A New Class of Pandiagonal Squares.

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Abstract

An interesting class of **purely pandiagonal, i.e. non-magic**, whole number (integer) squares of orders (row/column dimension) of the powers of two which are related to Gray codes and square Karnaugh maps has been identified. Treated as matrices these squares possess just two non-zero eigenvalues. The construction of these squares has been automated by writing Maple® code, which also performs tests on the results. A rather more trivial set of pandiagonal non-magic squares consisting of the monotonically ordered sequence of integers existing for all orders has also been found.

1 Definitions

A list of definitions [1][2] has been extended to facilitate the present study of integer squares and some closely related issues:

- A classical **magic square** [3] is the set of n^2 $(n > 2)$ sequential integers $1..n^2$ arranged in the form of an $n \times n$ array whose rows, columns and diagonals have the same sum (constant):

$$C_n = \frac{n}{2}\left(n^2 + 1\right) \tag{1}$$

  There is just a single magic square of the lowest order $n = 3$, the two millenia old Lo-shu square from ancient China [4], which is unique, apart from rotations and reflections:

$$LS = \boxed{\begin{array}{ccc} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{array}} \tag{2}$$

- A **semi-magic square** [3] has the same row and column sums as a magic square, but one or both diagonals do not share that sum. SM below may be obtained from LS in (2) by moving its first column to be the last column:

$$SM = \boxed{\begin{array}{ccc} 9 & 2 & 4 \\ 5 & 7 & 3 \\ 1 & 6 & 8 \end{array}} \tag{3}$$

- A **pandiagonal** (non-magic) **square** is a set of n^2 $(n > 1)$ sequential integers $1..n^2$ arranged in the form of an $n \times n$ array [5] with the sum C_n for its main diagonals and pandiagonals, *but not for its rows and columns.*

Pandiagonals, often referred to as **split** or **broken** diagonals, are a diagonal line of numbers parallel to the main or off-diagonal starting in the top row and continuing diagonally downward until reaching the side of the square. The split diagonal is then completed by wrapping around to the other side of the square and continuing in the next row. Another description of pandiagonals is the combination of any two parallel segments on each side of the main diagonals which total n elements.

- Simple examples of pandiagonal non-magic squares are **serial** squares, e.g. S_3 of the row-by-row sequence $1..3^2$:

$$S_3 = \begin{array}{|ccc|} \hline 1 & (2) & 3 \\ 4 & 5 & (6) \\ (7) & 8 & 9 \\ \hline \end{array} \tag{4}$$

  which has one of its pandiagonals, $2, 6, 7$, emphasized by parentheses. It may help to tile a copy of S_3 to an edge of itself to see the continuity of the n-element line, or even to wrap the square onto a torus [6] to join all opposite edges for the same effect. All serial squares are pandiagonal, but obviously non-magic.

- A **pandiagonal magic square** [7][3] combines the magic and pandiagonal properties.

- **Most-perfect pandiagonal magic squares** have the additional property that, (i) all 2×2 subsquares have the same sum [8], including those that run over the edges when tiled or when wrapped over a torus [6], and (ii) each element is complementary to the one distant from it $\frac{1}{2}n$ places in the same diagonal.

- An **orthogonal complement square** includes pandiagonal non-magic squares, but is more general. These have the magic sum for all sets of entries taken once from each row and column. Recently Mayoral [9] established a connection between these orthogonal complement squares and semi-magic squares, but gave only the example of serial squares.

2 Introduction

This paper examines the origin of a class of pandiagonal non-magic squares which arise from square Karnaugh maps used in digital logic design and establishes a pattern of eigenproperties which, while different from those of magic squares, have some parallels and may shed some light on the eigenproperties of certain special magic squares which share the pandiagonal property. In contrast to magic squares, the class of non-magic squares discussed here has barely been recognized in the literature although they appear in a variety of applications. These pandiagonal non-magic squares will be termed "logic squares" in this work.

During the summer of 1997 Loly [5] found a pandiagonal non-magic square of sequential integers, 1..16, by using a binary representation of the four dichotomous dimensions of the 4×4 Type Table of the sixteen personality types in the Myers-Briggs Type Indicator® (MBTI) [10]. A few months later in the Fall of 1997, electrical and computer engineering students in one of Loly's classes pointed out the similarity of the MBTI matrix with the structure of square Karnaugh maps [11] of digital logic design. When the entries (binary values) of a square Karnaugh map are replaced by their "minterm" indices the class of pandiagonal non-magic logic squares that interests us in the present paper were found.

After that initial work two papers by Besslich from 1983 were located. In one Besslich [12] explored the use of pandiagonal magic squares as dither matrices for image processing. However in a slightly earlier paper Besslich [13] used a "dyadic indexed array" for a number of applications, but although they are pandiagonal non-magic squares, he did not comment on this striking property. This earlier work, referencing also Karnaugh maps, should be noted in the present study.

If a reflected binary Gray code [14][6], sequenced by changing just one bit at a time between adjacent entries, is used to span each edge of the square, it is soon clear that a family of squares with order $N = 2^n$ (for $n = 1, 2, 3, 4..$; $N = 2, 4, 8, 16, ..$) can be constructed. It has been verified that when the entries are converted to decimal numbers using the standard [15] scheme that the squares are indeed pandiagonal and non-magic. The construction of these squares has been automated by writing Maple® [16] procedures which are given in the Appendix.

A rather more trivial set of pandiagonal non-magic squares consisting of the ordered sequence of integers filled out serially line-by-line in the usual way that matrix elements are stored in computers was also found. These serial squares occur for all orders, including a 2×2 case which has no magic square counterpart, and one of these was included in Mayoral's [9] recent discussion of the orthogonal complements of semi-magic squares.

Just as magic squares have been generalized (magic rectangles, magic cubes, magic hypercubes, etc.) [17], so too can the squares that were derived from Karnaugh maps, albeit with a structure that derives from the Gray coding for each edge of the Karnaugh maps. Only the square subset was retained in the present paper for studies of eigenproperties.

Many of the questions that might be asked about pandiagonal non-magic squares are related to issues that have arisen for magic squares. This is particularly so in the light of Mayoral's study [9]. In particular, the eigenvalue and eigenfunction pairs (eigenproperties) [18][19] of these pandiagonal squares are studied in this paper and show some interesting features.

In the remainder of this study consideration consideration was restricted to squares with sequential integers starting from 0, instead of the 1 usually used in the recreational mathematics literature, since there is a strong connection with binary coding where zero is a natural starting point.

3 Reflected Number Codes and a Gray Code Sequence

In 1953 Gray patented [6][14] a reflected number code for error reduction in digitizing shaft rotations, and subsequently Gray codes have found many applications. Also in 1953, Karnaugh [11] published his "map" method, and later the connection between the edge labels of Karnaugh maps and Gray codes was appreciated, so that soon texts in electronic engineering had chapters concerning Gray codes followed by others on Karnaugh maps [20].

To begin, take the sequence of $0, 1$ and reflect it to the sequence: $0, 1; 1, 0$, where the reflection point (midpoint) is indicated by a semicolon, and then add a preceeding bit of 0 to the first half, and of 1 to the second half, for four 2-bit strings: $00, 01; 11, 10$. Finally go beyond Gray and fold the second half below the first half while maintaining the order of bits as indicated in the sequence below:

- initial linear 2-bit Gray code:

00	01	11	10

(5)

- intermediate state: rotate second half by 90° (think of the three adjacent nearest neighbour (NN) pairs of 2−bit strings in (5) as the links of a bicycle chain)

00	01	
		11
		10

(6)

- rotate finally to a 2×2 square:

00	01
10	11

(7)

The links of a bicycle chain are a useful metaphor for folding longer Gray chains. Now observe that horizontal and vertical moves in (7) change just one bit, but that diagonal moves change two bits. This is a simpler example of the property that first piqued Loly's interest in the Myers-Briggs Type Table [10], and it extends to interesting cases of larger folded Gray code strings. Karnaugh [11] described his reorganization of Veitch's [21] ideas in terms of the reverse of this construction, i.e., unfolding the square to a line to find what would shortly become known as the Gray code. The "1-bit" property is a key feature of Karnaugh maps.

Gray [14] assigned to the four 2-bit strings in (5) the sequential integer values:

$$0, 1; 2, 3 \tag{8}$$

Instead, translate the bit-strings into integer values by the original (standard, older) Leibniz scheme [15]:

$$\sum_{i=0} a_i 2^i \tag{9}$$

In the present 2×2 case, a_1 is the left bit and a_0 is the right bit, to find (with semicolons showing the reflection points):

$$0, 1; 3, 2 \tag{10}$$

This way of translating Gray's bit-strings is a critical step for what follows in the rest of this paper.

The 8-element 3-bit Gray code sequence:

000	001	011	010	110	111	101	100

(11)

translates to the integer sequence:

$$0, 1; 3, 2; ; 6, 7; 5, 4 \tag{12}$$

(again with semicolons showing the reflection points).

Observe now that "opposite" or antipodal pairs, e.g. 000 and 111, are not halfway along the Gray sequence, as were 00 and 11 in the 2D case in (5) [22]. They are however at opposite corners of a binary labelled cube [6].

The 16-element 4-bit Gray sequence has the first half:

0000	0001	0011	0010	0110	0111	0101	0100

(13)

followed by:

1100	1101	1111	1110	1010	1011	1001	1000

(14)

with translation to the integer sequence:

$$0, 1; 3, 2; ; 6, 7; 5, 4; ; ; 12, 13; 15, 14; ; 10, 11; 9, 8 \tag{15}$$

For the line-by-line folding of these 16 elements into a 4×4 matrix a purely pandiagonal non-magic square is found:

$$L_4 = \begin{bmatrix} 0 & 1 & 3 & 2 \\ 4 & 5 & 7 & 6 \\ 12 & 13 & 15 & 14 \\ 8 & 9 & 11 & 10 \end{bmatrix} \tag{16}$$

Since just one bit changes between adjacent elements in the Gray code **bit-strings**, including the ends, it is immediately clear that they can be joined into a closed loop (or circle) whilst preserving the one-bit property.

In Appendix A.1 a Maple procedure is given for translating the Gray codes into **whole numbers (integers)** according to the Leibniz algorithm.

4 Karnaugh Maps

The smallest is the 1-bit, 2-element case of the binary digits $0, 1$:

K_{12}	0	1

(17)

N.B. The designation K_{12} simply stands as an abbreviation of a Karnaugh map of 1 row and 2 columns.

The 2-bit case has twice as many elements as K_{12} (17) and encodes two bits in a single cartesian dimension:

K_{14}	00	01	11	10

(18)

This is equivalent to the square in (7) by spanning each edge with the binary indices $0, 1$ of K_{12} from the upper left corner:

K_{22}	0	1
0	00	01
1	10	11

(19)

K_{42} and K_{24} are rectangular but may be folded into a $2 \times 2 \times 2$ cube. Since these have no resulting square, the rest of this paper focuses on the square Karnaugh maps of order 2^n which do result in a square.

Each edge of a square is now spanned with the Gray code of K_{14} in (18) , or K_{41}, to find K_{44}:

K_{44}	00	01	11	10
00	0000	0001	0011	0010
01	0100	0101	0111	0110
11	1100	1101	1111	1110
10	1000	1001	1011	1010

(20)

Note that (20) corresponds to the folding of the 16-element 4-bit Gray sequence in (13, 14) across successive rows. The antipodes (the pair consisting of the entry of a cell and its bit reversed opposite) lie spaced $p = (N = 2^n)/2$ steps along a pandiagonal, e.g. in (20) the antipodes are separated by 2 steps as $n = 2$, $N = 4$. The square K_{44} could also be thought of as a $2 \times 2 \times 2 \times 2$ four-dimensional hypercube [6].

4.1 The Basic Logic Square, $n = 1$, $N = 2$

Using (9), K_{22} in (19) becomes the "logic" square L_2:

$$L_2 = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \tag{21}$$

which is pandiagonal with a sum of 3, but clearly non-magic. This agrees with the magic square line sum formula for a sequence beginning with zero. For the sequence $0..(n^2 - 1)$ this sum is:

$$c_n = \frac{1}{2}n(n^2 - 1) \tag{22}$$

c.f. C_n in (1) which is written in the alternate case.

4.1.1 Determinant

The determinant of L_2 is -2, whereas it has been explicitly verified that the determinant vanishes for the larger L_4, L_8, L_{16}, and L_{32}, meaning that they are singular matrices, and it is conjectured that this holds for all larger squares.

4.1.2 Eigenproperties of L_2

Suppose that these logic squares are treated as matrices so that their eigenproperties can be investigated as Hruska [18], Trenkler [23] and Thompson [24] have done for the 3×3 magic square.

The eigenvector (the column in curly brackets below) and eigenvalue pairs of L_2 are:

$$\left\{ \begin{matrix} 1 \\ \frac{3}{2}+\frac{1}{2}\sqrt{17} \end{matrix} \right\} \leftrightarrow \frac{3}{2}+\frac{1}{2}\sqrt{17} \rightarrow 3.561\,55, \tag{23}$$

$$\left\{ \begin{matrix} 1 \\ \frac{3}{2}-\frac{1}{2}\sqrt{17} \end{matrix} \right\} \leftrightarrow \frac{3}{2}-\frac{1}{2}\sqrt{17} \rightarrow -.561\,55 \tag{24}$$

This document was prepared using a version of Maple®in Scientific Workplace 5.0 [25], a LaTeX-based system for consistent output in all calculations. Since Maple gives unreduced arithmetic answers in surds, which help understand the origins of the results, it is often useful to add a numerical rendering as given above.

The eigenvalues also follow from the characteristic equation obtained by setting the following determinant equal to zero:

$$\det(L_2 - xI) = \left| \begin{matrix} 0 & 1 \\ 2 & 3 \end{matrix} \right| - x \left| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right| = \left| \begin{matrix} -x & 1 \\ 2 & 3-x \end{matrix} \right| = 0$$

The solutions of this quadratic:

$$x^2 - 3x - 2 = 0$$

clearly give the eigenvalues above. The sum of the two eigenvalues is the diagonal sum (trace) of 3.

4.2 The Second Logic Square, $n = 2$, $N = 4$

Now take K_{44} and translate the bit-strings to integers with (9) to find the logic square, L_4, which was already given above in (16). Note in (16) that the antipodes of (20), 0 and 15, 5 and 10, 3 and 12, 9 and 6, etc., all sum to 15, which is half of c_4.

L_4 has a vanishing determinant and the following eigenproperties:

$$\left\{\begin{matrix} 1 \\ \frac{23}{14}+\frac{1}{14}\sqrt{305} \\ \frac{41}{14}+\frac{3}{14}\sqrt{305} \\ \frac{16}{7}+\frac{1}{7}\sqrt{305} \end{matrix}\right\} \leftrightarrow 15+\sqrt{305}, \left\{\begin{matrix} 1 \\ \frac{23}{14}-\frac{1}{14}\sqrt{305} \\ \frac{41}{14}-\frac{3}{14}\sqrt{305} \\ \frac{16}{7}-\frac{1}{7}\sqrt{305} \end{matrix}\right\} \leftrightarrow 15-\sqrt{305}, \qquad (25)$$

$$\left\{\begin{matrix} 2 \\ -3 \\ 1 \\ 0 \end{matrix}, \begin{matrix} 1 \\ -2 \\ 0 \\ 1 \end{matrix}\right\} \leftrightarrow 0 \qquad (26)$$

The sum of the eigenvalues is the diagonal sum of 30, or the magic line sum (22). Note the two zero eigenvalues and the integer elements of their eigenfunctions which sum in each case to zero. **This pattern has been verified for all larger logic squares up to L_{32}, and is conjectured to hold generally.**

The characteristic equation is:

$$\det(L_4 - xI) = x^2(x^2 - 30x - 80) = 0 \qquad (27)$$

where the double zero is obvious, and the remaining quadratic gives the other pair of solutions above.

4.2.1 The Original Pandiagonal MBTI matrix

At the time that Loly [5] originally digitized the MBTI table he did not know about the Gray code, nor about Karnaugh maps, but the resulting type matrix can now be compared with K_{44} in (20) by indexing the 4-bit strings in a modified way:

MBTI	·11·	·10·	·00·	·01·
$0 \cdot \cdot 1$	0111	0101	0001	0011
$0 \cdot \cdot 0$	0110	0100	0000	0010
$1 \cdot \cdot 0$	1110	1100	1000	1010
$1 \cdot \cdot 1$	1111	1101	1001	1011

(28)

In retrospect the binary labelling used in (28) corresponds to a different folding of the 16 element Gray string where the middle bits of the 4-bit strings are chosen to be a_2 and a_1 in (9). [5]. In the language of the MBTI, type 0011 corresponds to INTJ (Introverted iNtuitive Thinking Judging), whereas the antipodal type 1100 is ESFP (Extroverted Sensing Feeling Perceiving).

In (28) the rows are indexed by a version of K_{14} in (18) running backwards from the second cell and continuing from the other end at the right: $01, 00, 10, 11$, AND then being used for the extreme left and right components of the 4-bit strings. The columns use another version of K_{14} for the two middle bits running forward from the third cell and continuing from the other end at the left. The translation to integers running from 1..16 was afforded by adding 1 to each result from the algorithm given in (9):

$$MB = \begin{bmatrix} 8 & 6 & 2 & 4 \\ 7 & 5 & 1 & 3 \\ 15 & 13 & 9 & 11 \\ 16 & 14 & 10 & 12 \end{bmatrix} \tag{29}$$

This has eigenproperties:

$$\left\{ \begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 2 & 1 \\ -3 & -2 \end{array} \right\} \leftrightarrow 0, \left\{ \begin{array}{c} 1 \\ \frac{7}{5} \\ -\frac{9}{5} \\ -\frac{11}{5} \end{array} \right\} \leftrightarrow 4, \left\{ \begin{array}{c} \frac{4}{3} \\ 1 \\ \frac{11}{3} \\ 4 \end{array} \right\} \leftrightarrow 30 \tag{30}$$

MB has a characteristic equation, $x^2\left(120 - 34x + x^2\right) = 0$, showing the double singularity, and thus a zero determinant, in parallel with L_4 in (16). The eigenvalue sum is 34, which is the same as for the corresponding pandiagonal magic square.

4.3 The Third Logic Square, $n = 3$, $N = 8$

Now the 3-bit Gray code (11) is used to span each edge, followed by the translation of the 6-bit strings to integers (9) to find the 64-element L_8:

$$L_8 = \begin{bmatrix} 0 & 1 & 3 & 2 & 6 & 7 & 5 & 4 \\ 8 & 9 & 11 & 10 & 14 & 15 & 13 & 12 \\ 24 & 25 & 27 & 26 & 30 & 31 & 29 & 28 \\ 16 & 17 & 19 & 18 & 22 & 23 & 21 & 20 \\ 48 & 49 & 51 & 50 & 54 & 55 & 53 & 52 \\ 56 & 57 & 59 & 58 & 62 & 63 & 61 & 60 \\ 40 & 41 & 43 & 42 & 46 & 47 & 45 & 44 \\ 32 & 33 & 35 & 34 & 38 & 39 & 37 & 36 \end{bmatrix} \tag{31}$$

L_8 also has zero determinant and non-zero eigenvalues: $126+2\sqrt{4641}, 2\sqrt{4641}-126$ which give the pandiagonal sum of 252. The eigenvectors of the degenerate zero eigenvalues all have simple integers summing to zero, as was the case for L_4.

Besslich [13] gave 4×4 and 8×8 pandiagonal logic squares in 1983 in a paper on dyadic indexed data arrays which, though not identical to L_4 and L_8, are clearly related to them. In Appendix A.2 Maple procedures are given for the construction of these logic squares. When these are run for the construction and checking of L_{16} and L_{32}, the same pattern of determinant and eigenproperties are found, and it is conjectured that this pattern holds for all larger logic squares.

4.4 Proof of the Pandiagonality Non-Magic Property

Since the bit strings of a square Karnaugh map consist of equal length left and right bit strings, and since each half is a standard Gray code bit string, then it is clear that for a logic square of order 2^n there are $\frac{1}{2}2^n$ bits in each place leading to a total of:

$$\frac{1}{2}2^n\sum_{p=1}^{2n}(1+2+...2^p)=\frac{1}{2}2^n\left(2^{2n}-1\right) \tag{32}$$

This is just c_N in (22) for $N=2^n$, for any combination of N elements which uses each row and column exactly once, e.g. a pandiagonal, or one of Mayoral's more general orthogonal complement square sums [9], but obviously never a row or column.

5 Sequential Pandiagonal Squares: S_N

After being drawn from the 4×4 MBTI matrix to Karnaugh maps, it seems appropriate to go back and examine the "ordered sequence" of L_2 in (21) extended to higher orders. Consider S_3 in (4) which has zero determinant and eigenproperties:

$$\left\{\begin{array}{c} -\frac{1}{2}+\frac{3}{22}\sqrt{33} \\ \frac{1}{4}+\frac{3}{44}\sqrt{33} \\ 1 \end{array}\right\} \leftrightarrow \frac{15}{2}+\frac{3}{2}\sqrt{33}, \left\{\begin{array}{c} -\frac{1}{2}-\frac{3}{22}\sqrt{33} \\ \frac{1}{4}-\frac{3}{44}\sqrt{33} \\ 1 \end{array}\right\} \leftrightarrow \frac{15}{2}-\frac{3}{2}\sqrt{33}, \quad (33)$$

$$\left\{\begin{array}{c} 1 \\ -2 \\ 1 \end{array}\right\} \leftrightarrow 0 \quad (34)$$

Note the zero eigenvalue. Then look at S_4:

$$S_4 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \quad (35)$$

Again with zero determinant, and 2 non-zero eigenvalues:

$$\left\{\begin{array}{c} -\frac{17}{8}+\frac{3}{8}\sqrt{41} \\ -\frac{9}{16}+\frac{3}{16}\sqrt{41} \\ 1 \\ \frac{41}{16}-\frac{3}{16}\sqrt{41} \end{array}\right\} \leftrightarrow 17+3\sqrt{41}, \left\{\begin{array}{c} -\frac{17}{8}-\frac{3}{8}\sqrt{41} \\ -\frac{9}{16}-\frac{3}{16}\sqrt{41} \\ 1 \\ \frac{41}{16}+\frac{3}{16}\sqrt{41} \end{array}\right\} \leftrightarrow 17-3\sqrt{41}, \quad (36)$$

$$\left\{\begin{array}{c} 2 \\ -3 \\ 0 \\ 1 \end{array}, \begin{array}{c} 1 \\ -2 \\ 1 \\ 0 \end{array}\right\} \leftrightarrow 0 \quad (37)$$

Even in the absence of a general proof, it seems reasonable to conjecture that the entire S_N family has only two non-zero eigenvalues.

6 General Features of the Eigenspectra

In all cases studied here there are two non-zero eigenvalues, the sum of which gives the diagonal magic sum. The eigenvectors of the zero eigenvalues all have elements that sum to zero. These are intriguing properties that are not yet fully understood.

As a step in that direction the characteristic equation for a general 4×4 matrix may be expressed as:

$$x^4 + \alpha x^3 + \beta x^2 + \gamma x + \delta = 0 \tag{38}$$

in which the constant term δ is the determinant of the coefficients, here quadruple products; γ consists of triple products; β consists of quadratic terms, six products of pairs of diagonal coefficients and six off-diagonal reflection pair products, and α is the negative trace (the sum of the diagonal terms AND the magic line constant). For L_4 both δ and γ vanish.

For L_8 in (31) terms up to and including the fifth power must also vanish, and so on for larger logic squares. There is perhaps some hope with the logic squares, $L_4,\ L_8,\ L_{16}, ...$, that their intrinsic binary structure may lead to an understanding of both the non-vanishing of the coefficient of highest three powers, x^N, x^{N-1}, x^{N-2} in the generalisation of (38) and the vanishing of all lower powers of x. It may also be possible in a future study to use some algebra given by Andress [7] to help with that effort.

7 Conclusion

In this paper parallels have been discovered between techniques used in the design of digital electronic logic circuits and logical arrays used in other fields. The essential bridge has been the 4×4 case which is closely related to the MBTI [5], the original starting point for these investigations. Small square Karnaugh maps have been explored as a basis for logic squares of orders $N = 2^n$, $n = 1, 2, 3, ..$ for $N = 2, 4, 8, ...$ Since larger pandiagonal, $N = .., 64, ..1024, ..4096, ..$, non-magic matrices can be easily constructed with the Maple procedures provided in the Appendix, it may be worthwhile to revisit some of Besslich's applications [13], e.g. his use of pandiagonal squares as dither matrices [12]. In a separate work Loly [26] has shown how the Gray code and Karnaugh map concepts may be used to logically reorganize ancient Chinese patterns based on the combinations "opposite" pairs called yin and yang, ranging from the 4-fold 2×2 compass motif based on 2-bit stacks of yin and yang lines (tantamount to L_2), through the eight trigrams of the Pa-kua based on 3-bit stacks, which is used in the practice of Feng Shui, to the 8×8 which offers a logical rearrangement of the 64 hexagrams (6-bit stacks tantamount to L_8) of the I Ching [6]. A board game also maps onto the type table [27].

Chan and Loly [1] have recently updated some old ideas for compounding (multiplying) a pair of number squares to construct product squares whose orders are the product of the orders of the input pair, resulting in very large magic squares. Compounded squares have the lowest set of common global properties, e.g. a pandiagonal magic square and one of our pandiagonal logic squares gives a larger pandiagonal square which is not magic. In the present context very large pandiagonal non-magic squares can be constructed by compounding, but even if they are both logic squares it is not clear that the resulting square would be another logic square with the 1-bit property.

Finally, the existence of only two non-zero eigenvalues, for both the logic and the serial squares, calls for a deeper understanding of the roots of this property. In a parallel study of magic squares [19] an 8×8 most-perfect pandiagonal magic square [8] was found to have only three non-zero eigenvalues, one from the n-agonal $[1, 1, .., 1]$ eigenvector giving the magic constant as must always exist from the semimagic property, and the other pair adding to zero. Also, in another recent study, Schindler and Loly [28] have found maximally singular regular (associative) magic squares in orders four and five, as well as compounded multiples of those orders, which have just one non-zero eigenvalue (for the magic line sum).

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A Maple Codes for Order 2^n Logic Squares

Maple codes to automatically generate square Karnaugh maps, perform binary translation and test resulting properties are listed below.

A.1 Generating Gray Strings

```
gray := proc(m)
    local A,M,x,w;
    A := matrix(1,2^m):A[1,1] := 0:
    for w from 1 to m do
        for x from 1 to 2^(w-1) do
            A[1,2^(w-1) +x] := (2^(w-1) + A[1,2^(w-1) -x +1]):
        od:
    od:
    RETURN(evalm(A));
    end:
```

Example of 3-bit Gray coded integer sequence:

```
>gray(3);
```

output:

```
[0 1 3 2 6 7 5 4]
```

A.2 Square Karnaugh Matrices

```
karnaugh := proc(m)
    local M,A,x,y;
    A := gray(m);
    M := matrix(2^m, 2^m):
    for x from 1 to 2^m do
        for y from 1 to 2^m do
            M[x,y] := (2^m)*A[1,x] + A[1,y]:
        od:
    od:
```

```
    RETURN(evalm(M));
    end:
```

Example of a logic square:

```
    >karnaugh(2);
```

output:

```
    [[ 0 1 3 2][ 4 5 7 6][12 13 15 14][ 8 9 11 10]]
```

A.3 Test Pandiagonality of Square Karnaugh Matrices

```
diagonal := proc(M)
    local diag,n,u,s,k,v;
    n := rowdim(M);
    s := sum(M[k,k],k=1..n):
    for u from 1 to n do
        if (sum(M['v','modp((v+u)-2,n)+1'], v=1..n) <>s)
            then diag := 'no'; fi;
        if (sum(M['n+1-v','modp((v+u)-2,n)+1'], v=1..n) <>s)
        then diag := 'no'; fi;
    od:
        if (diag <>'no')
            then diag :='yes': fi:
        if (diag = 'yes')
            then RETURN(yes);
        else RETURN(no);
    fi;
    end:
```

Example of testing the matrix in A.3:

>diagonal(karnaugh(2));

`output:`

yes