [f] 880 set minimax compression; [g] degenerate R pairs.

N.B. clan α in Table 1 (also boldened) hosts the ancient Jupiter magic square, Dürer's famous 1514 square, as well as one of Franklin's magic squares [44]. To our surprise the well known pandiagonals and associatives in Dudeney Groups 1-3 do not exhibit the smallest values of H. The minimum entropy occurs in Groups 4, 5 and 6, while the maximum entropy occurs in Groups 7, 8, 9 and 10.

Figure 3 shows the Shannon entropy plotted against the index R for the 63 order four clans. The lower envelope contains all the singular clans in Groups I through VI-P and -S. These singular clans lie on a particular curve due to an implicit relationship between σ_2 and σ_3 for rank 3 since σ_1 is constant, 34, and σ_4 vanishes, as discussed in Appendix A. The scattered points above this curve are all non-singular clans of Groups 7 through 12.

5.3.1 Are σ_i^2 always integers for magic squares?

The σ_i values for $F25$: 34, $\sqrt{2(85+\sqrt{6697})}$, $\sqrt{2(85-\sqrt{6697})}$, answer negatively the Question: Are the Gramian eigenvalues, σ_i^2, always integers for magic squares? Clearly not.

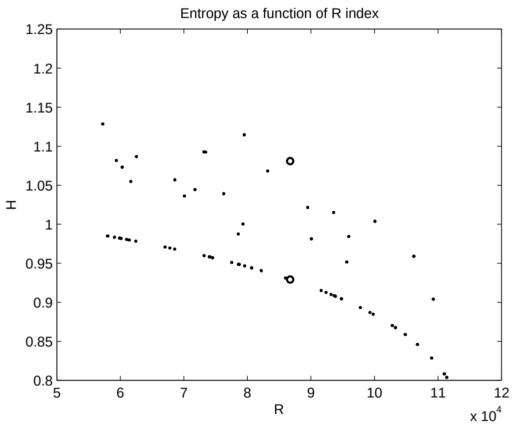

Figure 3: Entropy as a function of R. The data is double valued at $R = 86728$, and marked with circles.

5.3.2 Two SV sets, one non-singular, the other singular, with same index R, but different H

Note that in Figure 3 for the index value of $R = 86728$ there are two distinct clans, one from the singular Group 6-S on the lower envelope, clan 26, $F1$, Group 6.5, with $H = 0.9292...$, $C = 33\%$, and the other with a much higher entropy, clan 25, $F278$, Group 8, with $H = 1.0809..., C = 22\%$, from the non-singular Groups $7 - 10$. This unique feature arises from the difference in rank of these clans.

$$F1, c26 = \begin{bmatrix} \mathbf{1} & \mathbf{2} & 15 & 16 \\ 12 & 14 & \mathbf{3} & 5 \\ 13 & 7 & 10 & \mathbf{4} \\ 8 & 11 & 6 & 9 \end{bmatrix} ; F278, c25 = \begin{bmatrix} \mathbf{2} & 6 & 15 & 11 \\ 16 & 13 & \mathbf{4} & \mathbf{1} \\ 9 & 12 & 5 & 8 \\ 7 & \mathbf{3} & 10 & 14 \end{bmatrix} . \qquad (22)$$

The SV characteristic polynomial for the rank three $F1$ is:

$$\sigma^8 - 1496\sigma^6 + 407\,476\sigma^4 - 16\,688\,016\sigma^2 = 0 \ , \qquad (23)$$

while for the rank four $F271$ there is a constant term, as well as a changed quadratic term:

$$\sigma^8 - 1496\sigma^6 + 407\,476\sigma^4 - 16\,835\,472\sigma^2 + 170\,459\,136 = 0. \qquad (24)$$

Thus there are 63 different SV sets, but only 62 different index values. Many such degeneracies occur for order 6 Latin squares, as shown later in Section 7.3.

5.3.3 Index degeneracy split with Q

Since $P_1 = \sum_{i=1}^{n} \sigma_i^2$ in (12) is invariant for $n = 4$, and $P_2 = \sum_{i=1}^{n} \sigma_i^4$ in (12) are the same for these squares, the degeneracies are first resolved by using Q

in (17). In this case, since b_1 and b_2 are the same, this is due entirely to the different b_3's, for an amount 442368 (for both L, R).

5.4 Complete Set of order 5 Magic Squares

A similar analysis has been carried out for the $275,305,224$ order 5 magic squares where there is no comparable classification scheme so that the distribution of H and C obtained gives new insight into that large complete set. There are no rank 3 fifth order magic squares, only the singular rank fours and full rank fives. A few examples are given in Table 2, together with the bounds outlined in section 4.2.

Eight clan pairs of ultramagic squares [55] are included in Table 2. To save on display space the matrix elements are relegated to the Electronic Supplement, *shannonData.txt*.

r	square	H	$C\,(\%)$	R	$erank$
4	$bound2$	0.91420	43.2	$1,690,000$	(2)
4	$lcts43m2$ [a]	1.05067	34.72	$1,218,640$	2.85955
5	$suz6,9$ [b]	1.12526	30.08	$954,480$	3.08101
5	$suz2,13$ [b]	1.12706	29.97	$904,500$	3.08658
4	$lcts44m4$ [a]	1.20122	25.36	$706,000$	3.32418
5	$suz5,10$ [b]	1.20354	25.22	$822,000$	3.33188
5	$lcts45$ [a], $suz1,14$ [b]	1.23161	23.48	$772,980$	3.42674
$4,5$	**full set**	1.2827	20.3		
5	$suz7,12$ [b]	1.37667	14.46	$522,480$	3.96171
5	$suz3,16$ [b]	1.3783	14.36	$544,500$	3.96813
5	$suz8,11$ [b]	1.38085	14.20	$582,000$	3.97827
5	$suz4,15$ [b]	1.38229	14.11	$604,980$	3.98403
5	$magic5$ [c] $(Mars)$	1.38932	13.68	$532,000$	4.01212
5	$boundn$	1.442	10.4	$422,500$	(5)

Table 2: $n = 5$ [entropy increases down column 5, $\ln(5) = 1.6094...$]

[a] [33]; [b] [55]; [c] [37].

The maximum range of R values in Table 2 is only $1\,690\,000 - 422\,500 = 1\,267\,500$, and this must accommodate $275,305,224$ squares with $22,598,324$ different clans, so while R might still be useful as a preliminary index for order five, further index splitting with Q in (17) is eventually needed because of inevitable degeneracies in R. Also the ratio of extreme compressions for order five is about $4:1$, compared to $2.25:1$ for the order fours, but well within the $5:1$ of the order five bounds.

For the higher compression $suz6,9$ the respective contributions of the descending SVs to the entropy are: $\{0.3308, 0.3468, 0.2989, 0.08561, 0.0632\}$, where as usual σ_2 is the largest contribution, and the last two are much smaller than first three. The lower compression $suz4,15$ exhibits an effect not so far found in our studies of any other magic squares, namely the first SV, σ_1, gives the largest contribution to the entropy: $\{0.3507, 0.3195, 0.2563, 0.2323, 0.2234\}$, with the last three SVs all substantial.

Walter Trump (2007) Trump [61] found the eigenvalue characteristic polynomials of $245,824$ different singular cases, which is less than 0.1% of $275,305,224$ squares. He also found 10 different determinants for the non-singular cases.

6 Higher orders

The situation for higher orders is more difficult because there are only statistical estimates of the very large populations for order six and higher [45, 61]. We skip to order 8 and 9 for which there are many interesting cases. For order 8 some examples closely approach the bounds outlined in section 4.2, especially for the lower entropy bound of order 8. Our data for orders 6 and 7 fall well short of the lower entropy bounds and in neither case do we have a rank three example. Order 9 affords a link to NDS which prompted the present work. Euler [17] had shown how Latin squares could be used to construct some natural magic and semimagic squares.

6.1 Order 8 magic squares

Trump [61] estimated that there are about 5.2×10^{54} magic squares for order eight, while we have a range of R about 409×10^6 from the bounds in Table 3. Amongst other data which we are able to include at order 8 is the complete set of Ben Franklin's 8th order squares constructed by Schindel, Rempel and Loly [50]. That study did not examine their eigenproperties, but subsequently they were all found to have rank 3. These Franklin squares exhibit 64 clans. In Table 3, we show several order 8 magic squares with rank 3, the minimum rank for magic squares (Drury [13]) [see Appendix A for simplifications for rank 3, which we discussed earlier for order four].

r	square	H	$C\,(\%)$	R	*erank*
2	*bound2*	0.65479	68.51	476, 985, 600	(2)
3	$bf\min$ [a,b]	0.70720	65.99	476, 125, 440	2.02830
3	**magic8** [c]	**0.80335**	**61.37**	**462, 534, 912**	**2.23301**
3	*dko86* [d]	0.87072	58.13	431, 200, 512	2.38862
4	*wht8a* [e]	0.91569	55.96	431, 560, 960	2.49850
3	$bf\max$ [a]	0.97144	53.28	299, 964.672	2.64173
7	*gaspalou13* [f]	1.2867	38.12	224, 354, 880	3.62073
5	*euler8* [g,b]	1.4157	31.92	136, 171, 776	4.11942
7	*bimagic* [h] (bi8a)	1.5251	26.66	152, 375, 424	4.59565
8	*i88e8* [i]	1.5480	25.56	283, 664, 192	4.70214
7	*wtREG841BI* [j]	1.6829	19.07	102, 971, 488	5.38120
8	*eulerKnight* [k,b]	1.6887	18.79	146, 064, 640	5.41233
8	*boundn*	1.8415	11.44	68, 140, 800	(8)

Table 3: $n = 8$, $[\ln(8) = 2.0794...]$

[a] Franklin 4320 set [50] $bf\min(H), [\max(C)]$; [b] semimagic; [c] [37] and many other squares; [d] [41]; [e] [60]; [f] [21]; [g] [17]; [h] [5]; [i] Ian8 [lowest H]; [j] [61]; [k] [17].

The most highly compressed clan, (Franklin) $bf\min$, is shown at the top of Table 3 ($C = 66\%$, which is 96% of our upper bound compression) followed closely by the *magic8* MATLAB square. These are displayed below:

$$bf\min = \begin{bmatrix} 1 & 63 & 8 & 58 & 3 & 61 & 6 & 60 \\ 56 & 10 & 49 & 15 & 54 & 12 & 51 & 13 \\ 57 & 7 & 64 & 2 & 59 & 5 & 62 & 4 \\ 16 & 50 & 9 & 55 & 14 & 52 & 11 & 53 \\ 17 & 47 & 24 & 42 & 19 & 45 & 22 & 44 \\ 40 & 26 & 33 & 31 & 38 & 28 & 35 & 29 \\ 41 & 23 & 48 & 18 & 43 & 21 & 46 & 20 \\ 32 & 34 & 25 & 39 & 30 & 36 & 27 & 37 \end{bmatrix}, magic8 = \begin{bmatrix} 64 & 2 & 3 & 61 & 60 & 6 & 7 & 57 \\ 9 & 55 & 54 & 12 & 13 & 51 & 50 & 16 \\ 17 & 47 & 46 & 20 & 21 & 43 & 42 & 24 \\ 40 & 26 & 27 & 37 & 36 & 30 & 31 & 33 \\ 32 & 34 & 35 & 29 & 28 & 38 & 39 & 25 \\ 41 & 23 & 22 & 44 & 45 & 19 & 18 & 48 \\ 49 & 15 & 14 & 52 & 53 & 11 & 10 & 56 \\ 8 & 58 & 59 & 5 & 4 & 62 & 63 & 1 \end{bmatrix}. \tag{25}$$

The recurrent MATLAB *magic8* clan (boldened in Table 3) is also shared by many other squares: the ancient *Mercury* square, $laa(42)$[33], $i8$, as well as #1127 of the 4320 set of *Franklin* squares [50]. LCTS [33] found the SVs for MATLAB *magic8* to be $\sigma_i = \{260,\, 32\sqrt{21},\, 4\sqrt{21}\}$, with five zeroes (rank 3). We now find that his clan has $R = 462, 534, 912$, and is shared with one of the 10 pandiagonal Franklin clans (64 in total), with effective rank 2.233. We suggest that these are all related by SV conserving transforms of one another.

It may be worth noting the SVs of a few of the simpler clans: $bf\min$: $\sigma_i = \{260,\, 2\sqrt{2(1365 + \sqrt{1856505})},\, 16\sqrt{210/(1365 + \sqrt{1856505})}\}$, and $bf\max$: $\sigma_i = \{260,\, 28\sqrt{21},\, 16\sqrt{21}\}$. Clearly these order 8 rank three magic squares have low entropy. "Franklin" type squares of doubly even order with rank three are known to $n = 48$ [38], and they have progressively lower entropy.

George Styan Of special interest in connection with our celebration of George Styan's 75th, we examine a set of magic squares with knight's paths, especially a type called Caïssan [54]. We note that the *magic*8 clan is also shared by his Caïssan "beauty" square, caiss8Q4, as well as his *ursus* and *bcm*3. Earlier Beverly (1848) [59] gave a semi-magic knight tour square *bev*8, which we now find has the same clan as *euler*8 in Table 3.

Applying our methods to the Caïssan squares reveals that they have the same signatura as the Franklin pandiagonal squares and the 8th order most perfect squares of Ollerenshaw, indicating that these groups are related if indeed not the same.

Another interesting feature which Styan studies concerns knight-Nasik order 8 pandiagonal magic squares and the geometric patterns formed by lines drawn through successive elements connected by moves of a chess knight (Styan's CSP2,3). Related line patterns are quite old [6, 59].

6.2 Order 9

To afford a closer comparison with the order 9 study of NDS, we have included a few sample magic squares for order 9 in Table 4.

r	square	H	$C\,(\%)$	R	$erank$
2	$bound2$	0.65522	70.18	$1,960,718,400$	(2)
5	ta, td[a]	1.12999	48.57	$1,301,165,856$	3.09562
9	tx[b]	1.20501	45.16	$1,307,982,296$	3.33679
5	$dko9a$[c]	1.33486	39.25	$788,778,000$	3.79948
9	$bimagic9$[d]	1.55931	29.03	$783,193,032$	4.75553
7	$dko9b$[c]	1.69577	22.82	$413,322,912$	5.45082
9	$luna$[e]	1.85005	15.80	$472,695,264$	6.36014
9	$magic9$[f]	1.85479	15.69	$455,689,152$	6.37560
9	$boundn$	1.9490	11.30	$245,089,800$	(9)

Table 4: $n = 9$ [$\ln(9) = 2.1972...$]

[a] ta, td [51]; [b] tx [16]; [c] $dko9a, b$ [43]; [d] $bimagic9$ [5]; [e] $luna$ (magic square of the Moon); [f] $magic9$ [37].

Comparison of rank 9 tx with rank 5 ta shows that SVs of the latter decrease faster for a slightly smaller R:

tx: $\{369, 187.213, 94.4392, 11.0905, 9.8952, 6.9375, 5.7113, 3.2663, 0.47874\}$.

ta: $\{369, 187.061, 93.5307, 20.7846, 10.3923, 0, 0, 0, 0\}$.

Pfefferman's *bimagic*9 of 1891 [43] has a degenerate pair (boldened), a much smaller R-index and higher entropy than tx which has the same full rank:

*bimagic*9: $\{369, \mathbf{140.296}, \mathbf{140.296}, 46.765, 42.042, 24.823, 15.589, 8.559, 5.196\}$

Compound magic squares (CMSs) of order $n = pq$; $p, q \geq 3$, which begin at order 9, have the same 3-partitioned substructure as the Sudokus studied by NDS, albeit now with distinct sequential matrix elements, e.g. for $n = 9$ with elements 1...81. Rogers, Loly and Styan [47] have shown that order 9 CMSs are highly singular with rank 5 at order 9, whereas NDS found only Sudokus with ranks 8 and 9. The CMSs ta, and a permuted partner td, date respectively from 983 CE and 1275 CE. They share the same eigenvalues and singular values. For comparison observe that van den Essen's X-sudoku magic square [16], tx, has the same numbers in each subsquare as does ta, but without the magic subsquares which are increments of the Loshu:

$$ta = \begin{bmatrix} 71 & 64 & 69 & 8 & 1 & 6 & 53 & 46 & 51 \\ 66 & 68 & 70 & 3 & 5 & 7 & 48 & 50 & 52 \\ 67 & 72 & 65 & 4 & 9 & 2 & 49 & 54 & 47 \\ 26 & 19 & 24 & 44 & 37 & 42 & 62 & 55 & 60 \\ 21 & 23 & 25 & 39 & 41 & 43 & 57 & 59 & 61 \\ 22 & 27 & 20 & 40 & 45 & 38 & 58 & 63 & 56 \\ 35 & 28 & 33 & 80 & 73 & 78 & 17 & 10 & 15 \\ 30 & 32 & 34 & 75 & 77 & 79 & 12 & 14 & 16 \\ 31 & 36 & 29 & 76 & 81 & 74 & 13 & 18 & 11 \end{bmatrix}, tx = \begin{bmatrix} 68 & 72 & 70 & 6 & 4 & 3 & 47 & 46 & 53 \\ 65 & 67 & 66 & 1 & 8 & 7 & 54 & 51 & 50 \\ 71 & 69 & 64 & 2 & 9 & 5 & 49 & 52 & 48 \\ 21 & 26 & 23 & 43 & 37 & 45 & 60 & 56 & 58 \\ 27 & 20 & 24 & 44 & 39 & 40 & 61 & 59 & 55 \\ 19 & 25 & 22 & 41 & 38 & 42 & 57 & 62 & 63 \\ 33 & 30 & 29 & 81 & 77 & 73 & 17 & 13 & 16 \\ 31 & 28 & 35 & 75 & 79 & 74 & 14 & 18 & 15 \\ 34 & 32 & 36 & 76 & 78 & 80 & 10 & 12 & 11 \end{bmatrix}. \tag{26}$$

Ollerenshaw's 2006 rank 5 square [43] $dko9a$ was constructed from a magic Sudoku square, and while 3-partitioned, is not a CMS since the subsquares are not magic, though their elements do sum to nine times the magic sum (369).

7 Latin squares and Sudoku solutions

Now we look at the Latin squares for the insight that these simpler objects give into magic squares, including the R-degeneracy already found for order 4 magic squares, the connection to NDS, and knowing that Euler [17] showed that some magical squares can be constructed from them. Numerical (integer) Latin squares (elements $1...n$ repeated n times with each once in every row and column) have the same line sum of elements in all Rows and Columns (RC squares) but not necessarily for the diagonals ($D1$ and $D2$), specifically:

$$\text{Latin line sum: } s_n = (1 + 2 + ... + n) = n\,(1 + n)\,/2;\ n > 1. \tag{27}$$

Some Latin squares are RCD, but with repeated elements. Ninth order Sudokus have the additional constraint that each integer appears only once in each of the nine adjoining 3×3 subsquares, also called 3-partitioned [15]. Latin squares may be classified in several ways, e.g. Weisstein [62] gives 12 for order 3, before noting that there is just one normalized square with the first row and column given by $\{1, 2, ..., n\}$ at this order. Our focus on distinct SV sets uses these normalized Latin squares.

The smallest Latin square, $lat2$, is bisymmetric, has $\lambda_i = \{3, -1\}$, $\sigma_i = \{3, 1\}$, $D1 = 2 \neq D2 = 4$. Then $L = 82$, $R = 1$, $H = 0.562...$, $erank = 1.755...$, $C = 19\%$:

$$lat2 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix};\ lat3 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} \tag{28}$$

The sole order three $lat3$ is shown above and has $\sigma_1 = \{6, \sqrt{3}, \sqrt{3}\}$, $H = 0.9105...$, $C = 17\%$, $erank = 2.4856...$, $L = 1314$ and $R = 18$.

7.1 Order 4 Latin squares have 4 clans

Here there are 4 normalized reduced Latin squares [62] which show some differences in their spectral properties, especially the SVs. $lat4a$ is a bisymmetric case (about $D1$ and $D2$) [$D1 = 4 \neq D2 = 16$; $\lambda_i = \{10, 0, -4, -2\}$; $\sigma_i = \{10, 4, 2, 0\}$]. Its Gramian characteristic polynomial is $X^4 - 4X^3 - 52X^2 - 80X$. It is also a compound Latin square (CLS) [47] of rank 3 in which the upper right and lower left quadrants are augmented by two from the upper left and lower right quadrants. Loly and Styan (2010a) [54] identified a symbolic example in a study of sheets of postage stamps.

It follows from appendix A, that the sums of the diagonal elements of the Gramian matrices for all order four Latin squares are the same (here 30 for $n = 4$), and also that the Gramian linesums are 100 (from σ_1^2). The results for these Latin squares are summarized in Table 5:

r	SVclan	H	$C(\%)$	R	$erank$	σ_i
2	$bound2$	0.6183	55.40	400	1.8558	
3	$lat4a$[a]	0.9003	35.06	272	2.4602	$10, 4, 2, 0$
3	$lat4d$[d]	0.9361	32.47	200	2.5501	$10, \sqrt{10}, \sqrt{10}, 0$
4	$lat4b$[b]	1.0670	23.03	264	2.9068	$10, 4, \sqrt{2}, \sqrt{2}$
4	$lat4c$[c]	1.1554	16.65	144	3.1754	$10, 2\sqrt{2}, 2\sqrt{2}, 2$
4	$boundn$	1.1646	15.99	$133\frac{1}{3}$	3.2046	

Table 5: The four order 4 Latin clans $\ln 4 = 1.38629...$]

[a]First main class [34]: $lat4a = \{\{1, 2, 3, 4\}, \{2, 1, 4, 3\}, \{3, 4, 1, 2\}, \{4, 3, 2, 1\}\}$; Second main class members [34]: [d]$lat4d = \{\{1, 2, 3, 4\}, \{2, 4, 1, 3\}, \{3, 1, 4, 2\}, \{4, 3, 2, 1\}\}$,
[b]$lat4b = \{\{1, 2, 3, 4\}, \{2, 1, 4, 3\}, \{3, 4, 2, 1\}, \{4, 3, 1, 2\}\}$,
[c]$lat4c = \{\{1, 2, 3, 4\}, \{2, 3, 4, 1\}, \{3, 4, 1, 2\}, \{4, 1, 2, 3\}\}$.

Note that the two singular squares ($lat4a, d$) are bisymmetric, and the other two ($lat4b, c$) just symmetric. The lower rank pair have significantly higher compression (lower entropy) than the higher rank pair. $lat4b$ is a symmetric variant of $lat4a$ in which the lower right quadrant is rotated a quarter turn, reverting to rank four, and thus non-singular [47] [$D1 = 6 \neq D2 = 16$]; note two equal (degenerate) SVs in $lat4b, c, d$. $lat4c$ is a second non-singular symmetric case where $D1 = 8 \neq D2 = 16$, (and backward circulant matrix $circ4$). Then finally note $lat4d$ with rank 3 gives a second singular bisymmetric case that is also a magic Latin square as $D1 = D2 = 10$.

Observe that the normalized $lat4a, b, c, d$ are by definition never in mini-Sudoku form, i.e. a $2-$partitioned structure with one each of $1, 2, 3, 4$ in each quadrant, however they may be so transformed by moving rows, e.g. from $lat4a$ by moving row 2 to the bottom row (new row 4), with no change of SVs, e.g. $lat4a$ yielding:

$$sud4a = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \end{bmatrix}; \sigma_i = \{10, 4, 2, 0\}, \tag{29}$$

which is not symmetric, but is "magic" as $D1 = D2 = 10$. This square has $\lambda_i = \{10, 0, 0, 0\}$, where it is worth noting the single eigenvalue, previously observed for some order 4 natural magic squares by Loly et al [33]. The $lat4b, c$ squares transform into mini-Sudokus by the same row changes as above, while $lat4d$ transforms to $sud4d$ by moving row 4 to become the new row 2. In all these cases the SVs are unchanged, and so H, C, L, R and $erank$ are also unchanged.

The entries in Table 5 were ordered according to increasing H (and $erank$, but decreasing C), while R shows some deviation (seen also in some of our later tables). Since all measures depend on the SVs, the different (descending) magnitudes of these are responsible for any variations, e.g. only $lat4c$ has the largest contribution to H from the leading SV, σ_1, while the others have their largest contribution from σ_2 (and for the degenerate $\sigma_2 = \sigma_3$ for $lat4d$).

7.2 Order 5 Latins have 7 clans

For this prime order the 56 normalized Latin squares (Weisstein 2011) [62] all have rank 5, i.e. non-singular, having only seven clans (different entropies, compressions, effective ranks, etc.), as reported in Table 6 with the number of normalized squares in each clan in brackets (from 1 to 20), each with a distinct R index in order of the descending compressions, and all within the bounds of 625 (just) to 2500. None are magic in standard form.

square	H	$C(\%)$	R	$erank$
$bound2$	0.6272	61.03	2500	1.8723
$lat5f(1)$[a]	1.0911	32.20	1230	2.9776
$lat5b(20)$[b]	1.1741	27.05	1038	3.2351
$lat5e(2)$[a]	1.2618	21.60	1030	3.5319
$lat5a(20)$[b]	1.2973	19.39	978	3.6594
$lat5d(10)$[b]	1.3259	17.62	798	3.7654
$lat5c(1)$[a]	1.3421	16.61	750	3.8272
$lat5g(2)$[a]	1.3646	15.21	630	3.9142
$boundn$	1.3655	15.16	625	3.9175

Table 6: The seven order 5 Latin clans - all rank 5 [$\ln(5) = 1.6094...$]

[a] First main class [34]; [b] 2nd main class [34].

Again the maximum compression, $C = 32.20...\%$, of this set falls well short of the upper bound, $C = 61.03...\%$. Note also that the surd-in-surd structure of the $\sigma_{2...4}$ for $lat5f$, namely $\sqrt{(25 \pm 11\sqrt{5})/2}$ (twice), shows that not all Latin squares have integer σ_i^2, as did $n = 4$ Latins. Also $lat5b...g$ in Table 3 have the largest contribution to H from the leading SV, σ_1, while $lat5f$ has two equal contributions from its degenerate $\sigma_2 = \sigma_3$, with almost as much from σ_1.

7.3 Order 6 Latins have 599 clans

Of the 9408 normalized squares [34, 57] (none magic), we find 158 of rank 4 in 22 clans with 20 R-indices; 1568 of rank 5 in 91 clans with 73 keys (R-indices);

and 7682 of full rank in 486 clans with 282 keys (R-indices). Two pairs of rank 4 squares have R-indices 5409 and 4689 respectively, but distinct entropies. There are multiple clans with the same R-index but distinct entropies in the range of $R: 2229...6849$, within the bounds 2205 (just) to $11,025$. Clearly order six Latin squares are more complicated than the preceding orders.

We show just the pair of compound Latin squares [47] for order 6 and rank 4, which are formed from the two combinations of $lat2$ and $lat3$, the first 2-ply, the second 3-ply [15]:

$$cls6a = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \\ 3 & 1 & 2 & 6 & 4 & 5 \\ 4 & 5 & 6 & 1 & 2 & 3 \\ 5 & 6 & 4 & 2 & 3 & 1 \\ 6 & 4 & 5 & 3 & 1 & 2 \end{bmatrix}, \; cls6b = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \\ 3 & 4 & 5 & 6 & 1 & 2 \\ 4 & 3 & 6 & 5 & 2 & 1 \\ 5 & 6 & 1 & 2 & 3 & 4 \\ 6 & 5 & 2 & 1 & 4 & 3 \end{bmatrix}. \quad (30)$$

Their properties are listed in Table 7, along with a selection of other examples. Note that there are also corresponding order six Sudokus with respectively 2×3 and 3×2 blocks with the same SVs and other Shannon properties as $cls6a$ and $cls6b$ respectively.

r	square	H	$C(\%)$	R	$erank$
2	$bound2$	0.6327	64.69	11025	1.8827
4	$pair1$[a]	1.0575	40.98	5409	2.8792
4	$pair2$	1.0942	38.93	5409	2.9868
4	$cls6a$[b], $sud6a$, $bailey4$	**1.1090**	**38.10**	**6849**	**3.0315**
4	$cls6b$[c], $sud6b$, $quartet1$	1.1494	35.85	4689	3.1562
4	$quartet2$	1.1660	34.92	4689	3.2091
5	$quartet3$	1.1773	34.29	4689	3.2456
5	$sextet1$	1.3445	24.96	3537	3.8363
6	$quartet4$	1.3621	23.98	4689	3.9044
6	$sextet2$	1.3699	23.54	3537	3.9350
6	$sextet3$	1.4067	21.49	3537	4.0825
6	$sextet4$	1.4271	20.35	3537	4.1666
6	$sextet5$	1.4298	20.20	3537	4.1779
6	$sextet6$	1.4302	20.18	3537	4.1795
6	$costas6$[d], $circ6$, $albuni6$	**1.4920**	**16.73**	**2961**	**4.4461**
6	$singlet$[a]	1.5308	14.56	2229	4.6219
6	$boundn$	1.5320	14.50	2205	4.6273

Table 7: Selected order 6 Latin clans [$\ln(6) = 1.7918...$]

[a]set limits; [b] $\sigma_i = \{21, 4\sqrt{3}, 4\sqrt{3}, 3, 0, 0\}$; [c] $\sigma_i = \{21, 9, 2\sqrt{3}, 2\sqrt{3}, 0, 0\}$;[d] $\sigma_i = \{21, 6, 6, 2\sqrt{3}, 2\sqrt{3}, 3\}$.

Note the degenerate SVs for $clsa, b$ and $costas6$. Table 7 includes a quartet with $R = 4689$, a sextet with $R = 3537$, a square for a Costas array [12] with the same clan as an order six circulant, *circ*6, and a remarkable XII-XIII century square from Al-Buni (Descombes 2000) [11] in the Arabic symbols for $1, 2, 4, 6, 8, 9$ which we relabelled to 1...6, denoted by *albuni*6 some time before Euler.

7.4 Order 9 Latin squares and low rank order 9 Sudokus

There are $377, 597, 570, 964, 258, 816$ normalized Latin squares [62]. NDS suspected that there were no Sudokus with algebraic multiplicity more than 1, i.e. just rank 8 and 9, but were not able to prove that. We have now been able to generate Sudokus of ranks 5, 6 and 7 as well as a few others with ranks 8 and 9, and include their properties in Table 8.

r	square	H	$C\,(\%)$	R	*erank*
2	*bound*2	0.6414	70.81	291, 600	2.89913
9	*idc*15*m*	1.2288	44.08	142, 236	3.41712
5	***cls*9, *idc*5*p*[a]**	**1.2952**	**41.05**	**119, 556**	**3.65182**
6	*idc*6*p*	1.4296	34.94	100, 692	4.17710
7	*idc*7*p*	1.5389	29.96	94, 644	4.65925
9	*knecht*	1.5610	28.96	101, 028	4. 76339
8	*idc*8*p*	1.6406	25.33	82, 752	5.15814
9	*bailey*12	1.6700	23.99	76, 208	5.31231
8	*nds*21[b]	1.7119	22.09	66, 864	5.53922
9	*idc*9*p*[c]	1.7127	22.05	76, 936	5. 54363
9	*bailey*11	1.7610	19.85	62, 636	5.81851
9	*ndsfig*1	1.7789	19.04	63, 408	5.92335
9	*nds*22	1.8163	17.34	67, 068	6.14895
9	*idc*69[d]	1.8878	14.05	40, 824	6.60506
9	*albuni*9	1.9080	13.16	36, 972	6.73980
9	*idc*14*m*[e]	1.9083	13.15	36, 936	6.74167
9	*boundn*	1.9099	13.08	36, 450	7.75220

Table 8: Order 9 Latins and Sudokus from rank 5 to rank 9.

[a] $\sigma_i = \{45, 9\sqrt{3}, 9\sqrt{3}, 3\sqrt{3}, 3\sqrt{3}, 0, 0, 0, 0\}$;
[b] $\sigma_i = \{45, 13.968, 10.672, 9.337, 9.041, 5.628, 4.6463, 2.971, 0\}$;
[c] $\sigma_i = \{45, 14.953, 11.180, 8.644, 8.337, 4.973, 4.033, 2.331, 0.874\}$;
[d] $\sigma_i = \{45, 9, 9, 9, 9, 9, 9, 3\sqrt{3}, 3\sqrt{3}\}$; [e] $\sigma_i = \{45, 9, 9, 3\sqrt{7}, 3\sqrt{7}, 3\sqrt{7}, 3\sqrt{7}, 3\sqrt{7}, 3\sqrt{7}\}$.

In Table 8 there is a CLS iterated from *lat*3 with 3×3-ply, *cls*9, which has the same clan properties as *idc*5*p* in Table 9 above. It can also be transformed to Sudoku form by row and column permutations. Since these matrices take a lot of display space, their matrix elements are listed in an Electronic Supplement, *shannonData.txt*. We found that the clan represented by *cls*9, *idc*5*p* (boldened) has recurred frequently, probably because of ease of construction. These must be clan members related by transformations which preserve H and these serve to illustrate the usefulness of our clan concept in identifying related squares. While we initially found that lower rank Sudokus had lower entropy, further examples of full rank but lower entropy prompted the introduction of the effective rank measure. This is dramatically illustrated by the top entry in Table 8 where we found a full rank Sudoku with $C = 44\%$ compression. Another full rank Sudoku, *knecht*, provided by Craig Knecht [26] also has low effective rank but was constructed by criteria involving water retention in 3D models of Sudokus. Note that $cls9$ and $idc5p$, with high compression, and $idc69$ and $idc14m$, with very low compressions, each have rather pretty SVs in surd form. As well almost the smallest compression found so far is for a remarkable Arabic Latin square of Al-Buni from the XII-XIII century [11], $albuni9$, which has properties very close to $idc14m$ in Table 9. We now know from a suggestion by Lih [27] that that a Korean magic square constructed by Choe (1646–1715) from a pair of Latin squares, $choeL$ and $choeR$, shares the same clan as the recurring $idc5p$.

8 Summary

While the key issue is the usefulness of isentropic clans, we found that the effective rank measure $erank$ is a better guide for comparing different orders of both magic and Latin squares than the entropies, which have an upper bound of $\ln(n)$. As this work progressed, an early view that the low entropy of magical squares was associated with low rank was dispelled by finding some full rank examples with very low entropy. Very high compressions and low effective ranks were found for doubly even magic squares which often exhibit rank 3, as well as for compound magic squares which also exhibit much less than full rank. Aside from a pair of order four magic clans (see 22), order six Latin squares give examples of different clans with duplicate R indices.

Our link to Girard's work [35] followed from a critical review of Andrews' book [1] by G.A. Miller [36]. As the magnitudes of the indices R and Q grow with increasing order n, it would eventually be necessary to use higher precision arithmetic to clarify differences in H, C and $erank$. These various measures constitute our SIGNATURA.

Contrasting our new results with those of NDS's order nine rank 8 and 9 Latin and Sudoku random ensembles suggests that our lower rank examples are relatively rare and so less likely to be found in those ensembles, but their apparent ease of construction means that these often recur historically. Studying whether there is any correlation between the difficulty of Sudoku puzzles and the measures introduced here might make an interesting student project.

We have shown that the clan concept is useful in classifying historical magical squares, as well as new examples as they are generated. We propose a "Library of Magical Squares", where R is used to find the correct shelf, and Q the proper position on that shelf.

The use of eigenvalue and singular value considerations in this work should not be a deterrent to wider use of the measures advocated here. For advice in implementing these ideas contact Ian.Cameron@ad.umanitoba.ca for FreeMat, Macsyma, Python, Octave; rogers@physics.umanitoba.ca for MATLAB and

Maple; or loly@cc.umanitoba.ca for Mathematica, Maple, Wolfram Alpha via web browser and iPad.

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A General invariant for sum of σ_i^2 for full cover of integers

Since the trace of $AA^T = b_1(n)$, given in (12), is a useful check on calculations we consider the general third order matrix:

$$Z3 = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}. \tag{31}$$

If $A = Z3$, the diagonal elements of the Gramian matrix are immediately seen to be $a^2 + b^2 + c^2$, $d^2 + e^2 + f^2$, and $g^2 + h^2 + i^2$. In general these diagonal elements sum the squares of all the matrix elements, so that the trace of the Gramian matrix for order n squares with elements from $1...n^2$ is:

$$\sum_{i=1}^{n^2} i^2 = b_1(n) = n^2\left(n^2+1\right)\left(2n^2+1\right)/6, \tag{32}$$

which is an invariant for a given order, and asymptotic to $O\left(n^6\right)$ for large n. This adds to the other two invariants: the well-known linesum and the moment of inertia invariances of natural magic squares [28].

N.B. Sloane's On-Line Encyclopedia of Integer Sequences [53] already contains the present integer series, Sloane A113754: "Number of possible squares on an $n^2 \times n^2$ grid": $(1, 30,)\ 285, 1496, 5525, 16206, 40425, 89440, ...$, noting that the first two (bracketed) do not exist for magic squares which begin at order three.

Curiously, the difference, $b_1(n) - \sigma_1^2$, e.g. n=3: $285 = 15^2 + 60$, n=4: $1496 = 34^2 + 340$, n=5: $5525 = 65^2 + 1300$, etc., is just the moment of inertia of magic squares on a unit lattice [28], $n^2(n^4-1)/12$, Sloane A126275: Moment of inertia of all magic squares of order n: $5, 60, 340, 1300, 3885, 9800, ...$

A.1 Latin squares with symbols $1...n$

The trace of the Gramian matrix is the sum of the squares of the SVs, which we will call P_1 or $b_1(n)$ for later convenience [see (12)], now has n sets of the squares of $1...n$:

$$b_1(n) = n \sum_{i=1}^{n} i^2 = n^2(1+n)(1+2n)/6, \tag{33}$$

which is an invariant for a given order:

$b_1(2) = 10$; $b_1(3) = 42$; $b_1(4) = 120$; $b_1(4) = 275$; etc.

Sloane A108678 [53] gives: $(n+1)^2(n+2)(2n+3)/6$: $1, 10, 42, 120, 275, 546, ...$

[Note that the first term does not exist for Latin squares which begin at order two (after $n \to n-1$).]

A.2 Rank 3 simplification

Since many interesting cases (especially some examples for $n = 3, 4, 8$) have the minimum rank of 3 [13], the value of σ_2^2 or σ_2, is a simple guide to variations since σ_3^2 is fixed by:

$$\sigma_3^2 = b_1(n) - \left(\sigma_1^2 + \sigma_2^2\right), \tag{34}$$

which explains the lower envelope in Figure 3.

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A Electronic files

1. shannonData.txt: Matrix elements for squares cited in tables, where not easily found in a reference.