

Scientific Studies of Magic Squares

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September 29, 2003

Abstract

While magic squares are recreational in grade school, they may be treated somewhat more seriously in linear algebra courses. The smallest and earliest is the ancient Chinese 3×3 Lo-shu magic square of the numbers 1..9, which has a magic line constant of 15 for rows, column, and diagonals. This square motif appears in the same time frame as yin-yang schema (I Ching, Feng Shui) in early Chinese cosmology and philosophy. Over the past few years I have made several novel contributions, often through work with undergraduate summer students, finding pandiagonal non-magic squares, turning the compounding method into a computer code to produce the world's largest magic square, obtaining theorems for the moment of inertia and electric quadrupole tensors, and Monte Carlo estimates of populations of these squares. I will show very fruitful interdisciplinary interactions between mathematics, computing (digital design), physics (mechanics and electricity) and philosophy (Carl Jung's depth psychology).

1 Introduction

Small magic squares are often encountered in early grades as an arithmetic game on square arrays (patterns or motifs). Classical magic squares of the whole numbers, $1..n^2$, have the same line sum (magic constant) for each row, column, and main diagonals:

$$C_n = n(n^2 + 1)/2 \tag{1}$$

This line sum invariance depends only on the order, n , of the magic square. The ancient 3×3 Chinese Lo-shu square of the first nine consecutive integers is the smallest magic square and apart from rotations and reflections there are no others this size or smaller:

4	9	2
3	5	7
8	1	6

(2)

where the magic sum is 15. For $n = 3$, $C_3 = 15$, as expected. The best statement that can be made about its age is 2500 years old, give or take 1500 years! While the middle figure may be most appropriate, the hardest evidence gives just the lesser (Ollerenshaw and Brée, 1998), while legends (Swetz, 2002) claim the older.

We briefly review the recreational aspect of magic squares before drawing attention to the simpler semi-magic squares, and to pandiagonal non-magic squares. Then we examine the scientific aspects of all these squares through applications in classical physics and matrix analysis. In fact it is partly through the coupled oscillator problem that the mathematics of matrices was developed. Through an elementary example in matrix-vector multiplication, which can be done at the high school level, we demonstrate a simple eigenvalue-eigenvector problem. Magic squares can then play a valuable role in modern courses in linear algebra.

2 Recreational Mathematics

At the recreational level magic squares are fun for all ages, as I found when introducing them to visitors during the summer of 2000 whilst volunteering at the "Arithmetricks" travelling exhibit at the Museum of Man and Nature in Winnipeg, Manitoba. Various types of magic squares have become a recreational pastime of amateurs, often very gifted individuals e.g. Albrecht Dürer, and Ben Franklin. There are several journals which publish results in this area of recreational mathematics. However magic squares present difficult challenges for mathematicians and over the past few hundred years many famous mathematicians have contributed to our knowledge of them, including Euler.

3 Semi-Magic and Pandiagonal Magic Squares

For a number of purposes it is important to recognize that there are two specially important variations on the theme of magic squares.

Firstly, **semi-magic squares**, which do not necessarily have the diagonals summing to the row-column line sum, some of which may be obtained simply by moving an edge row and/or column to the opposite side, e.g.

$$\begin{array}{|c|c|c|} \hline 9 & 2 & 4 \\ \hline 5 & 7 & 3 \\ \hline 1 & 6 & 8 \\ \hline \end{array} \tag{3}$$

The removal of the diagonal constraints means more squares due to the smaller number of constraints, in this case there are eight more. Secondly, **pandiagonal non-magic squares**, which have the same magic line sum for all the split lines parallel to the main diagonals. We can illustrate pandiagonals by taking a non-magic serial square and tiling a copy to its right (or left, or top or bottom):

1	2	3	1	2	3
4	5	6	4	5	6
7	8	9	7	8	9

(4)

The pandiagonals are $(1, 5, 9)$, $(2, 6, 7)$, $(3, 4, 8)$, $(3, 5, 7)$, $(1, 6, 8)$, $(2, 4, 9)$, together with the main diagonals. Observe that for a given order the number of row and column constraints is the same as the number of pandiagonal constraints.

A pandiagonal magic square is the combination of this pandiagonal property with the requirements of a magic square. These first occur in order 4, and of these 48 are found amongst the 4th order squares, with none existing for singly even orders $(6, 10, \dots)$:

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

(5)

4 Counting Magic Squares

There are 880 distinct 4×4 magic squares of the first 16 integers, and the 275, 305, 224 distinct 5×5 of the first 25 integers, the latter were first counted by computer in 1973 (Schroeppel). Already by order six they have become uncountable, and as a result only statistical estimates are then possible. Pinn and Wieczerkowski (1998) performed a Monte Carlo simulated annealing computation for an estimate of $(0.17745 \pm 0.00016) \times 10^{20}$ for the 6×6 , and a less accurate estimate of the 7×7 .

Walter Trump (2003) has developed a more efficient hybrid backtracking Monte Carlo method and has improved the accuracy of these estimates. He also has good estimates of the number of magic squares up to 10×10 , a remarkable achievement. Our group of 2003 summer undergraduate students has recently taken Trump's 6×6 GB32 code, which uses 13 random cells, and converted it to C++. Matt Rempel finds that the C++ code runs about 10% faster than GB32 on the same PC in producing a sample of some 700,000 squares, and has begun to remove random cells, finding longer run times, but more accurate results, and a larger sample of magic squares. Dan Schindel has taken the ideas in the 6×6 code and constructed pure backtracking codes without random cells to count the known numbers of magic squares for the 4×4 and 5×5 cases. While we have previously been able to analyse the complete set of 4×4 's, we can now begin to analyse the 5×5 's. This gives us the ability to study a variety of interesting questions, e.g. their eigenproperties. Dan Schindel has also amended the 4×4 code to count the number of pandiagonal non-magic 4×4 squares, finding some three million.

5 Compound Squares

An undergraduate project with Wayne Chan took an old Chinese idea for compounding a 3×3 magic square with itself to construct a 9×9 magic square, or a 3×3 with a 4×4 to make a pair of 12×12 magic squares, and devised a computer program (Chan and Loly, 2002) to extend this to very large squares. One of the squares is used as a frame and the other is incremented on placement in the appropriate position in the frame. As a result we were able to set a new world record sized magic square at $12,544 \times 12,544$.

6 Where is the Science?

Why did a theoretical physicist get involved with magic squares? This was not a linear process. Having been blissfully unaware of them for my first five decades, an encounter with the Myers-Briggs Type Indicator (MBTI, Myers and Myers, 1980, 1993) scheme of personalities resonated with my background in mathematical structures, partly from research in solid state physics. Coordinate rotation matrices in classical and relativistic mechanics, combined with periodic boundary conditions for finite crystals soon lead to links with magic squares, while the psychological nature of the MBTI eventually made connections with early work of Carl Jung, which finally connected with Jung's interest in Chinese patterns, especially the dichotomous yin-yang schemes of Feng Shui (a.k.a. The Golden Flower) and the I Ching {von Franz, 1974}.

The science begins whenever we go beyond the square patterns, e.g. treating magic squares as matrices, or as arrays of point masses or electric charges. There are more examples, but our own work concentrates on the remarkable results associated with matrix issues in the context of linear algebra, i.e. solving sets of simultaneous equations, as well as some topics in classical physics.

7 A Mechanical Application - Moment of Inertia

Another success growing out of teaching undergraduate classical mechanics for many years was the discovery of a new invariance, or universal property, for magic squares through calculations of their "moments of inertia" (essentially the inertia of the square through an axis perpendicular to its centre), which eventually turned out to depend only on the order of the square, i.e. the number of rows or columns (Loly, 2004). The numbers in the magic square are replaced by corresponding multiples of a unit mass placed on a square unit lattice. In fact this was truly a "Eureka" moment for quite spontaneously I had the idea which fused long activity in teaching moment of inertia in introductory courses with a more recent activity in magic squares.

The moment of inertia, I_n , of a magic square of order n about an axis perpendicular to its centre is obtained by summing mr^2 for each cell, where

m is the number centred in a cell and r is the distance of the centre of that cell from the centre of the square measured in units of the nearest neighbour distance. For the Lo-shu square the corner cells then have their centres at a distance of $\sqrt{2}$ from the axis. We can now calculate the sum for the 3×3 :

$$I_3 = [1 + 3 + 7 + 9] (1)^2 + [2 + 4 + 6 + 8] (\sqrt{2})^2 = 60 \quad (6)$$

The moments of inertia about the horizontal and vertical axes through the centre are each 30, reminding us of the perpendicular axis theorem, which says that their sum gives the moment of inertia about the axis through the centre and perpendicular to the plane.

When I used a data file for the complete set of the 4×4 's (by courtesy of Harvey Heinz) it was a surprise to find that they all gave $I_4 = 340$. I was then motivated to attempt a derivation, which was easy since the calculations only depended on the semi-magic property so that the parallel axis theorem and the perpendicular axis theorem could be used. In retrospect this could have been set as an examination problem in a sophomore course on classical mechanics!

$$I_n = \frac{1}{12} n^2 (n^4 - 1) \quad (7)$$

This remarkably simple formula recovers the results for $n = 3, 4$ above and is valid for arbitrary order. The derivation of the formula only depends on the row and column properties, and not on the diagonals of magic squares, so that it actually applies to the larger class of **semi-magic** squares which lack one or both diagonal magic sums of magic squares.

Adam Rogers has recently extended these ideas to the calculation of the full inertia tensor of magic cubes (Rogers and Loly, 2003).

8 Electric Quadrupoles

A new magic square topic has just emerged from my renewed involvement with our honours electromagnetism course. The idea is to treat the numerical value of each element as an electric charge. It is soon clear for small squares that the dipole moment vanishes, so we then proceed to study the quadrupole moment. As a first thought I neutralized magic squares so that their elements ran from $-(n^2 - 1)/2$ to $+(n^2 - 1)/2$, but later Adam Rogers analyzed the multipole expansion for a normal magic square, finding that it takes care of many of the details. The full story involves calculating the quadrupole tensor, something beyond the scope of the present article. The short story is that the quadrupole tensor vanishes, so one then proceeds to the octupole!

9 Pandiagonal Non-Magic Squares (n=4, 8, ...)

Here the highlight is the discovery (Loly, 2002-2003) of a small new class of purely pandiagonal non-magic, number squares having dimensions of the powers

of 2, with the additional property that the binary representation changes by just 1-bit in horizontal and vertical moves (they are counted from 0..15 in the example below):

0	1	3	2
4	5	7	6
12	13	15	14
8	9	11	10

(8)

These squares derive from the Myers-Briggs dichotomous scheme of personality types. Then work with summer undergraduate Marcus Steeds (Loly and Steeds 2003), who devised a number of useful tools in Maple mathematical software, explored how generalizations of these squares are related to the Gray code and square Karnaugh maps of digital logic design, a connection made by some of my third year engineering students a few years ago. I have also applied these ideas to ancient Chinese patterns based on the yin-yang duality (Loly, 2002).

Recently Dan Schindel found that of the 3,465,216 pandiagonal non-magic squares, 48 have the 1-bit property. This agrees with a symmetry argument made recently by Ian Cameron (2003).

10 From Coupled Oscillators to Modern Linear Algebra

Undergraduate physics provides several examples in mechanics and wave motion (from coupled oscillators) where semi-magic **matrices** arise with algebraic or non-integer elements. The investigation of the mechanical problems had a vibrant interplay with mathematics for two centuries from the time of Huyghens and Newton. Huyghens, of course, is well known for his study of the isochrony of pendulum motion. An excellent chronology is found in Brillouin (1946), whose study of waves in periodic systems is a tour-de-force.

A central mathematical theme in physical science and engineering concerns what are known as eigenvalue problems. These involve homogeneous linear equations which only have non-trivial solutions if the **determinant** of the coefficients vanishes, with as many solutions as the number of equations. These issues can be clarified by using a specific example for which the coupled oscillator is ideal. At the same time we can prepare the ground for studying magic square matrices in their own right.

10.1 Homogeneous Simultaneous Equations for the Coupled Oscillator

In the usual description of this one-dimensional problem (Marion and Thornton, 1995) the equations of motion for masses M displaced along the x -direction from their equilibrium positions by x_1 and x_2 , a coupling spring of force constant γ ,

and with each tied to fixed posts at opposite ends by springs of force constant κ are:

$$M\ddot{x}_1 + (\kappa + \gamma)x_1 - \gamma x_2 = 0 \quad (9)$$

$$M\ddot{x}_2 + (\kappa + \gamma)x_2 - \gamma x_1 = 0 \quad (10)$$

These are simplified by taking out a simple time-dependence: $x(t) = B \exp(it\omega)$ for:

$$(\kappa + \gamma - M\omega^2)B_1 - \gamma B_2 = 0 \quad (11)$$

$$-\gamma B_1 + (\kappa + \gamma - M\omega^2)B_2 = 0 \quad (12)$$

Instead of the general approach of setting the determinant of the coefficients of these simultaneous equations to zero, this simple problem may be solved simply by forming ratios of the variables:

$$\frac{B_1}{B_2} = \frac{\gamma}{(\kappa + \gamma) - M\omega^2} = \frac{(\kappa + \gamma) - M\omega^2}{\gamma} \quad (13)$$

Cross multiplication of the second equality results in a quadratic equation in ω^2 , with two solutions, one, $\omega^2 = \frac{\kappa}{M}$, just the frequency of each oscillator without coupling, and the other higher, $\omega^2 = \frac{\kappa + 2\gamma}{M}$. Extended to a chain of two or more alternating masses and springs, we have the origin of the gaps in the spectrum which are a characteristic feature of solid state physics. We continue this example after introducing some essential matrix operations, which are easy enough to cover in high school.

11 Determinants and Matrices

Cayley initiated matrix theory in 1846, followed by contributions from Peirce, Hamilton, Poincaré, and Sylvester. We highlight the issues of interest with respect to magic squares, the semi-magic matrices and tensors arising in mechanics, and the related interest for pandiagonal non-magic squares with a brief discussion using 2×2 matrices.

11.1 Matrix Multiplication

Consider the usual matrix-vector multiplication:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} aP + bQ \\ cP + dQ \end{bmatrix} \quad (14)$$

Clearly if $P = Q = 1$, this sums the rows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a + b \\ c + d \end{bmatrix} \quad (15)$$

This $[1, 1]$ vector will be referred to as a diagonal (or 2-agonal) vector, and generalizes to higher orders. However if one takes the original matrix operator to act from the right onto a row vector on the left, as a "left-hand" problem, then one finds the column sums of the original matrix:

$$\begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} Pa + Qc & Pb + Qd \end{bmatrix} \quad (16)$$

if $P = Q = 1$, this sums the columns.

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a + c & b + d \end{bmatrix} \quad (17)$$

This may also be achieved by left multiplication of the transposed matrix with a row vector as illustrated next.

11.2 Eigenproblems - Eigenvectors and Eigenvalues

An illustration of the utility of the " n -agonal" eigenvector $[1, 1, 1, \dots]$ of the n -cube is seen by showing how the rows sum in the Lo-shu magic square :

$$\begin{bmatrix} 3 & 5 & 7 \\ 4 & 9 & 2 \\ 8 & 1 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 15 \\ 15 \end{bmatrix} = 15 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad (18)$$

where we have factored out the eigenvalue, 15, to show the action of the matrix operator in leaving the eigenvector unchanged (Hruska, 1991). The columns have the same eigenvalue as follows from the transposed matrix:

$$\begin{bmatrix} 3 & 4 & 8 \\ 5 & 9 & 1 \\ 7 & 2 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 15 \\ 15 \end{bmatrix} = 15 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad (19)$$

Alternatively we can work with a row eigenvector on the left with the original matrix:

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \left[\begin{bmatrix} 3 & 5 & 7 \\ 4 & 9 & 2 \\ 8 & 1 & 6 \end{bmatrix} \right] = \begin{bmatrix} 15 & 15 & 15 \end{bmatrix} = 15 \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \quad (20)$$

We must note that the n -agonal eigenvector is both a left and a right eigenvector, and that this property depends only on the semi-magic property. An immediate application is now afforded by the coupled oscillator.

12 Coupled Oscillator Eigenvectors

When written in modern matrix notation, the semi-magic nature of the characteristic equation (11, 12) is apparent:

$$\begin{bmatrix} \kappa + \gamma & -\gamma \\ -\gamma & \kappa + \gamma \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = M\omega^2 \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad (21)$$

It is immediately clear that a solution is the $[1, 1]$ eigenvector, both as a right, and as a left eigenvector. It has the lower eigenvalue of $\omega^2 = \frac{\kappa}{M}$. The other eigenvector is $[1, -1]$, which corresponds to the higher solution of $\omega^2 = \frac{\kappa+2\gamma}{M}$.

A similar semi-magic property is also found for the full moment of inertia tensor of magic cubes (Rogers and Loly, 2003).

12.1 Left and right eigenvectors

Of significant interest for our studies of magic squares, this topic is important for teaching linear algebra beyond the introductory course. The use of magic square examples already occurs in such courses, but it can be used even more seriously. The existence of identical left and right eigenvectors implies deeper properties giving rise to the theorem of biorthogonality, and the theorem of Perron (Mattingly, 2001). These theorems develop deep links between the left and right eigenvectors and eigenvalues. As such, magic squares are insightful examples for advanced linear algebra courses.

Software such as Maple, Mathematica and MATLAB can be profitably employed in such studies. Indeed, MATLAB has initiated some of this already by including a function, *magic(n)*, which returns a magic square from one of three algorithms, one each for odd, even and doubly-even cases. A drawback with *magic(n)* is that the single squares which result are not representative of the richness of the spectrum of magic squares of a given order, save for $n > 3$.

12.2 Eigenvalues of Magic Squares

In another topic with Marcus Steeds as a significant participant (Loly, Hruska, Williams and Steeds, 2002), we have nearly finished a study of the eigenvalues of magic square matrices. For the 880 distinct 4×4 's in the 12 Dudeney groups, we find that members of the first six (singular) groups have three distinct eigenvalue patterns, with a subset of the first three groups having three zero eigenvalues, while the last six (non-singular) groups have two further eigenvalue patterns. Also if the 1-bit pandiagonal non-magic squares discussed earlier are treated as matrices they possess examples with just two non-zero eigenvalues for any order (Loly and Steeds, 2003).

13 Conclusion

With the conference theme of "Reforming School Science through the History, Philosophy, and Sociology of Science", we conclude with a few closing thoughts about each of these issues.

Some of the history has already been addressed, but further information may be found in recent books (Swetz, 2000) by Frank Swetz, a mathematics

educator, by René Descombes (2001), as well as in Clifford Pickover's recent book (Pickover, 2002). Those sources also enlarge on the philosophical aspects, which began in China as a cosmology, or organizing scheme.

There are opportunities to enrich teaching in classical physics, and likely in quantum physics as well. Certainly more can be done in the context of teaching linear algebra, which can begin in high school. As for sociology, I have found wonderful opportunities for students to cooperate in some group work as summer research assistants, indeed their enthusiasm and initiative in tackling problems has been gratifying.

14 Acknowledgments

David Lavis, Frank Hruska, Joe Williams, Walter Trump, John Hendricks, and Harvey Heinz have helped with advice on various aspects of this study, which engaged the following undergraduates: Marcus Steeds, Wayne Chan, Adam Rogers, Matt Rempel and Dan Schindel.

This research was funded in 2003 by the Winnipeg Foundation Post Secondary Grants Program, especially for support to Adam Rogers and Matt Rempel.

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