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The electric multipole expansion for a magic cube

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Abstract
Recent studies of the rotational properties of magic cubes are extended to a corresponding electrical problem. While expecting the dipole moments of magic cubes to vanish, it was a surprise to find their quadrupole moments also vanished. These properties are shown to follow from the RCP (row, column, pillar) symmetry of the orthogonal line sums, namely the semi-magic property of the cubes, which opens up the conclusions to a much wider class of charge distributions.

1. Introduction

Recently Rogers and Loly [1] showed that the inertia tensor of a discrete rigid mass distribution, whose masses reflect the magnitudes and positions of numbers in a magic cube, has maximal symmetry. Here we consider the corresponding electrical problem with charges replacing masses. In this case, we find that the dipole and quadrupole terms in the multipole expansion of the electric potential vanish.

2. Magic squares and magic cubes

Magic square arrays have the property that the elements of all rows, columns and main diagonals add to the same value, known as the magic constant. Normal magic squares of order $N(N > 2)$ use all the integers 1 to N^2 for their elements. The $N = 3$ magic square is shown below:

$$\begin{array}{|ccc|} \hline 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \\ \hline \end{array}\,. \tag{1}$$

It is easily shown that the line sum of normal magic squares is [2]

$$C_{\text{square}} = \frac{N}{2}(N^2 + 1). \tag{2}$$

Part of the fascination of magic squares is that while there are only 880 distinct normal magic squares in fourth order, by sixth order their population is so large that methods of statistical physics have to be employed [3].

Replacing the numbers in (1) by corresponding multiples of a unit mass arranged on a square unit lattice, it is easy to show that the centre of mass is at the centre of the square. Loly [4] found that the moment of inertia of these magic squares, taken about an axis perpendicular to the square and through the centre of mass, gave a simple closed form result:

$$I_{\text{square}} = \frac{N}{6}(N^2 - 1)C_{\text{square}} \tag{3}$$

for all squares of any order N.

If we interpret the numbers as corresponding multiples of a unit charge, the dipole moment of the magic square in (1) vanishes when taken about the centre of the square.

Normal magic cubes [5] of numbers $1 \ldots N^3$ have the sums of all rows, columns, pillars and the four main body diagonals given by [2]:

$$C_{\text{cube}} = \frac{N}{2}(N^3 + 1). \tag{4}$$

An $N = 3$ normal magic cube is displayed below in layers:

$$\begin{array}{ccc} \boxed{\begin{array}{rrr} 7 & 11 & 24 \\ 23 & 9 & 10 \\ 12 & 22 & 8 \end{array}} & \boxed{\begin{array}{rrr} 15 & 25 & 2 \\ 1 & 14 & 27 \\ 26 & 3 & 13 \end{array}} & \boxed{\begin{array}{rrr} 20 & 6 & 16 \\ 18 & 19 & 5 \\ 4 & 17 & 21 \end{array}} \\ \text{1st Layer} & \text{2nd Layer} & \text{3rd Layer} \end{array}. \tag{5}$$

While there are only four fundamental order three cubes [6], Trump recently estimated the number of fourth-order normal magic cubes at approximately 7 trillion [7]. The full inertia tensor of a magic cube can also be written in a closed form [1] and applies to this large set of distributions.

It is easy to calculate the dipole moment of the magic cube in (5) and verify that it vanishes. So the question arises—do the dipole moments of all magic cubes vanish and what about the higher multipole moments, for example the quadrupole and octupole moments?

3. The multipole expansion

In classical electromagnetism, one expands the potential V of a charge distribution in a series of terms that decrease as powers of inverse distance from the origin, which we take as the centre of the cube. The series of terms is known as the multipole expansion of the electric potential [8]:

$$V(\mathbf{x}) = \frac{1}{4\pi\varepsilon_o}\left(\frac{Q}{r} + \frac{\sum_i x_i p_i}{r^3} + \frac{1}{2}\frac{\sum_{i,j} x_i x_j Q_{ij}}{r^5} + \frac{1}{2}\frac{\sum_{i,j,k} x_i x_j x_k O_{ijk}}{r^7} + \cdots\right) \tag{6}$$

with $r^2 = \sum_i x_i^2$ and with the various multipole elements defined below.

The first term in (6) varies inversely as distance and is characterized by the monopole moment, Q [9].

$$Q = \sum_{\alpha=1}^{N^3} q_\alpha, \tag{7}$$

where q_α represents the value of the αth charge. In the case of a normal magic cube $Q = N^2 C_{\text{cube}}$, and is non-zero.

The second term in (6) decreases as $1/r^2$ (since the x_i are components of $\mathbf{r}$) and is characterized by the dipole moment with components:

$$p_i = \sum_\alpha q_\alpha x_{\alpha,i} \tag{8}$$

in which the $x_{\alpha,i}$ give the displacement of charge q_α from the origin.

Each of the N layers of a normal magic cube has identical total charge by equation (4), and since pairs of these layers have equal and opposite displacements from the origin, the dipole components vanish. This argument depends only on the equality of row, column and pillar sums (RCP symmetry).

4. The quadrupole tensor of a magic cube

The third term in (6) is characterized by the quadrupole tensor, with a potential decreasing as $1/r^3$:

$$Q_{ij} = \sum_\alpha q_\alpha \left(3x_{\alpha,i}x_{\alpha,j} - \delta_{i,j}r_\alpha^2\right). \tag{9}$$

It is easily verified for the magic cube in equation (5) that $Q_{ij} = 0$, but how general is this result? For a normal magic cube of order N, cubic symmetry means that we only need to consider one of the diagonal elements of the quadrupole tensor. Taking $i = j = 1$ in equation (9):

$$Q_{11} = \sum_\alpha q_\alpha \left(2x_\alpha^2 - y_\alpha^2 - z_\alpha^2\right). \tag{10}$$

Examining one of the terms as a sum of layers:

$$\sum_\alpha q_\alpha x_\alpha^2 = \sum_{\text{layer } x} x_\alpha^2 \left(\sum_{\text{plane at } x} q_\alpha = NC_{\text{cube}} \right) = NC_{\text{cube}} \sum_{\text{layer } x} x_\alpha^2 \tag{11}$$

so that since this sum also equals $\sum_\alpha q_\alpha y_\alpha^2$ and $\sum_\alpha q_\alpha z_\alpha^2$, the Q_{ii} all vanish.

For one of the off-diagonal $(i \neq j)$ elements:

$$Q_{12} = 3\sum_\alpha q_\alpha x_\alpha y_\alpha \tag{12}$$

then performing the same layer analysis:

$$Q_{12} = 3\sum_{\text{layer } x} x_\alpha \left(\sum_{\text{plane at } x} q_\alpha y_\alpha = C_{\text{cube}} \sum_{y \text{ at } x} y_\alpha \right) \tag{13}$$

so that finally using the y and x values which range from $-(N-1)/2$ to $(N-1)/2$ on a unit lattice, we have

$$Q_{12} = 3C_{\text{cube}} \left(\sum_{\text{layer } x} x_\alpha \right) \left(\sum_{y \text{ at } x} y_\alpha \right) = 0. \tag{14}$$

So all $Q_{ij} = 0$ for a vanishing quadrupole tensor. Again this result depends only on the RCP symmetry of the cube.

5. The octupole moment of a magic cube

Finally, the fourth term in the multipole expansion is characterized by the octupole tensor of the distribution. This term provides a potential which goes as $1/r^4$ and is a third-rank tensor, composed of 27 elements defined by

$$O_{ijk} = \sum_{\alpha} q_\alpha [5x_{\alpha,i}x_{\alpha,j}x_{\alpha,k} - r^2(x_{\alpha,i}\delta_{j,k} + x_{\alpha,j}\delta_{i,k} + x_{\alpha,k}\delta_{i,j})]. \quad (15)$$

For the magic cube in (3) we can find the elements of this octupole tensor. When i, j, k are all equal, the triagonal elements vanish. However, the other elements do not vanish in general. For example, the six elements for which i, j and k are all different depend on the magic cube under consideration. These elements have the form

$$O_{ijk} = \sum_{\alpha} q_\alpha (5x_\alpha y_\alpha z_\alpha), \quad (16)$$

but do not vanish along a pillar because while the x and y co-ordinates stay constant the z co-ordinate changes so that in general the pillar contributions will not cancel and we cannot factor the magic sum from the product. Moreover, since the values in each pillar are different these octupole products do not have a general form.

6. Neutral magic cubes—a coursework problem

In his textbook, *Introduction to Electrodynamics* [9], Griffiths presents the following problem (problem 3.45, part c): 'Show that the quadrupole moment is independent of origin if the monopole and dipole moments both vanish. (This works all the way up the hierarchy—the lowest non-zero multipole moment is always independent of origin.)'

Clearly this general theorem has application to any combination of simple $\pm q$ dipoles, since they have vanishing monopole moments, but are there other distributions for which it applies to the higher order terms? For the normal magic squares and cubes in this paper, the monopole moment did not vanish. However, any magic square can be written as the sum of a constant square and a neutral magic square with line sum zero, e.g.,

$$\begin{array}{|ccc|}\hline 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \\ \hline\end{array} = \begin{array}{|ccc|}\hline 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \\ \hline\end{array} + \begin{array}{|rrr|}\hline -1 & 4 & -3 \\ -2 & 0 & 2 \\ 3 & -4 & 1 \\ \hline\end{array} \quad (17)$$

and a neutral magic cube can be obtained from a normal magic cube by subtracting the average element from all the entries of the cube. Then the charges of the neutral cube run from $-\left(\frac{N^3+1}{2}\right)$ to $\left(\frac{N^3+1}{2}\right)$, and the monopole moment vanishes. The dipole and quadrupole contributions to the potential vanish as before. Again the octupole tensor does not in general vanish and again depends on the specific cube used to perform the calculation. However, the neutral cube octupole tensor should now be independent of origin. In a direct calculation for the magic cube in (5) this is easily verified.

7. Conclusion

While this study was prompted by Griffiths' problem of the previous section, the principal results obtained here for the vanishing of dipole moments and quadrupole tensors extend far beyond the normal magic cubes with which we started to any distribution which possesses RCP symmetry. For example, they also apply to cubic scaffolds of uniformly charged rods.

The reader may also consider evaluating the quadrupole tensor for the magic square in (1). (Hint: this charge distribution does not possess RCP symmetry.)

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